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Research Article

Instantons and the Point Particle Field Theory Derived From Strings

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Abstract. String theories with supersymmetry have perturbation series that are finite at each order with exponential bounds and do not reflect the presence of nonperturbative effects. Worldsheet instantons in superstring theory are surfaces which support fields with a finite Euclidean action. Dirichlet boundaries can be added to compact surfaces to represent the coupling of open and closed strings and yield an exponential term with a dependence on the coupling characteristic of strings rather than point-particle field theory. An additional set of worldsheet occurs in $N = 2$ string theory after the quantization of a $U(1)$ symmetry. The $N = 2$ open string amplitudes with $U(1)$ instantons may be derived from a cubic Yang-Mills theory. Nevertheless, the summation over the genus and $U(1)$ instanton number includes other amplitudes without an exponential nonperturbative term. The expansion of $N = 2$ closed string amplitudes similarly consists of many vanishing terms, and couplings with the open string are required initially for the introduction of nonperturbative string effects. It is necessary to evaluate the action of a nontrivial solution to the effective field equations to find an exponential term with the dependence on the coupling of point-particle field theories. The theory then can be developed into a description of a model for elementary particles at larger distance scales. The theory of Eisenstein series and cusp forms is developed for the delineation between strings and point particles in the effective action.

Keywords. Modular invariance, Effective action, Modular forms, Cusp forms

Mathematics Subject Classification (2020). 11F11, 11M36, 83E30

1. Introduction

The selection of a vacuum in a field theory is necessary for the existence of a space of quantum states defined by operators that are coefficients in an orthonormal expansions of variables which have commutation relations derived from the classical Poisson brackets and the series expansion of a scattering matrix in the region of the interactions. There exists configurations, however, which cannot be described by perturbation theory. Nonperturbative effects are required for the characterization of certain states in the theory. It may be demonstrated that these instantons arise in field theory, within the series expansion of correlation functions equal to functional derivatives of the partition function, as terms of $\mathcal{O}(e^{-\frac{1}{\lambda^2}})$, where λ is the coupling. If perturbation theory is developed around a new vacuum representing the nontrivial state, the special configurations would occur with probability $\mathcal{O}(1)$. Therefore, the selection of the vacuum and the classification of instantons is essential for a complete understanding of the model.

Nonperturbative effects in string theory initially have been related to terms of $\mathcal{O}(e^{-\frac{1}{g_s}})$, where g_s is the coupling. The point particle limit of the string partition function must reproduce the nonperturbative term, which would require g_s to be proportional to g_{YM}^2. This condition follows from the equality of the closed string coupling to the square of the open string coupling [10]. By contrast with field theory, however, string theories have been formulated initially through the path integrals for two-dimensionally conformally invarant Lagrangians. Therefore, the instantons that are nontrivial finite action solutions to the Euclidean equations can be defined at first only in two dimensions. Introducing a form of coupling to the open string through the addition of disks with Dirichlet boundary conditions on the fields, localized at a point, an exponential term of this order can be added to the perturbation series [23]. The nature of the coupling, however, will determine if this effect is separate from the nonperturbative physics of point particle. The geometry of these worldsheet instantons do not extend over the entire space-time, and therefore, do not represent instantons of point particle field theories which are functions of the coordinates of the embedding manifold. A second class of instantons, being solutions to the Euclideanized string effective field equations, must be introduced. The string effective action is a finite-dimensional integral over the embedding space-time. The contribution of a space-time instanton to the effective action also may be evaluated to be $e^{-\frac{|Q|}{g_s}}$, where Q is the topological charge. The coupling here is that of a closed string, and this exponential matches the instanton terms in gauge field theories [8].

The equations of closed bosonic string theory include harmonic conditions on the coordinate variables that can have nonconstant solutions with finite Euclidean action only if the surfaces have finite genus and a boundary or infinite genus and do not belonging to the class O_{HD}, defined by the absence of harmonic functions with finite Dirichlet norm. The equations of the superstring sigma model have other terms with fermion fields. The $N = 2$ string model has a finite particle spectrum and, in its primary form, it consists of a single scalar field. Therefore, the $N = 2$ string theory has some similarities with that bosonic strings.

It is known that $N = 2$ superstring amplitudes may be derived from an effective field theory for closed strings

$$\mathcal{I}_{\text{eff.}} = \int d^4x \left[\frac{1}{2}\phi \Box \phi + \frac{2}{3\mu^3}\phi(\partial_+^{\dot{a}}\partial_-^{\dot{b}}\phi)(\partial_{+\dot{a}}\partial_{-\dot{b}}\phi) \right] \tag{1.1}$$

for the zero-instanton sector and

$$\mathcal{I}_{\text{eff., inst.}} = \int d^4x \left[\frac{1}{2}\phi \Box \phi + \frac{2}{3\mu^3}\phi(\partial_+^{\dot{a}}\partial_+^{\dot{a}}\phi)(\partial_{+\dot{a}}\partial_{+\dot{b}}\phi) \right] \tag{1.2}$$

