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Research Article

Method of Construction and Some Properties of 4-Row-Regular Circulant Partial Hadamard Matrices of Order $(k \times 2k)$

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Abstract. In this paper, some properties of circulant partial Hadamard matrices of the form $4-H(k\times 2k)$ have been obtained together with a method of construction with the help of Toeplitz matrices.

Keywords. Hadamard matrices, Partial Hadamard matrices, Circulant partial Hadamard matrices, Toeplitz matrix, Orthogonal design

Mathematics Subject Classification (2020). 05B15, 05B30

1. Introduction

A square matrix H of order n is called Hadamard matrix if $HH' = nI_n$. Hadamard matrices are row orthogonal matrices. The order of a Hadamard matrix is 1 or 2 or $4t$ where t is a positive integer. Whether there exists a Hadamard matrix of order n for a given positive integer n which is divisible by 4? This is a challenging question for the mathematical community and is an open problem in mathematics known as the Hadamard matrix conjecture. Despite a large number of methods for constructing Hadamard matrices, the Hadamard matrix conjecture remains one of the most long-standing problems [4, 8]. Row orthogonal rectangular matrices of order $k\times n$ are known as partial Hadamard matrices. Partial Hadamard matrices are a generalization of Hadamard matrices. The partial Hadamard matrices which are formed by the right cyclic shift of its row are known as circulant partial Hadamard matrices. The known circulant Hadamard matrices are of order 4. Ryser [12] conjectured that there does not exist any circulant partial Hadamard matrix of order greater than 4 (see also, Euler [2]). This led to concentrating the study on circulant partial Hadamard matrices. In 2013, Craigen *et al*. [1] have been obtained many new results on circulant partial Hadamard matrices including a table which gives the maximum number of rows possible in a circulant partial Hadamard matrices up to order 64. The table obtained by Craigen *et al*. [1] is by computer search. Kao [5, 6] identified applications of circulant partial Hadamard matrices in the construction of functional design resonance imaging (fMRI) experimental designs. This article gives a method of construction of circulant partial Hadamard matrices of type $4-H(k\times 2k)$. The only known circulant partial Hadamard matrices of this type are for $k=6$ and $k=14$. Recently, some new examples of this type of circulant partial Hadamard matrices are given by Manjhi and Rana [9], but not of new unknown orders. The first method of construction of this type of circulant partial Hadamard matrices is given by Manjhi and Rana [10]. This article also provides some new properties of circulant partial Hadamard matrices including a new method of construction with the help of Toeplitz matrices.

2. Preliminaries

A square matrix H of order n and with entries ± 1, is called a Hadamard matrix if $HH'=nI_n$. Some examples of Hadamard matrices are given below:

$$(1),\ \begin{pmatrix} 1 & 1\\ -1 & 1\end{pmatrix},\ \begin{pmatrix} 1 & 1 & 1 & -1\\ 1 & 1 & -1 & 1\\ 1 & -1 & 1 & 1\\ -1 & 1 & 1 & 1\end{pmatrix},\ \begin{pmatrix} -1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1\end{pmatrix}\ \text{etc.}$$

The latest construction of the new Hadamard matrix of order 428 is given by Kharaghani and Tayfeh-Rezaie in 2004 ([7]). Orders of Hadamard matrices less than 1000 that are still unknown are 668, 716, 892 and 956.

Let $x=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{n-1} & \alpha_n\end{pmatrix}\in\mathbb{R}^n$, then circulant matrix $C(x)$, semicirculant matrix $S(x)$ and negacyclic matrix $N(x)$ with the first-row $x=\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\end{pmatrix}$ are defined as follows:

$$C(x)=\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\\ \alpha_n & \alpha_1 & \dots & \alpha_{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_2 & \alpha_3 & \dots & \alpha_1\end{pmatrix},\ S(x)=\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\\ 0 & \alpha_1 & \dots & \alpha_{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \alpha_1\end{pmatrix}\ \text{and}\ N(x)=\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\\ -\alpha_n & \alpha_1 & \dots & \alpha_{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ -\alpha_2 & -\alpha_3 & \dots & \alpha_1\end{pmatrix},$$

respectively.

