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# Solutions for Some Elliptic Problems with Double Resonance*

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**Abstract** In this paper, we prove the existence results and multiplicity results of nontrivial solutions for some elliptic problems with double resonance by using Morse theory.

## 1. Introduction

In this paper, we consider the nontrivial solutions for the Dirichlet boundary value problem by using Morse theory,

$$\begin{cases} -\Delta u = p(x,u), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $p \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$, such that $p(x,0) = 0$.

We assume that

$$p_0 := \lim_{u\to 0} \frac{p(x,u)}{u} = \lambda_m, \qquad p_\infty := \lim_{|u|\to\infty} \frac{p(x,u)}{u} = \lambda_k \tag{1.2}$$

which characterizes (1.1) as double resonance at both zero and infinity. Denote $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$ to be the distinct eigenvalues sequence of $-\Delta$ in $H_0^1(\Omega)$. The resonant problem has been widely studied by many authors using various methods under various assumptions on nonlinearity $p$ and its primitive $P$. See [3, 4, 5, 6, 7, 8, 9, 10, 11] and the references therein. We will give conditions under which the problem (1.1) has nontrivial solution. We also allow the case when $\lambda_m = \lambda_k$. For some special cases we consider its multiple solutions.

In section 2, we give some preliminaries for our paper, which are preliminary to the computations of critical groups at degenerate critical points. In section 3,

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we prove our main theorems, which result in the existence and multiplicity of nontrivial solutions.

## 2. Preliminaries

Let $X := H_0^1(\Omega)$ is the usual Sobolev space with the inner product and the norm

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v, \quad \|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

Define the functional $f : H_0^1(\Omega) \to \mathbb{R}$ as

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega P(x,u)dx,$$

where $P(x,u) = \int_0^u p(x,t)dt$. Thus solutions of the problem (1.1) are critical point of the functional $f$. Corresponding to the eigenvalue $\lambda_n$, $H_0^1(\Omega)$ can be splitted as

$$H_0^1(\Omega) = W^- \oplus V \oplus W^+$$

where

$$W^- = \oplus_{j<n} \ker(-\Delta - \lambda_j), \quad V = \ker(-\Delta - \lambda_n), \quad W^+ = (W^- \oplus V)^\perp.$$

Denote by

$$H_0^1(\Omega) = W_*^- \oplus V_* \oplus W_*^+ \quad (* = 0, \infty)$$

the decomposition corresponding to $\lambda_m$, $\lambda_k$, respectively.

Set

$$q_\infty(x,t) = p(x,t) - p_\infty t, \qquad Q_\infty(x,t) = \int_0^t q_\infty(x,\tau)d\tau$$

$$q_0(x,t) = p(x,t) - p_0 t, \qquad Q_0(x,t) = \int_0^t q_0(x,\tau)d\tau$$

We shall use the following assumptions:

$(p_0)_\pm$, $\dfrac{Q_\infty(x,t)}{|t|} \to \pm\infty$, $|t| \to \infty$;

$(p_1)$ $\exists\, c > 0$, $\beta > 0$ such that $|q_0(x,u)| \le c|u|^\beta$, $|u| < 1$, $x \in \Omega$;

$(p_2)_\pm$ $\dfrac{Q_0(x,u)}{u^{2\beta}} \to \pm\infty$, as $|u| \to 0$ uniformly in $x \in \overline{\Omega}$.

It will be seen that critical groups and Morse theory are the main tools we use to solve our problems. Now let us to recall some results used below. We refer the readers to the books [1] for more information on Morse theory.

Let $X$ be the Banach space and $f \in C^1(X, \mathbb{R})$ be a functional satisfying the compactness condition $(PS)$, and $H_q(A, B)$ be the $q$th singular relative homology group with integer coefficients. Let $u_0$ be an isolated critical point of $f$ with $f(u_0) = c \in \mathbb{R}$, and $U$ be a neighborhood of $u_0$. The group

$$C_q(f,u_0) := H_q(f^c \cap U, f^c \cap U \backslash \{u_0\}), \quad q \in N_0 := \{0,1,2,\cdots\}$$

is called the $q$th critical group of $f$ at $u_0$, where $f^c = \{u \in X : f(u) \leq c\}$. Let $K := \{u \in X : f'(u) = 0\}$ be the set of critical points of $f$ and $\alpha < \inf f(K)$. The critical groups of $f$ at infinity are formally defined as [3]

$$C_q(f,\infty) := H_q(X, f^\alpha), \quad q \in N_0.$$

The following results are used to prove the results in our paper.