after summing over $U(1)$ instantons for some constant c [16]. This result follows from perturbative expansion of the Lorentzian model with the equations consisting of the differential operator $\partial_+\partial_- = (\partial_\tau + \partial_\sigma)(\partial_\tau - \partial_\sigma) = \partial_\tau^2 - \partial_\sigma^2$. The addition of $U(1)$ instantons, related to a rotational invariance of set of currents for the superconformal algebra, and therefore worldsheet supersymmetry, can be achieved by including solutions to Euclidean equations that contain the differential operator $\partial_\tau^2 + \partial_\sigma^2$. The operator $\partial_+\partial_+$ equals $\partial_\tau^2 + \partial_\sigma^2 + 2\partial_\tau\partial_\sigma$. To first approximation, the solutions to the Lorentzian equations have the form $f(\tau+\sigma) + g(\tau-\sigma)$, which can be Wick rotated to $\frac{1}{2}[f(\tau+i\sigma) + f(\tau-i\sigma) + g(\tau+i\sigma) + g(\tau-i\sigma)]$, and $\frac{1}{2}\partial_\tau\partial_\sigma[f(\tau+i\sigma) + f(\tau-i\sigma) + g(\tau+i\sigma) + g(\tau-i\sigma)] = \frac{1}{2}[f''(\tau+i\sigma) - f''(\tau-i\sigma) + g''(\tau+i\sigma) - g''(\tau-i\sigma)]$, which has vanishing real part and does not contribute to the action of a real instanton. The choice of the derivatives in the action (1.2) is consistent with generation of these instantons.

A Chern-Simons action is derived with the identification of φ with a string field. The change from the derivative ∂_- to ∂_+ allows a generalization to a theory with a mixed derivative $\frac{\partial_- + \partial_+}{2}$ invariant under the coordinate parameterizations in the orthogonal direction. Reduction over this coordinate would yield the three-dimensional theory. Since the Chern-Simons theory is represented by the boundary action of a self-dual Yang-Mills theory, it may be deduced that there is a derivation of the point particle gauge theories describing the elementary particle interactions. It may be noted that the $N = 2$ string theory is formulated initially on a manifold with Kleinian signature $(-+-+)$. With the reduction to three dimensions, the signature would be $(-++)$. The Yang-Mills theory in four dimensions then can be reintroduced in a space-time with Lorentzian signature $(-+++)$.

Scattering amplitudes have been evaluated both in a flat space-time, which has ten dimensions for superstring and heterotic string theory and four for $N = 2$ string theory, and compactifications over the torus T^d. Since an $SL(2;\mathbb{Z})$ modular invariance occurs generally in string theory, the Eisenstein series would be expected to be a factor of the couplings in the effective action. It may be demonstrated, however, that the three-point amplitude vanishes beyond first order in the perturbation series for open superstring theory. The coefficient in the string effective action is found to have a conventional value not determined by Eisenstein series. The closed string amplitude at genus one is found to be an integral of an Eisenstein series that includes a divergent first term. This term is rendered finite through a modular-invariant zeta-function regularization method in Section 2. Therefore, the final formula for the coupling is not precisely equal to the expansion of an Eisenstein series, and this characteristic would be valid for the couplings in the string effective action. The toroidal compactification introduces $SO(d,d;\mathbb{Z})$ Eisenstein series, and the generalization of the formula to arbitrary genus requires the proof of the independence of the region in the Siegel upper half plane of period matrices with respect to symplectic modular transformations in Section 3.

The expressions for these amplitudes form either exponential or convergent series. The occurrence of instantons, other than those arising from the quantization of a $U(1)$ gauge field in the $N = 2$ theory, is not predicted by the perturbation expansion. By analogy with the superstring theory, either D-instantons must be added to the path integral or space-time instantons are found from the four-dimensional effective action.

The $N = 2$ open string is classically equivalent to self-dual Yang-Mills theory [19], [17], with the coupling $g_{s,\text{open}}$ identified with the gauge coupling g_{YM}. The instanton $A_\mu^a(x) = \frac{2}{g_{YM}} \frac{\eta^a_{\mu\nu} x_0^\nu}{(x-x_0)^2+\rho^2}$, where x_0 is the center of the solution and ρ is a scale, with $\eta^a_{\mu\nu}$ being the 't Hooft symbol, has the action $\frac{8\pi^2}{g_{YM^2}}$ [2] . The contribution of this instanton action to the effective action then equals $e^{-\frac{8\pi^2}{g_{YM}^2}}$. Therefore, it is again the bosonic component of the space-time instanton in $N = 2$ string theory that yields the exponential term representing nonperturbative effects in point particle field theory.

The dimensions of the space of modular forms and cusp forms of weight k, where k is even, are derived in Section 4. It is found that cusp forms of half-integer weight also do exist with the same codimension in the space of modular of forms of the same weight. These forms would provide a basis for a series that has an expansion with a coupling $\frac{1}{g_s^2}$ rather than $\frac{1}{g_s}$ when the alternating coefficients can be set equal to zero. The cusp forms then would represent a component of the series expansion of higher-dimensional effective actions with U-duality that defines the nonperturbative effects characteristic only of a point particle field theory.