That is, if $C(x)=(c_{ij})$, $S(x)=(s_{ij})$ and $N(x)=(n_{ij})$, then $c_{ij}=\alpha_{(j-i)(\bmod n)+1}$,

$$s_{ij}=\begin{cases}\alpha_{(j-i)(\bmod n)+1}, & \text{for all } j\ge i\\ 0, & \text{otherwise}\end{cases}\quad\text{and}\quad n_{ij}=\begin{cases}\alpha_{(j-i)(\bmod n)+1}, & \text{if } j\ge i\\ -\alpha_{(j-i)(\bmod n)+1}, & \text{if } j<i\end{cases},\ \text{respectively.}$$

A matrix $T=(t_{i,j})$ is called Toeplitz matrix if $t_{i,j}=t_{i-1,j-1}$, for all i,j. Circulant, semicirculant and negacyclic matrices are particular types of Toeplitz matrices. Following are some examples of Toeplitz matrices:

$$\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},\ \begin{pmatrix}a & b & c\\ d & a & b\\ e & d & a\end{pmatrix},\ \begin{pmatrix}a & b & c & d & e & f\\ g & a & b & c & d & e\\ h & g & a & b & c & d\\ i & h & g & a & b & c\\ j & i & h & g & a & b\\ k & j & i & h & g & a\end{pmatrix}.$$

A matrix M of order k and with entries 0, ±1 is called a conference matrix if $MM' = (k-1)I_k$. Following are some examples of conference matrices:

$$\begin{pmatrix}0 & 1 & 1 & 1 & -1 & 1\\ -1 & 0 & 1 & 1 & 1 & -1\\ 1 & -1 & 0 & 1 & 1 & 1\\ -1 & 1 & -1 & 0 & 1 & 1\\ -1 & -1 & 1 & -1 & 0 & 1\\ -1 & -1 & -1 & 1 & -1 & 0\end{pmatrix} \text{ and } \begin{pmatrix}0 & 1 & 1 & 1 & -1 & 1\\ 1 & 0 & 1 & -1 & -1 & 1\\ 1 & 1 & 0 & 1 & -1 & -1\\ 1 & -1 & 1 & 0 & 1 & -1\\ 1 & -1 & -1 & -1 & 0 & 1\\ 1 & 1 & -1 & -1 & 1 & 0\end{pmatrix}$$

([3], [13] and [14]).

Following are the only known circulant matrices which are Hadamard matrices: $C(1)$, $C(-1)$, $C(-1,1,1,1)$, $C(1,-1,-1,-1)$, $C(1,-1,1,1)$, $C(-1,1,-1,-1)$, $C(1,1,-1,1)$, $C(-1,-1,1,-1)$, $C(1,1,1,-1)$, $C(-1,-1,-1,1)$ [2].

The circulant Hadamard matrices obtained so far are of order 1 or 4. H.J. Ryser [12] conjectured that there is no circulant partial Hadamard matrix of order greater than 4.

A rectangular circulant matrix H of order $k \times n$ with row sum r is called circulant partial Hadamard matrix if $HH' = nI_k$, and is denoted by $r - H(k \times n)$.

Craigen *et al*. [1] used the notation $r - H(k \times n)$ for the class of all r-row-regular circulant Hadamard matrices of order $(k \times n)$. In this paper, the symbol $r - H(k \times n)$ is used for a particular member of the family of all r-row-regular circulant Hadamard matrices of order $(k \times n)$. Following are some examples of circulant partial Hadamard matrices:

$$\begin{pmatrix}-1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\end{pmatrix},\ \begin{pmatrix}1 & 1 & 1\end{pmatrix},\ \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\end{pmatrix} \text{ etc.}$$

The properties of circulant partial Hadamard matrix $r - H(k \times n)$ related to proposed research work are given below:

(i) $k \le n$.

(ii) $k > 1 \Longrightarrow n$ is even.

(iii) $k > 2 \Longrightarrow n$ is divisible by 4.

(iv) $k > 1 \Longrightarrow r \le \frac{n}{2}$.

(v) $k = n \Longrightarrow n = r^2$.