**Proposition 1** ([3]). *Assume that $X$ is a Banach space $X = X^- \oplus X^+$, $\dim X^- = l < \infty$ and $f \in C^1(X,\mathbb{R})$ satisfies the $(PS)$ condition. If $f$ is bounded form below on $X^+$ and*

$$f(u) \to -\infty, \quad as\ \|u\| \to \infty,\ u \in X^-.$$

*Then*

$$C_l(f,\infty) \neq 0.$$

**Proposition 2** ([3]). *Assume that $f \in C^1(X,\mathbb{R})$ satisfies the $(PS)$ condition,*

(i) *If there exists some $k \in N_0$, s.t. $C_k(f,\infty) \neq 0$. Then $f$ has a critical point $u$ satisfying $C_k(f,u) \neq 0$.*

(ii) *Assume 0 is and isolated critical point. If there exists some $k \in N_0$, s.t. $C_k(f,\infty) \neq C_k(f,0)$. Then $f$ has a nontrivial critical points.*

**Proposition 3** ([1]). *Assume that $X$ is a Hilbert space, $f \in C^2(X,\mathbb{R})$ and $u_0$ is an isolated critical point with Morse index $\mu$ and the nullity $\nu$. If $f''(u_0)$ is a Fredholm operator. Then for $q \notin [\mu,\mu+\nu]$, $C_q(f,u_0) \cong 0$. Furthermore*

(i) *$C_\mu(f,u) \neq 0$ implies $C_q(f,u) \cong \delta_{q,\mu}\mathcal{G}$;*

(ii) *$C_{\mu+\nu}(f,u) \neq 0$ implies $C_q(f,u) \cong \delta_{q,\mu+\nu}\mathcal{G}$.*

In section 3, we will give the proof of our main theorems and give more existence results.

## 3. Proof of our main results

**Theorem 3.1.** *Let $(p_0)_+$ and $(p_1)$ hold. Then the problem (1.1) has at least one nontrivial solution in each of the following cases:*

(a) *$(p_2)_+$ and $m \neq k$;*

(b) *$(p_2)_-$ and $m \neq k+1$.*

***Proof.*** We just consider the critical points of the functional $f : X = H_0^1(\Omega) \to \mathbb{R}$

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega P(x,u)dx.$$

It is obvious that $f \in C^2$. As the proof of [9], under $(p_0)_+$ the functional $f$ satisfies the $(PS)$ condition and

$$f(u) \to -\infty, \quad \text{as } \|u\| \to \infty \text{ with } u \in V_\infty \oplus W_\infty^-,$$

$$f(u) \to +\infty, \quad \text{as } \|u\| \to \infty \text{ with } u \in W_\infty^+.$$

By Proposition 1,

$$C_\mu(f,\infty) \neq 0, \quad \mu = \dim(W_\infty^- \oplus V_\infty) = \sum_{j=1}^{k} \dim\ker(-\Delta - \lambda_j).$$

It follows Proposition 2 or the Morse inequality that $f$ has a critical point $u^*$, such that

$$C_\mu(f,u^*) \neq 0. \tag{3.1}$$

(a) If $(p_1)$ and $(p_2)_+$ hold, then by Proposition 2 [5] we have

$$C_q(f,0) = \delta_{q,\mu_0}\mathcal{G} \tag{3.2}$$

where $\mu_0 = \sum_{i=1}^{m} \dim\ker(-\Delta - \lambda_i)$.

Now $m \neq k$ implies $\mu \neq \mu_0$. It follows (3.1) and (3.2) that

$$C_q(f,0) \neq C_q(f,u^*). \tag{3.3}$$

Hence $u^* \neq 0$ is a nontrivial solution of (1.1).

(b) If $(p_1)$ and $(p_2)_-$ hold, then by Proposition 2 [5] we have

$$C_q(f,0) = \delta_{q,\mu_0}\mathcal{G} \tag{3.4}$$

where $\mu_0 = \sum_{j=1}^{m-1} \dim\ker(-\Delta - \lambda_j)$.