2. The Effective String Coupling with Worldsheet Instantons

The $N = 2$ superstring amplitudes without gauge instantons are given by

$$A^{(n)}_{\text{no instantons}}(k_1,\ldots,k_n) = \sum_g \kappa^{2g-2+n} \int_{s\mathcal{M}_g} dm_g dm_A \langle V(k_1)\ldots V(k_n)\mathcal{A}\mathcal{P}_+^j \mathcal{P}_-^j\rangle, \tag{2.1}$$

where $k_1,\ldots,k_n$ are the momenta of the n vertex operators and $\mathcal{P}_+$ and $\mathcal{P}_-$ are picture-changing operators. The dimension of the punctured moduli space is $3g-3+n$ and $j = 2g-2+n$. The addition of gauge instantons requires a second coupling related given by a phase $e^{i\theta}$, and summing over the instanton number [16]

$$A^{(n)}(k_1,\ldots,k_n) = \sum_g \sum_c \binom{2j}{j+c} \kappa^j \sin^{j-c}\frac{\theta}{2} \cos^{j+c}\frac{\theta}{2} \int_{s\mathcal{M}_g} dm_g dm_A \langle V(k_1)\ldots V(k_n)\mathcal{A}\mathcal{P}_+^{j+c}\mathcal{P}_-^{j-c}\rangle. \tag{2.2}$$

The product of the picture-changing operators in the expectation value allows its evaluation in terms of states with a specific ghost number. The amplitude also could be expressed in the form

$$\sum_g \int_{s\mathcal{M}_g} dm_g dm_A \langle V(k_1)\ldots V(k_n)\mathcal{A}[v^a \widetilde{\mathcal{P}}_a]^{2j}\rangle \tag{2.3}$$

with

$$\begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \sqrt{\kappa} \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix} \tag{2.4}$$

and $\widetilde{\mathcal{P}}_+ = \epsilon_{+b}\widetilde{\mathcal{P}}^b = \widetilde{\mathcal{P}}^- = \mathcal{P}_+\mathcal{S}\left(-\frac{1}{2}\right)$ and $\widetilde{\mathcal{P}}_- = \epsilon_{-b}\widetilde{\mathcal{P}}^b = -\widetilde{\mathcal{P}}^+ = -\mathcal{P}_-\mathcal{S}\left(\frac{1}{2}\right)$, since $v^a\widetilde{\mathcal{P}}_a = \sqrt{\kappa}[\cos\frac{\theta}{2}\widetilde{\mathcal{P}}_+ - \sin\frac{\theta}{2}\widetilde{\mathcal{P}}_-] = \sqrt{\kappa}[\cos\frac{\theta}{2}\mathcal{P}_+\mathcal{S}\left(-\frac{1}{2}\right) + \sin\frac{\theta}{2}\mathcal{P}_-\mathcal{S}\left(\frac{1}{2}\right)]$, where $\mathcal{S}(\alpha)$ is a spectral flow operator [16]. Then the n-point amplitude at $g=0$

$$
\begin{aligned}
\mathcal{A}_0^{(n)} &= \int dm_h dm_A \kappa^j \left\langle V(k_1)\ldots V(k_n)\mathcal{A}\left(\cos\frac{\theta}{2}\mathcal{P}_+\mathcal{S}\left(-\frac{1}{2}\right) + \sin\frac{\theta}{2}\mathcal{P}_-\mathcal{S}\left(\frac{1}{2}\right)\right)^{2j}\right\rangle \\
&= \sum_c \binom{2j}{j+c}\kappa^j \sin^{2j-c}\frac{\theta}{2}\cos^{2j+c}\frac{\theta}{2}(-1)^{j^2-c^2} \\
&\quad \cdot \int dm_h dm_A \left\langle V(k_1)\ldots V(k_n)\mathcal{A}\mathcal{S}\left(-\frac{1}{2}\right)^{j+c}\mathcal{S}\left(\frac{1}{2}\right)^{j-c}\mathcal{P}_+^{j+c}\mathcal{P}_-^{j-c}\right\rangle. \qquad (2.5)
\end{aligned}
$$

Given that $\mathcal{S}(\alpha)|\pi_+,\pi_-,q\rangle = |\pi_+ + \alpha, \pi_- - \alpha, q+\alpha\rangle$, where $(\pi_+,\pi--)$ are picture charges and q is the global boost charge with respect to the $SL(2;\mathbb{R})$ generator L_{+-} [16], and $\mathcal{S}\left(-\frac{1}{2}\right)\mathcal{S}\left(\frac{1}{2}\right) = \mathbb{I}$. The three-point amplitude