(vi) $r\sqrt{k} \le n$ (Cauchy-Schwartz inequality).

(vii) The sum of all column sums in H is equal to rk.

(viii) The sum of squares of column sums in H is equal to kn.

(see [1]).

Particularly, for the circulant partial Hadamard matrices $2-H(k\times 2k)$, Craigen *et al*. [1] have given the following two important theorems and one conjecture:

Theorem 1. *Existence of $2-H(k\times 2k)$ is equivalent to the existence of negacyclic conference matrix of order k.*

Theorem 2. *For every prime power q, there exist a $2-H((q+1)\times 2(q+1))$.*

Conjecture 1. *A $2-H(k\times 2k)$ exist if and only if $k-1$ is an odd prime power.*

3. Merging of Toeplitz Matrices to form Circulants

In this section some properties of Toeplitz matrices are discussed together with the sufficient conditions for merging two of more Toeplitz matrices to form a circulant matrix.

Consider a Toeplitz matrix T given by

$$T=\begin{pmatrix} a & x_1 & x_2 & \dots & x_{n-2} & x_{n-1}\\ y_1 & a & x_1 & \dots & x_{n-3} & x_{n-2}\\ y_2 & y_1 & a & \cdots & x_{n-4} & x_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ y_{n-2} & y_{n-3} & y_{n-4} & \cdots & a & x_1\\ y_{n-1} & y_{n-2} & y_{n-3} & \cdots & y_1 & a \end{pmatrix}.$$

Observe that

$$T=\begin{pmatrix} a & x_1 & x_2 & \dots & x_{n-2} & x_{n-1}\\ 0 & a & x_1 & \dots & x_{n-3} & x_{n-2}\\ 0 & 0 & a & \cdots & x_{n-4} & x_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a & x_1\\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0\\ y_1 & 0 & 0 & \dots & 0 & 0\\ y_2 & y_1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ y_{n-2} & y_{n-3} & y_{n-4} & \cdots & 0 & 0\\ y_{n-1} & y_{n-2} & y_{n-3} & \cdots & y_1 & 0 \end{pmatrix}$$

and

$$\begin{aligned} T&=\begin{pmatrix} a & x_1 & x_2 & \dots & x_{n-2} & x_{n-1}\\ 0 & a & x_1 & \dots & x_{n-3} & x_{n-2}\\ 0 & 0 & a & \cdots & x_{n-4} & x_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a & x_1\\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}+\begin{pmatrix} 0 & y_1 & y_2 & \dots & y_{n-2} & y_{n-1}\\ 0 & 0 & y_1 & \dots & y_{n-3} & y_{n-2}\\ 0 & 0 & 0 & \cdots & y_{n-4} & y_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & y_1\\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}'\\ &=S(a,x_1,x_2,\dots,x_{n-1})+S(0,y_1,y_2,\dots,y_{n-1})', \end{aligned}$$

where $S(x)$ is semicirculant of vector $x\in\mathbb{R}^n$.

From the above observation, we have the following theorem:

Theorem 3. *Every Toeplitz matrix can be expressed as a sum of a semicirculant matrix and a transpose of a semicirculant matrix.*

We note that If T is any Toeplitz matrix of order n then $T = \alpha I_n + S_1 + S_2'$, where S_1 and S_2 are semicirculant matrices with zero diagonal. Let us call this representation of T, *the formal representation of* T.

Let $S = S(0,x_1,x_2,\dots,x_{n-1})$ be a semicirculant matrix with zero diagonal, then the semicirculant matrix $S(0,x_{n-1},x_{n-2},\dots,x_1)$ is denoted by $S^{[-1]}$.

With formal representation of Toeplitz matrix, we are able to state the following theorems:

Theorem 4. *A Toeplitz matrix T with formal representation $T = \alpha I_n + S_1 + S_2'$ is a circulant if and only if $S_2 = S_1^{[-1]}$.*

Proof. Let $S_1 = S(0,x_1,x_2,\dots,x_{n-1})$ and $S_2 = S(0,y_1,y_2,\dots,y_{n-1})$. Then the matrix $T = \alpha I_n + S_1 + S_2'$ is a circulant if and only if $x_i = y_{n-i}$, for all $i = 1,2,\dots(n-1)$, and $x_i = y_{n-i}$, for all $i = 1,2,\dots(n-1)$ if and only if $S_2 = S_1^{[-1]}$. This proves the theorem. □

Next, theorem gives a sufficient condition that the matrix $(A|B)$ is to be a circulant matrix for any two Toeplitz matrices A and B.