Now $m \neq k+1$ implies $\mu \neq \mu_0$. It follows (3.1) and (3.4) that (3.3) holds. Again $u^* \neq 0$ is a nontrivial solution of (1.1). □

**Remark 3.1**. In (a), we still get the same results under the condition $p_0$ with $s \geq 2$, $k \geq 2$ in paper [6].

Now, we give a dual version of Theorem 3.1.

**Theorem 3.2.** *Let $(p_0)_-$ and $(p_1)$ hold. Then the problem (1.1) has at least one nontrivial solution in each of the following cases:*

(a) $(p_2)_+$ *and* $m \neq k-1$;

(b) $(p_2)_-$ *and* $m \neq k$.

***Proof.*** As the proof of Theorem 3.1, under $(p_0)_-$ the functional $f$ still satisfies the $(PS)$ condition. Moreover, $f$ has the following properties:

$$f(u) \to -\infty, \quad \text{as } \|u\| \to \infty \text{ with } u \in W_\infty^- \tag{3.5}$$

$$f(u) \to +\infty, \quad \text{as } \|u\| \to \infty \text{ with } u \in V_\infty \oplus W_\infty^+. \tag{3.6}$$

Thus by Proposition 1 we have

$$C_\mu(f,\infty) \neq 0, \quad \text{with } \mu = \sum_{j=1}^{k-1} \dim\ker(-\Delta - \lambda_j).$$

The rest of the proof is similar to the proof of Theorem 3.1. □

**Remark 3.2.** Under more conditions, we can get more solutions and more information about those solutions as in paper [6].

**Remark 3.3.** Paper [10] gets the similar results as ours, while those condition in [10] are stronger, such as $(f_{1-3})$, which are necessary to the $(PS)$ of the functional $J$.

Multiplicity results:

**Theorem 3.3.** *Let* $(p_0)_-$, $(p_1)$ *and* $k = 1$ *hold. Then the problem* (1.1) *has at least two nontrivial solution in each of the following cases*:

(a) $(p_2)_+$ *and* $m \geq 1$;
(b) $(p_2)_-$ *and* $m > 1$.

***Proof.*** Since $k = 1$ we see that $W_\infty^- = \emptyset$. By $(p_0)_-$ and (3.5), $f$ is bounded from below. Therefore

$$C_q(f,\infty) \cong \delta_{q,0}\mathscr{G}. \tag{3.7}$$

By Proposition 2, $f$ has a critical point $u_0$, s.t.

$$C_0(f,u_0) \ncong .$$

In fact, $u_0$ is the global minimum of $f$. Hence

$$C_q(f,u_0) \cong \delta_{q,0}\mathscr{G}. \tag{3.8}$$

Now we know that $u = 0$ is a degenerate critical point of $f$ and the critical groups of $f$ at $u = 0$ are

in case (a)

$$C_q(f,0) \cong \delta_{q,\mu_0}\mathscr{G} \ \text{ with } \mu_0 = \sum_{j=1}^{m} \dim\ker(-\Delta - \lambda_j);$$

in case (b)

$$C_q(f,0) \cong \delta_{q,\mu_0}\mathscr{G} \ \text{ with } \mu_0 = \sum_{j=1}^{m-1} \dim\ker(-\Delta - \lambda_j).$$

It follows from case (a) $m \geq 1$ or case (b) $m > 1$ that $u_0 \neq 0$. If the critical set $K = \{u_0, 0\}$, then by the More inequality we have

$$(-1)^0 + (-1)^n = (-1)^0, \quad \text{with } n = \mu_0.$$

This is impossible. Therefore $f$ must have another critical point $u_1$ different from $u_0$ and 0. Moreover, by Proposition 3 the critical groups of $f$ at $u_1$ satisfy

$$\text{either} \quad C_{n-1}(f,u_1) \ncong 0 \quad \text{or} \quad C_{n+1}(f,u_1) \ncong 0. \tag{3.9}$$

Then the Morse index $\mu_1$ and the nullity $\nu_1$ of $u_1$ satisfy

$$\text{either} \quad \mu_1 \le n-1 \quad \text{or} \quad \mu_1 + \nu_1 \ge n+1. \tag{3.10}$$

Hence the proof is complete. □

**Remark 3.4**. We would like to point out that based on the Morse theory, we get more information about the second nontrivial solution than the three critical points theorem [5, Theorem 1].