$$
\begin{aligned}
\mathcal{A}_0^{(3)} &= \int dm_h dm_A \langle V^A(k_1)V^B(k_2)V^C(k_3)\mathcal{A}[v^a\widetilde{\mathcal{P}}_a]^2\rangle \\
&= \int dm_h dm_A \Big\langle V^A(k_1)V^B(k_2)V^C(k_3) \\
&\quad \cdot \left[(v^+)^2\mathcal{P}_+^2\mathcal{S}\left(-\frac{1}{2}\right)^2 - v^+v^-\mathcal{P}_+\mathcal{P}_-\left(\mathcal{S}\left(-\frac{1}{2}\right)\mathcal{S}\left(\frac{1}{2}\right) + \mathcal{S}\left(\frac{1}{2}\right)\mathcal{S}\left(-\frac{1}{2}\right)\right) + (v^-)^2\mathcal{P}_-(\mathcal{S}\left(\frac{1}{2}\right)^2\right]\Big\rangle \\
&= i\kappa f^{ABC}\left[\cos^2\frac{\theta}{2}k_1^-\wedge k_2^- + \sin\frac{\theta}{2}\cos\frac{\theta}{2}k_1^{(+}\wedge k_2^{-)} + \sin^{\frac{\theta}{2}}k_1^+\wedge k_2^+\right], \qquad (2.6)
\end{aligned}
$$

with $\mathcal{A}_0^{(3)}|_{\theta=0,\kappa=\frac{1}{\mu}} = \frac{i}{\mu}f^{ABC}k_1^-\wedge k_2^-$, and the vanishing higher-point amplitudes at leading order can be derived from a cubic action [16].

Generalizing the formula to genus g,

$$
\begin{aligned}
\mathcal{A}_g^{(n)}(k_1,\ldots,k_n) &= \sum_c \binom{4g-4+2n}{2g-2+n+c}\kappa^{2g-2+n}\sin^{2g-2+n-c}\frac{\theta}{2}\cos^{j+c}\frac{\theta}{2} \qquad (2.7) \\
&\quad \cdot \int dm_h dm_A \Big\langle V(k_1)\ldots V(k_n)\mathcal{A}\mathcal{P}_+^{2g-2+n+c}\mathcal{P}_-^{2g-2+n-c}\Big\rangle \\
&= \sum_{-2g+2-n\le c\le 2g-2+n}\int dm_h dm_A\langle V(k_1)\ldots V(k_n)\mathcal{A}[v^a\widetilde{\mathcal{P}}_a]^{2(2g-2+n)}\rangle_{g,c},
\end{aligned}
$$

where the binomial have been absorbed in the power $[v^a\widetilde{\mathcal{P}}_a]^{4g-4+2n}$, and the amplitude is evaluated in the $SL(2;\mathbb{R})$ gauge by fixing the vector v^a.

When $g=1$, $n=3$ and $j=3$

$$
\begin{aligned}
[v^+\widetilde{\mathcal{P}}_+ + v^-\widetilde{\mathcal{P}}_-]^6 = \Big[&(v^+)^6(\mathcal{P}_+)^6\mathcal{S}\left(-\frac{1}{2}\right)^6 - 6(v^+)^5v^-(\mathcal{P}_+)^5\mathcal{P}_-\mathcal{S}\left(-\frac{1}{2}\right)^5\mathcal{S}\left(\frac{1}{2}\right) \\
&+ 15(v^+)^4(v^-)^2(\mathcal{P}_+)^4(\mathcal{P}_-)^2\mathcal{S}\left(-\frac{1}{2}\right)^4\mathcal{S}\left(\frac{1}{2}\right)^2 \\
&- 20(v^+)^3(v^-)^3(\mathcal{P}_+)^3(\mathcal{P}_-)^3\mathcal{S}\left(-\frac{1}{2}\right)^3\mathcal{S}\left(\frac{1}{2}\right)^3
\end{aligned}
$$

$$+15(v^+)^2(v^-)^4(\mathcal{P}_+)^2(\mathcal{P}_-)^4\mathcal{S}\left(-\frac{1}{2}\right)^2\mathcal{S}\left(\frac{1}{2}\right)^4$$
$$-6v^+(v^-)^5\mathcal{P}_+(\mathcal{P}_-)^5\mathcal{S}\left(-\frac{1}{2}\right)\mathcal{S}\left(\frac{1}{2}\right)^5+(v^-)^6(\mathcal{P}_-)^6\mathcal{S}\left(\frac{1}{2}\right)^6\Bigg]. \tag{2.8}$$

Restoring the $SO(2,2)$ symmetry and aligning v^+ and v^- with all three vectors k_1, k_2 and k_3 gives

$$A_1^{(3)}\Big|_{\kappa=\frac{1}{\mu}}=-\frac{20i}{\mu^3}f^{ABC}\sin^6\frac{\theta}{2}\cos^6\frac{\theta}{2}k_1^+\wedge k_2^+\wedge k_3^+\wedge k_1^-\wedge k_2^-\wedge k_3^-, \tag{2.9}$$

where $\{f^{ABC}\}$ is the set of structure constants of the vertex operator algebra. It vanishes when θ equals 0 or π and $A_1^{(3)}\Big|_{\kappa=\frac{1}{\mu},\theta=\frac{\pi}{2}}=-\frac{5i}{16\mu}f^{ABC}[k_1^+\wedge k_2^+\wedge k_3^+\wedge k_1^-\wedge k_2^-\wedge k_3^-]$. The open string three-point amplitude vanishes for $g\geq 2$. Consequently, the coefficient in the cubic action for the open superstring theory may be fixed without any further corrections from genus $g\geq 2$.