Theorem 5. *Let A and B be two Toeplitz matrices each order n with the formal representations $A = \alpha I_n + A_1 + A_2'$ and $B = \beta I_n + B_1 + B_2'$, respectively.*
Then the augmented matrix $(A|B)$ is a circulant matrix if and only if $A_1 = B_2^{[-1]}$ and $B_1 = A_2^{[-1]}$.

Proof. Let $A_1 = S(0,x_1,x_2,\dots,x_{n-1})$, $A_2 = S(0,x_1',x_2',\dots,x_{n-1}')$, $B_1 = S(0,y_1,y_2,\dots,y_{n-1})$ and $B_1 = S(0,y_1',y_2',\dots,y_{n-1}')$.
Then the augmented matrix $(A|B)$ is a circulant if and only if $x_{n-i} = y_i'$ and $y_{n-i} = x_i$, for all $i = 1,2,\dots,(n-1)$, and, $x_{n-i} = y_i'$ and $y_{n-i} = x_i'$, for all $i = 1,2,\dots,(n-1)$ if and only if $A_1 = B_2^{[-1]}$ and $B_1 = A_2^{[-1]}$. This proves the theorem. □

Notes. (i) In case of four Toeplitz matrices A, B, C and D with formal representations $A = \alpha I_n + A_1 + A_2'$, $B = \beta I_n + B_1 + B_2'$, $C = \gamma I_n + C_1 + C_2'$ and $D = \delta I_n + D_1 + D_2'$, respectively. The augmented matrix is a circulant if and $(A|D)$ only if $A_1 = B_2^{[-1]}$, $B_1 = C_2^{[-1]}$, $C_1 = D_2^{[-1]}$ and $D_1 = A_2^{[-1]}$.

(ii) In case of m Toeplitz matrices with formal representations $A^i = \alpha^i I_n + A_1^i + A_2^i$, for all $i = 1,2,\dots,m$. The augmented matrix $(A^1|A^2|A^3|\cdots|A^m)$ is a circulant matrix if and only if $A_1^i = (A_2^{i(\mathrm{mod}\, m)+1})^{[-1]}$, for all $i = 1,2,\dots,m$.

4. Construction of Circulant Partial Hadamard Matrices $4-H(k \times 2k)$

In 2013, R. Kragen *et al.* [1] forward a table for maximum values of k in a circulant partial Hadamard matrix $r-H(k \times 2k)$. With the help of this table, they got many interesting results on circulant partial Hadamard matrices. they also pointed out that there are only two known circulant partial Hadamard matrices in the form of $4-H(k\times 2k)$ which are $4-H(6\times 12)$ and $4-H(14\times 28)$. Phoa *et al.* [11] obtained the same circulant partial Hadamard matrices of the type $4-H(k\times 2k)$ by searching using swarm intelligence.

In this section a method of construction of circulant partial Hadamard matrices of the type $4-H(k\times 2k)$ is forwarded. This method merges four Toeplitz matrices of some special type to form a circular partial Hadamard matrix of the type $4-H(k\times 2k)$. The details are explained through the following theorem:

Theorem 6. *Let A, B, C and D be square Toeplitz matrices, each of order $\frac{k}{2}$ having diagonal entries zero and non-diagonal entries ± 1. In addition, the matrices A, B, C and D satisfy the following properties:*

(i) $A+B+C+D=0$.

(ii) *The augmented matrix $(A|B|C|D)$ is a circulant.*

(iii) $AA'+BB'+CC'+DD'=(2k-4)I_{\frac{k}{2}}$.

(iv) $AB'+BC'+CD'+DA'=-4I_{\frac{k}{2}}$.