**Theorem 3.4.** *Assume that $q_\infty$ is bounded, $k=1$ and*

$$\int_\Omega Q_\infty(x,u)dx \to +\infty \quad \text{as } \|u\| \to \infty \text{ with } u \in V_\infty. \tag{3.11}$$

*Then* (1.1) *has at least two nontrivial solutions in each of the following cases:*

(a) $(p_1),(p_2)_+$ *and* $m \ge 1$;

(b) $(p_1),(p_2)_-$ *and* $m \ne 2$.

***Proof.*** Since $q_\infty$ is bounded and (3.11) holds, $f$ satisfies the conditions of Lemma 5.2 [1, Chapter II]. It follows that

$$C_q(f,\infty) \cong \delta_{q,1}\mathcal{G}.$$

Thus by Proposition 2, $f$ has a critical point $u_0$ s.t.

$$C_1(f,u_0) \ncong 0.$$

Hence $u_0$ is a mountain pass point of $f$ [2] and then

$$C_q(f,u_0) \cong \delta_{q,1}\mathcal{G}. \tag{3.12}$$

Similar arguments as Theorem 3.3 show that $f$ has another critical point $u_1$ different from $u_0$ and 0. Moreover, the nontrivial solution of (1.1) still satisfies (3.9) and (3.10). □

Now we give a dual version of Theorem 3.4

**Theorem 3.5.** *Assume that $q_\infty$ is bounded, $k=2$ and*

$$\int_\Omega Q_\infty(x,u)dx \to -\infty \quad \text{as } \|u\| \to \infty \text{ with } u \in V_\infty. \tag{3.13}$$

*Then* (1.1) *has at least two nontrivial solutions in each of the following cases*:

(a) $(p_1)$, $(p_2)_+$ *and* $m \ge 1$;

(b) $(p_1)$, $(p_2)_-$ *and* $m \ne 2$.

***Proof.*** Since $q_\infty$ is bounded and (3.13) holds, $f$ satisfies the conditions of Theorem 1.2 [4]. It follows that

$$C_q(f,\infty) \cong \delta_{q,1}\mathscr{G}.$$

Thus $f$ has a critical point $u_0$ satisfying (3.12). Similar arguments as Theorem 3.3 show that $f$ has another critical point $u_1$ different from $u_0$ and 0. Moreover, the nontrivial critical point $u_1$ satisfies (3.9) and (3.10). □

**Remark 3.5**. In paper [6], they consider the solutions of (1.1) in the case $k \geq 2$ and $\lambda_m \neq \lambda_k$, while our results allow the case when $\lambda_m = \lambda_k$ including $k = 1$.

## References

[1] K.C. Chang, *Infinite Dimensional Morse Thoery and Multiple Solutios Problems*, Birkhauser, Boston, 1993.

[2] K.C. Chang, Morse theory in nonlinear analysis, *Proc. Sympos. ICTP* (1997).

[3] T. Bartsch and S.J. Li, Critical point theory for asymptotically quadratic functionals and applications to problems with resonance, *Nonl. Anal.* **28** (1997), 419–441.

[4] S.J. Li and J.Q. Liu, Computations of critical groups at degenerate ciritical point and applications to nonlinear differential equations with resonace, *Houson J. Math.* **25** (1999), 563–582.

[5] S.J. Li and M. Willem, Multiple solutions for asymptotically linear boundary value problems in which the nonlineariy cross at least one eigenvalue, *Nonl. Diff. Equa. Appl.* **5** (1998), 479–490.

[6] S.B. Liu, *Critical Groups at Infinite, Multiple Solutions for Nonlinear Elliptic Equations*, Doctorial Dissertation, 2003.

[7] R. Molle and D. Passaseo, Nonlinear elliptic equations with large supercritical exponents, *Calc. Var.* **26** (2006), 201–225.

[8] A.X. Qian, Neumann problem of elliptic equation with strong resonance, *Nonl Anal T.M.A.* **66** (2007), 1885–1898.

[9] M. Ramos, Remarks on resonance problems with unbounded perturbations, *Diff. Intg. Eqns.* **6** (1998), 215–223.

[10] J.B. Su, Existence and multiplicity results for classes of elliptic resonant problems, *J. Math. Anal. Appl.* **273** (2002), 565–579.

[11] Z.T. Zhang and S.J. Li, On sign-changing and multiple solutions of the $p$-laplacian, *J. Funt. Anal.* **197** (2003), 447–468

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