If $g=1$, $n=4$ and $j=4$,

$$[v^+\widetilde{\mathcal{P}}_++v^-\widetilde{\mathcal{P}}_-]^8=\Bigg[(v^+)^8(\mathcal{P}_+)^6\mathcal{S}\left(-\frac{1}{2}\right)^8-8(v^+)^7v^-(\mathcal{P}_+)^7\mathcal{P}_-\mathcal{S}\left(-\frac{1}{2}\right)^7\mathcal{S}\left(\frac{1}{2}\right)$$
$$+28(v^+)^6(v^-)^2(\mathcal{P}_+)^6(\mathcal{P}_-)^2\mathcal{S}\left(-\frac{1}{2}\right)^6\mathcal{S}\left(\frac{1}{2}\right)^2$$
$$-56(v^+)^5(v^-)^3(\mathcal{P}_+)^5(\mathcal{P}_-)^3\mathcal{S}\left(-\frac{1}{2}\right)^5\mathcal{S}\left(\frac{1}{2}\right)^3$$
$$+70(v^+)^4(v^-)^4(\mathcal{P}_+)^4(\mathcal{P}_-)^4\mathcal{S}\left(-\frac{1}{2}\right)^4\mathcal{S}\left(\frac{1}{2}\right)^4$$
$$-56(v^+)^3(v^-)^5(\mathcal{P}_+)^3(\mathcal{P}_-)^5\mathcal{S}\left(-\frac{1}{2}\right)^3\mathcal{S}\left(\frac{1}{2}\right)^5$$
$$+28(v^+)^2(v^-)^6(\mathcal{P}_+)^2(\mathcal{P}_-)^6\mathcal{S}\left(-\frac{1}{2}\right)^2\mathcal{S}\left(\frac{1}{2}\right)^6$$
$$-8v^+(v^-)^7\mathcal{P}_+(\mathcal{P}_-)^7\mathcal{S}\left(-\frac{1}{2}\right)\mathcal{S}\left(\frac{1}{2}\right)^7+(v^-)^8(\mathcal{P}_-)^8\mathcal{S}\left(\frac{1}{2}\right)^8. \tag{2.10}$$

Then, it would follow that $A_1^{(4)}=70\kappa^4d^{ABCD}\sin^8\frac{\theta}{2}\cos^8\frac{\theta}{2}k_1^+\wedge k_2^+\wedge k_3^+\wedge k_4^+\wedge k_1^-\wedge k_2^-\wedge k_3^-\wedge k_4^-$ and $A_1^{(4)}\Big|_{\kappa=\frac{1}{\mu},\theta=\frac{\pi}{2}}=\frac{35}{128\mu^4}d^{ABCD}k_1^+\wedge k_2^+\wedge k_3^+\wedge k_4^+\wedge k_1^-\wedge k_2^-\wedge k_3^-\wedge k_4^-$, with d^{ABCD} resulting from the integral of the expectation value the product of four vertex operators, picture changing and spectral flow operators over the modulus of the torus and the gauge moduli. Again, the $N=2$ four-point open-string amplitude vanishes for $g\geq 2$.