Then there exists a circulant partial Hadamard matrix $4-H(k\times 2k)$ of the form

$$H=\begin{pmatrix} I_{\frac{k}{2}}+A & I_{\frac{k}{2}}+B & I_{\frac{k}{2}}+C & I_{\frac{k}{2}}+D \\ I_{\frac{k}{2}}+D & I_{\frac{k}{2}}+A & I_{\frac{k}{2}}+B & I_{\frac{k}{2}}+C \end{pmatrix}.$$

Proof. (i) and (ii) implies that H is a circulant matrix with row sum 4. Next,

$$HH'=\begin{pmatrix} X & Y \\ Y & X \end{pmatrix},$$

where the block matrices X and Y are given by the following expressions:

$$X=4I_{\frac{k}{2}}+(A+B+C+D)+(A+B+C+D)'+AA'+BB'+CC'+DD',$$
$$Y=4I_{\frac{k}{2}}+(A+B+C+D)+(A+B+C+D)'+AB'+BC'+CD'+DA'.$$

Using (i), (iii) and (iv), we get $X=4I_{\frac{k}{2}}+0+0+(2k-4)I_{\frac{k}{2}}=2kI_{\frac{k}{2}}$ and $Y=0$.

Hence $HH'=\begin{pmatrix} 2kI_{\frac{k}{2}} & 0 \\ 0 & 2kI_{\frac{k}{2}} \end{pmatrix}=2kI_k$. This proves that H is a circulant partial Hadamard matrix of order $(k\times 2k)$ and with row sum 4. □

Illustration 1 (Construction of $4-H(4\times 8)$). Consider the following four zero-diagonal Toeplitz matrices each of order 2 with non-diagonal entries ± 1:

$$A=C=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B=D=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\implies A+B+C+D=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ AA'+BB'+CC'+DD'=\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}=4I_2,$$

and $AB'+BC'+CD'+DA' = \begin{pmatrix}-4 & 0\\ 0 & -4\end{pmatrix} = -4I_2$.

We observe that the augmented matrix $(A|B|C|D)$ is a circulant. Thus, the conditions of Theorem 6 are satisfied by the matrices A, B, C and D.

Hence

$$\begin{pmatrix} I_{\frac{4}{2}}+A & I_{\frac{4}{2}}+B & I_{\frac{4}{2}}+C & I_{\frac{4}{2}}+D\\ I_{\frac{4}{2}}+D & I_{\frac{4}{2}}+A & I_{\frac{4}{2}}+B & I_{\frac{4}{2}}+C \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1\\ -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1\\ -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

is a circulant partial Hadamard matrix of the type $4-H(4\times 8)$.

Illustration 2 (Construction of $4-H(6\times 12)$)**.** Consider the following four zero diagonal square Toeplitz matrices each of order 3 with non-diagonal entries ± 1:

$$A=\begin{pmatrix}0 & 1 & 1\\ -1 & 0 & 1\\ -1 & -1 & 0\end{pmatrix},\ B=\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ 1 & 1 & 0\end{pmatrix},\ C=\begin{pmatrix}0 & 1 & -1\\ 1 & 0 & 1\\ -1 & 1 & 0\end{pmatrix} \text{ and } D=\begin{pmatrix}0 & -1 & -1\\ -1 & 0 & -1\\ 1 & -1 & 0\end{pmatrix}$$

$$\implies \quad A+B+C+D = \begin{pmatrix}0 & 1 & 1\\ -1 & 0 & 1\\ -1 & -1 & 0\end{pmatrix} + \begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ 1 & 1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1 & -1\\ 1 & 0 & 1\\ -1 & 1 & 0\end{pmatrix} + \begin{pmatrix}0 & -1 & -1\\ -1 & 0 & -1\\ 1 & -1 & 0\end{pmatrix}$$
$$= \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$AA'+BB'+CC'+DD' = \begin{pmatrix}2 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 2\end{pmatrix} + \begin{pmatrix}2 & -1 & -1\\ -1 & 2 & 1\\ -1 & 1 & 2\end{pmatrix} + \begin{pmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{pmatrix} + \begin{pmatrix}2 & 1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}$$
$$= \begin{pmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{pmatrix} = 8I_3$$