The existence of instantons in a string perturbation series can be determined by a modular form that represents the amplitude. The coefficients would not occur in an Eisenstein series for the open string with $SL(2;\mathbb{Z})$ being replaced by a relative modular group [3]. The expansion of modular forms representing of the quartic sector of $SL(2;\mathbb{Z})$-invariant superstring effective action consists of a series nonvanishing at zeroth and first orders and an exponential term. Therefore, the string perturbation expansion would cause the quartic curvature term to be transformed from $\mathcal{L}_{4pt} = \frac{\zeta(3)}{3\cdot 2^6}(Im\ \tau)^{\frac{3}{2}}(t_8^{ABCDEFGH}t_8^{MNPQRSTU} + \frac{1}{8}\varepsilon_{10}^{ABCDEFGHIJ}\varepsilon^{MNPQRSTU}{}_{IJ})$ $\hat{R}_{ABMN}\hat{R}_{CDPQ}\hat{R}_{EFRS}\hat{R}_{GHTU}$, where $\hat{R}_{MN}{}^{PQ} = R_{MN}{}^{PQ} + \frac{1}{2}e^{-\frac{\phi}{2}}\nabla_{[M}H^1_{N]}{}^{PQ} - \frac{1}{4}g_{[M}{}^{[P}\nabla_{N]}\nabla^{Q]}\phi$ and $\tau = \chi + ie^{-\phi}$ is a complex scalar field, with χ belonging to the Ramond sector, ϕ being the dilaton in the Neveu-Schwarz sector and $H^{\alpha}{}_{KLM} = \partial_K B^{\alpha}_{LM} + \partial_M B^{\alpha}_{KL} + \partial_L B^{\alpha}_{MK}$, $\alpha = 1,2$, representing the field strengths of the antisymmetric tensor B^1_{LM} in the Neveu-Schwarz sector and $B^2{}_{LM}$ in the Ramond sector, to $\mathcal{L}_{R^4} = \frac{1}{3\cdot 2^7}f_0(\tau,\bar{\tau})(t_8^{ABCDEFGH}t_8^{MNPQRSTU} + \frac{1}{8}\varepsilon_{10}^{ABCDEFGHIJ}\varepsilon^{MNPQRSTU}{}_{IJ})\hat{R}_{ABMN}\hat{R}_{CDPQ}\hat{R}_{EFRS}\hat{R}_{GHTU}$ such that the coefficient is $f_0(\tau,\bar{\tau}) = \sum_{m,n}{}'\frac{(Im\ \tau)^{\frac{3}{2}}}{|m+n\tau|^3} = 2\zeta(3)(\mathrm{Im}\,\tau)^{\frac{3}{2}} + \frac{2\pi}{3}(\mathrm{Im}\,\tau)^{-\frac{1}{2}} + 8\pi(\mathrm{Im}\,\tau)^{\frac{1}{2}}\cdot\sum_{m\neq 0,n\geq 1}\left|\frac{m}{n}\right|e^{2\pi imn\,\mathrm{Re}\,\tau}\ \cdot K_1(2\pi|mn|\,\mathrm{Im}\,\tau)$ [14]. The value of $\zeta(3)$ is $\frac{7}{180}\pi^3 - 2\sum_{k=1}^{\infty}\frac{1}{k^3(e^{2\pi k}-1)}$ [1]. If the sum of exponential terms is separated, then the coefficients in the first two terms of the series include $\frac{7}{90}\pi^3(\mathrm{Im}\,\tau)^{\frac{3}{2}}$ and $\frac{2\pi}{3}(\mathrm{Im}\,\tau)^{-\frac{1}{2}}$, which are algebraically dependent over transcendental extension of the field of rational numbers, $\mathbb{Q}[\pi,\mathrm{Im}\,\tau]$. The coefficients in eq. (2.9) and (2.10) are both rational multiples of a power of the coupling and and the form constructed from the momentum vectors. Since the string coupling is proportional to $e^{-\phi}$, a similar series expansion may be deduced for the $N = 2$ string amplitudes without the separation of the sum over exponential functions in the second term. For the quartic curvature sector of the superstring effective action, the ratio of the first two terms is $\frac{60}{7}\pi^{-2}(\mathrm{Im}\,\tau)^{-2}$. Identifying κ with $\pi^{-2}(\mathrm{Im}\,\tau)^{-2}$, the numerical factor apart from the power of the coupling is $\frac{60}{7}$. Given that the three-point amplitude at genus zero arises from $\kappa[v^+\widetilde{\mathcal{P}}_+ + v^-\widetilde{\mathcal{P}}_-]^2|_{\kappa=\frac{1}{\mu},\theta=\frac{\pi}{2}}$, the coefficient multiplying the form is $-\frac{2}{\mu}\sin^2\frac{\pi}{4}\cos^2\frac{\pi}{4} = -\frac{1}{2\mu}$. The ratio of coefficients of the first two terms in the expansion of the three-point amplitude is $\frac{-\frac{5}{16\mu^3}}{-\frac{1}{2\mu}} = \frac{5}{8\mu^2}$. Similarly, the four-point amplitude at genus zero is given by $\kappa^2[v^+\widetilde{\mathcal{P}}_+ + v^-\widetilde{\mathcal{P}}_-]^4|_{\kappa=\frac{1}{\mu},\theta=\frac{\pi}{2}}$, which yields the coefficient $\frac{6}{\mu^2}\sin^4\frac{\pi}{4}\cos^4\frac{\pi}{4} = \frac{3}{8\mu^2}$. The ratio of the coefficients of the first two terms in the expansion of the open string amplitude is $\frac{\frac{35}{128\mu^4}}{\frac{3}{8\mu^2}} = \frac{35}{48\mu^2}$. Both of these ratios differ from that of the quartic curvature sector of the Type IIB superstring effective action. It may be noted, however, that quotient of the three-point ratio by the four-point ratio is $\frac{6}{7}$. Considering the degeneration limits of the genus-one amplitudes, the ratios will be given by the product of three-point genus-zero amplitude, combined with a ratio resulting from the removal of a four-point contribution, and the inverse of a superstring factor, and the product of a four-point amplitude and the inverse of a superstring factor. The calculation of the relative modular factor in $N = 2$ open string theory will include a quotient by a numerical coefficient derived from the breaking of the restriction of the $N = 2$ closed string worldsheet supersymmetry to that of the open string without the projection of Type IIB supersymmetry. The restriction to the relative modular group also introduces a factor of $\frac{1}{4}$. Using the original four-point genus-zero amplitude and the derived three-point genus-zero coefficient, the ratio is $\frac{\frac{3}{8\mu^2}}{\frac{5}{8\mu^2}} = \frac{3}{5}$. The complement of this ratio is $1 - \frac{3}{5} = \frac{2}{5}$. The product of this fraction with relative modular group and the closed

[1] S. Plouffe, *Identities inspired by Ramanujan Notebooks, part II and III*, (2009), URL: http://plouffe.fr/simon/articles/Identities%20inspired%20by%20Ramanujan%20Notebooks2.pdf.

superstring factors is $\frac{2}{5}\times\frac{1}{4}\times\frac{60}{7}=\frac{6}{7}$, which coincides with the quotient of the three-point and four-point ratios.