and

$$AB'+BC'+CD'+DA' = \begin{pmatrix}0 & -1 & 1\\ 1 & -2 & -1\\ 1 & -1 & -2\end{pmatrix} + \begin{pmatrix}-2 & 1 & -1\\ 1 & 0 & -1\\ 1 & 1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1 & -1\\ -1 & -2 & 1\\ -1 & 1 & -2\end{pmatrix} + \begin{pmatrix}-2 & -1 & 1\\ -1 & 0 & 1\\ -1 & -1 & 0\end{pmatrix}$$
$$= \begin{pmatrix}-4 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{pmatrix} = -4I_3.$$

Also, the augmented matrix $(A|B|C|D)$ is a circulant (vide note (ii) of Theorem 5).

Hence on the basis of the conditions given in Theorem 6, the matrix H given by

$$H = \begin{pmatrix} I_{\frac{6}{2}}+A & I_{\frac{6}{2}}+B & I_{\frac{6}{2}}+C & I_{\frac{6}{2}}+D\\ I_{\frac{6}{2}}+D & I_{\frac{6}{2}}+A & I_{\frac{6}{2}}+B & I_{\frac{6}{2}}+C \end{pmatrix}$$

is a circulant partial Hadamard matrix $4-H(6\times 12)$ and is given by

$$H=\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1\\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1\\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1\\ 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1\\ -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \end{pmatrix}.$$

5. More Results on Circulant Partial Hadamard Matrix $4-H(k\times 2k)$

In this section some theorems and examples on circulant partial Hadamard matrix $4-H(k\times 2k)$ are given with the help of column sum properties.

Theorem 7. *For $k>4$ there is no column in $4-H(k\times 2k)$ with sum $k-2i+2$ or $2i+2-k$ if $i>\frac{k+\sqrt{k^2-4k}}{2}$ or $i>\frac{k-\sqrt{k^2-4k}}{2}$.*

Proof. Let C_i be ith column sum in $4-H(k\times 2k)$, for all $i=1,2,\dots,2k$. Then, we must have $C_i+C_{k+i}=4$, for all $i=1,2,\dots,k$. Also, we know the parity of column sums and k in a circulant partial Hadamard matrix are same, therefore the possible column sums in H occurs in the unordered pairs $(k-2i+2,2i+2-k)$, for all $i=0,1,2,\dots,\frac{k}{2}$. Let x_i be the number of unordered pairs of column sums $(k-2i+2,2i+2-k)$ in H, for all $i=0,1,2,\dots,\frac{k}{2}$. Then, we must have the following two equations:

$$\sum_{i=0}^{\frac{k}{2}}[(k-2i+2)^2+(2i+2-k)^2]x_i=2k^2 \text{ and } \sum_{i=0}^{\frac{k}{2}}x_i=k \quad \text{(see (vii) and (viii) of page 615)}$$

$$\Longrightarrow \quad \sum_{i=0}^{\frac{k}{2}}(2k^2+8-8ki+8i^2))x_i=2k^2 \quad \text{(I)} \quad \text{and} \quad \sum_{i=1}^{\frac{k}{2}}x_i=k \quad \text{(II)}$$

(I)-8(II) gives

$$\sum_{i=1}^{\frac{k}{2}}(2k^2-8ki+8i^2)x_i=2k^2-8k.$$

Since $2k^2-8ki+8i^2>2k^2-8k$ implies $x_i=0$. Therefore, no column in $4-H(k\times 2k)$ with sum $k-2i+2$ or $2i+2-k$ if $2k^2-8ki+8i^2>2k^2-8k$.
Since $2k^2-8ki+8i^2>2k^2-8k \Longleftrightarrow \left(i-\frac{k}{2}\right)^2>\frac{k^2-4k}{4}$. Therefore, no column in H with sum $2i+2-k$ or $2i+2-k$ if $i>\frac{k+\sqrt{k^2-4k}}{2}$ or $i<\frac{k-\sqrt{k^2-4k}}{2}$. This proves the theorem. □