The closed string amplitudes $A_0^{(n)}$ also vanish for more than three vertices at leading order. The closed string three-point amplitude $A_0^{(3)}$ does not have singularities and $A_{1,\,cl}^{(3)}$ is evaluated through modular integration on the torus [4] to be

$$A^{(3)}_{1,\text{ cl, unreg}}=\frac{3}{16}c_{12}^6\int_{\mathcal{F}}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\sum_{(n,m)\neq(0,0)}\frac{(\mathrm{Im}\,\tau)^3}{|n+m\tau|^6} \tag{2.11}$$

where $c_{12}=\bar{p}_1\cdot p_2-p_1\cdot\bar{p}_2$, with p_1 and p_2 being the complex external momenta, and one half of the sum is the nonholomorphic Eisenstein series [31]

$$G(\tau,3)=\zeta(6)(\mathrm{Im}\,\tau)^3+\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)}\zeta(5)(\mathrm{Im}\,\tau)^{-2}+\frac{2\pi^3}{\Gamma(3)}(\mathrm{Im}\,\tau)^{\frac{1}{2}}\sum_{\substack{m\geq1\\ r\neq0}}m^{-\frac{5}{2}}|r|^{\frac{5}{2}}K_{\frac{5}{2}}(2\pi m|r|\,\mathrm{Im}\,\tau)e^{2\pi imr\,\mathrm{Re}\,\tau}. \tag{2.12}$$

The integral of the first term $\frac{2\pi^6}{7!}c_{12}^6\int_{\mathcal{F}}d^2\tau\ (\mathrm{Im}\,\tau)$ is attributed to infrared divergences. It can be traced to a divergent formula for the three-point vertex

$$-12\frac{\pi^3}{(\mathrm{Im}\,\tau)^3}c_{12}^6\int d^2z\,d^2w\,d^2x\,\Delta(w,z)\Delta(w,z)\Delta(z,x)$$

at the coincidences of the bosonic propagator $\Delta(z_1,z_2)=\log\left|\frac{\Theta\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right](z_{12},\tau)}{\Theta'\left[\begin{smallmatrix}\frac12\\ \frac12\end{smallmatrix}\right](0,\tau)}\right|^2-\frac{2\pi}{\mathrm{Im}\,\tau}(\mathrm{Im}\,z_{12})^2$ [4]. The inclusion of a regularized Green function in the amplitude would yield a finite integral for the three-point amplitude, which is nonsingular on the sphere. The removal of the first term is not invariant under modular transformations. Consider instead the volume of the fundamental region $\mathbb{H}^2/PSL(2;\mathbb{Z})$, which is equal to $V_1=2\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)|_{s=2}=\frac{\pi}{3}$. Given the invariance of zeta function regularization under the action of the generators $L_n=z^{n-1}\frac{d}{dz}$ on the complex plane including $sl(2;\mathbb{R})$ and $sl(2;\mathbb{Z})$ [24], this method yields the coefficient in the central extension of the Witt algebra to the Virasoro algebra. Analytic continuation to $s=-1$ gives $2\pi^{-\frac{1}{2}}\Gamma\left(-\frac{1}{2}\right)\zeta(-1)=\frac{1}{3}$. Setting the first integral equal to this constant,

$$\begin{aligned}A_{1,\text{cl},\text{reg}}&=\frac{2}{3}c_{12}^6+\frac{9\pi}{64}\zeta(5)c_{12}^6\int_{\mathcal{F}}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^4}\\&\quad+\frac{3\pi^{\frac{7}{2}}}{16}c_{12}^6\int_{\mathcal{F}}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^{\frac{3}{2}}}\sum_{\substack{m\geq1\\ r\neq0}}m^{-\frac{5}{2}}|r|^{\frac{5}{2}}e^{-2\pi m|r|\,\mathrm{Im}\,\tau}(3(2\pi m|r|\,\mathrm{Im}\,\tau)^{-\frac{1}{2}}\\&\quad+3(2\pi m|r|\,\mathrm{Im}\,\tau)^{-\frac{3}{2}}+(2\pi m|r|\,\mathrm{Im}\,\tau)^{-\frac{5}{2}})e^{2\pi imr\,\mathrm{Re}\,\tau}\\&=2\left[\frac{1}{3}+\frac{3\pi}{64\sqrt{3}}\zeta(5)\right]c_{12}^6+\frac{9\pi^3}{16\sqrt{2}}c_{12}^6\int_{\mathcal{F}}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\sum_{\substack{m\geq1\\ r\neq0}}m^{-3}|r|^2e^{-2\pi m|r|\,\mathrm{Im}\,\tau}\\&\quad+\frac{9\pi^2}{32\sqrt{2}}c_{12}^6\int_{\mathcal{F}}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^3}\sum_{\substack{m\geq1\\ r\neq0}}m^{-4}|r|e^{-2\pi m|r|\,\mathrm{Im}\,\tau}e^{2\pi imr\,\mathrm{Re}\,\tau}\\&\quad+\frac{3\pi}{32\sqrt{2}}c_{12}^6\int_{\mathcal{F}}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^4}\sum_{\substack{m\geq1\\ r\neq0}}m^{-5}e^{-2\pi m|r|\,\mathrm{Im}\,\tau}e^{2\pi imr\,\mathrm{Re}\,\tau}\end{aligned} \tag{2.13}$$

with $\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^4} = \frac{2}{3\sqrt{3}}$. This formula for the regularized three-point amplitude would be derived from an effective action for the closed $N = 2$ string theory with a coefficient of the coupling that does not exactly equal the integral of an Eisenstein series. The factors of $\mathrm{Im}\,\tau$ have been found to be essential for establishing the consistency of the point particle limit [5]. Other methods for integrating over the fundamental region of $SL(2;\mathbb{Z})$ with powers of $\mathrm{Im}\,\tau$ in the numerator include its partitioning and the evaluation of separate nonanalytic threshold terms, where all of the integrals must be evaluated to preserve modular invariance [9].