Theorem 8. *The number of columns in $4-H(k\times 2k)(k>2)$ with sum $(k+2i-2)$ or $(2+2i-k)$ is less than t if $i>\frac{k+\sqrt{\frac{k^2-4k}{t}}}{2}$ or $i<\frac{k-\sqrt{\frac{k^2-4k}{t}}}{2}$.*

Proof. Let C_i be ith column sum in $4-H(k\times 2k)$, for all $i=1,2,\dots,2k$. Then, we must have $C_i+C_{k+i}=4$, for all $i=1,2,\dots,k$. Therefore, the possible column sums in H occurs in the unordered pairs of $(k-2i+2,2i+2-k)$, for all $i=1,2,\dots,\frac{k}{2}$. Let x_i be the number of unordered pairs of column sums $(k-2i+2, 2i+2-k)$ in H, for all $i = 1, 2, \dots, \frac{k}{2}$. Then, we must have the following two equations:

$$\sum_{i=1}^{\frac{k}{2}} [(k-2i+2)^2 + (2i+2-k)^2] x_i = 2k^2 \quad \text{and} \quad \sum_{i=1}^{\frac{k}{2}} x_i = k$$

$$\implies \sum_{i=1}^{\frac{k}{2}} (2k^2 - 8ik + 8(i^2+1)) x_i = 2k^2 \ \text{(A)} \quad \text{and} \quad \sum_{i=1}^{\frac{k}{2}} x_i = k \ \text{(B)}$$

(A)-8(B) gives $\sum_{i=1}^{\frac{k}{2}} (2k^2 - 8ki + 8i^2) x_i = 2k^2 - 8k$. Since $x_i < t$ implied by $(2k^2 - 8ki + 8i^2)t > 2k^2 - 8k$. Therefore, the number of columns in $4 - H(k \times 2k)$ with sum $k - 2i + 2$ or $2i - k + 2$ is less than t if $\left(i - \frac{k}{2}\right)^2 > \frac{k^2-4k}{4t} \iff i > \frac{k + \sqrt{\frac{k^2-4k}{t}}}{2}$ or $i < \frac{k - \sqrt{\frac{k^2-4k}{t}}}{2}$. This proves the theorem. □

Corollary 1. *There is no any column in $4 - H(k \times 2k)$ with sum k or $4 - k$.*

Proof. The inequality $1 < \frac{k - \sqrt{k^2-4k}}{2} \implies x_1 = 0$ in Theorem 7. □

Theorem 9. *If all column sums in $4 - H(k \times 2k)$ are equal, then H will be equivalent to $((CH(4))|CH(4))$ where $CH(4)$ is circulant Hadamard matrix of order 4.*

Proof. Let each column sum in $4 - H(k \times 2k)$ be c, the we have $2kc = 4k$ and $2kc^2 = 2k^2$. Thus $k = 4$.

We note that for $k = 4$, $4 - H(k \times 2k)$ is equivalent to the following matrix:

$$\begin{pmatrix} -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix} = (CH(4)|CH(4)),$$

where $CH(4)$ is circulant Hadamard matrix of order 4. □

6. Thoughts and Conclusions

A general method of constructing a circulant partial Hadamard matrix of the form $4 - H(k \times 2k)$ with the help of four Toeplitz matrices is proposed in Theorem 6. To date there is no method to construct circulant partial Hadamard matrices of the type $4 - H(k \times 2k)$. Therefore, it will now become a routine task for researchers to search for Toeplitz matrices, from which we can construct some unknown circulant partial Hadamard matrices of the form $4 - H(k \times 2k)$. Theorem 8 and Theorem 9 give idea about the non-existence of column sums in a circulant partial Hadamard matrices of the type $4 - H(k \times 2k)$. Section 2 includes some merging results on Toeplitz matrices. The results in Section 2 may give some information about the search for matrices A, B, C and D given in Theorem 6. The results obtained in this paper are obtained analytically and without a computer search. I hope, that the results will help the researchers in advancement of knowledge.

7. Questions

On the basis of presented work herewith I am forwarding the following new questions:

Question 1. Does the existence of $4-H(k\times 2k)$ imply that $k-1$ is an odd prime number?

Question 2. Does the divisibility of k by 4 indicate the non-existence of the partial circulant Hadamard matrices $4-H(k\times 2k)$?

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] R. Craigen, G. Faucher, R. Low and T. Wares, Circulant partial Hadamard matrices, *Linear Algebra and its Applications* **439**(11) (2013), 3307 – 3317, DOI: 10.1016/j.laa.2013.09.004.

[2] R. Euler, L.H. Gallardo and O. Rahavandrainy, Eigenvalues of circulant matrices and a conjecture of Ryser, *Kragujevac Journal of Mathematics* **45**(5) (2021), 751 – 759, URL: https://imi.pmf.kg.ac.rs/kjm/pdf/accepted-finished/01600fd373df27b2cb120d21fe74373b_2536_08012019_032857/kjm_45_5-7.pdf.

[3] E.G. Kocer, Circulant, negacyclic and semicirculant matrices with the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers, *Hacettepe Journal of Mathematics and Statistics* **36**(2) (2007), 133 – 142, URL: http://www.hjms.hacettepe.edu.tr/uploads/ef6188e6-4891-4148-8ad4-0ddf384b1629.pdf.

[4] K.J. Horadam, *Hadamard Matrices and Their Application*, Princeton University Press, Princeton (2007), DOI: 10.1515/9781400842902.

[5] M.-H. Kao, A new type of experimental design for event-related fMRI via Hadamard matrices, *Statistics & Probability Letters* **84** (2014), 108 – 112, DOI: 10.1016/j.spl.2013.09.024.

[6] M.-H. Kao, Universally optimal fMRI design for comparing hemodynamic response functions, *Statistica Sinica* **25** (2015), 499 – 506, DOI: 10.5705/ss.2013.248.

[7] H. Kharaghani and B. Tayfeh-Rezaie, A Hadamard matrix of order 428, *Journal of Combinatorial Designs* **13**(6) (2005), 435 – 440, DOI: 10.1002/jcd.20043.

[8] P.K. Manjhi and A. Kumar, On the construction of Hadamard matrices, *International Journal of Pure and Applied Mathematics* **120** (04) (2017), 51 – 58, URL: http://ijpam.eu/contents/2018-120-1/4/4.pdf.

[9] P.K. Manjhi and M.K. Rana, Some new examples of circulant partial Hadamard matrices of type $4-H(k \times n)$, *Advances and Applications in Mathematical Sciences* **21**(05) (2022) 2559-2564, URL: https://www.mililink.com/upload/article/1387427361aams_vol_215_march_2022_a19_p2559-2564_pankaj_kumar_manjhi_and_mahendra_kumar_rana.pdf.

[10] P.K. Manjhi and M.K. Rana, Some results on r-row-regular circulant partial Hadamard matrices of order $(k \times 2k)$, *Communications in Mathematics and Applications* **13**(1) (2022), 129 – 136, DOI: 10.26713/cma.v13i1.1734.

[11] F.K.H. Phoa, Y.L. Lin and T.C. Wang, Using swarm intelligence to search for circulant partial Hadamard matrices, in: *Advances in Swarm Intelligence*, Y. Tan, Y. Shi and C.A.C. Coello (eds.), ICSI 2014, *Lecture Notes in Computer Science*, Vol. 8794, Springer, Cham, DOI: 10.1007/978-3-319-11857-4_18.

[12] H.J. Ryser, *Combinatorial Mathematics*, The Carus Mathematical Monographs, Vol. 14, AMS and The Mathematical Association of America, John Wiley and Sons Inc., New York, 154 pages (1963).

[13] J. Seberry, Remarks on negacyclic matrices, *Journal of Theoretical and Computational Mathematics* **2**(1) (2016), 71 – 79, URL: http://mathscinet.ru/files/2016JenniferSeberry_Nega-cyclic.pdf.

[14] J. Seberry, *Orthogonal Designs: Hadamard Matrices, Quadratic Forms and Algebra*, Springer International Publishing AG (2017), URL: https://documents.uow.edu.au/~jennie/WEB/WEB17/2017_Bookmatter_OrthogonalDesigns.pdf.