The four-point function vanishes on-shell, and the amplitude derived from a cubic action with off-shell cubic couplings still requires equations derived with the relativistic null momentum to be vanishing [21]. Since the off-shell four-point amplitude would not be zero, the higher-point amplitudes do not necessarily vanish at genus $g \geq 1$. For example, the factorization of the four-point torus amplitude into three-point sphere and torus amplitudes introduces kinematical factors [4] that vanish only on-shell, and therefore, integration over off-shell internal momenta could yield a non-zero result.

The effective action of the $N = 2$ string in four dimensions, evaluated from the non-zero three-point function, is found to be a self-dual Yang-Mills theory that can be written in either of two gauges [16]. The $N = 2$ heterotic string effective action can be reduced either to two or three dimensions after the momentum vector is chosen to belong to a four-dimensional subspace of the 26 dimensional embedding space, after $U(1)$ reduction from 28 dimensions, or a twenty-four dimensional subspace with another timelike coordinate, and the action will receive corrections beyond four-point terms for the scalar field [19].

The genus g amplitude of the topological string on $T^2 \times \mathbb{R}^2$ [20] is equal to

$$\begin{aligned}F^g(u_L, u_R) = C(g) &\sum_{(n,m)\neq(0,0)} |n + m\sigma|^{2g-4} \left(\frac{u_L^1 u_R^1}{n + m\sigma} + \frac{u_L^2 u_R^2}{n + m\bar{\sigma}} \right)^{4g-4} \\ &\cdot \sum_{-2g+2 \leq n_L \leq 2g-2} \binom{4g-4}{2g-2+n_L} \sum_{(n,m)\neq(0,0)} \frac{|n + m\sigma|^{2g-4}}{(n + m\sigma)^{2g-2+n_L}(n + m\bar{\sigma})^{2g-2-n_L}} \\ &\cdot (u_L^1)^{2g-2+n_L}(u_R^1)^{2g-2+n_L}(u_L^2)^{2g-2-n_L}(u_R^2)^{2g-2-n_L}, \end{aligned} \tag{2.14}$$

which follows from the series

$$\begin{aligned}F^g(u_L, u_R) = &\sum_{-2g+2 \leq n_L, n_R \leq 2g-2} \binom{4g-4}{2g-2+n_L} \binom{4g-4}{2g-2+n_R} F^g{}_{n_L,n_R} \\ &\cdot (u_L^1)^{2g-2+n_L}(u_R^1)^{2g-2+n_R}(u_L^2)^{2g-2-n_L}(u_R^2)^{2g-2-n_R}\end{aligned}$$

only if

$$\begin{aligned}&F^g_{n_L,n_R} = 0, \quad n_L \neq n_R, \\ &F^g{}_{n_L n_L} = C(g) \binom{4g-4}{2g-2+n_L}^{-1} \sum_{(n,m)\neq(0,0)} \frac{|n + m\sigma|^{2g-4}}{(n + m\sigma)^{2g-2+n_L}(n + m\bar{\sigma})^{2g-2-n_L}}\end{aligned} \tag{2.15}$$

and yields

$$F^2{}_{0,0} = C(2) \binom{4}{2}^{-1} \sum_{(n,m)\neq(0,0)} \frac{1}{(n + m\sigma)^2(n + m\bar{\sigma})^2},$$

$$F^2{}_{1,1} = C(2)\binom{4}{3}^{-1}\sum_{(n,m)\neq(0,0)}\frac{1}{(n_+m\sigma)^3(n+m\bar{\sigma})}$$

and

$$F^2{}_{2,2} = C(2)\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\sigma)^4},$$

which coincides with the Eisenstein series for genus 2 if $C(2) = 12$.

The contribution of the instantons is

$$\begin{aligned}\mathcal{A}^g(\theta,\sigma) &= \sum_m \binom{4g-4}{2g-2+m}^2 \mathcal{A}^g_m e^{im\vartheta}\\ &= C(g)\sum_{r,s}{}'|r+s\sigma|^{2g-4} + \left(\frac{e^{\frac{i\vartheta}{2}}}{r+s\sigma} + \frac{e^{-\frac{i\vartheta}{2}}}{r+s\bar{\sigma}}\right)^{4g-4}\\ &= 2^{4g-4}C(g)\sum_{r,s}{}'\frac{(\mathrm{Re}(r+s\sigma)\cos\frac{\vartheta}{2} + \mathrm{Im}(r+s\sigma)\sin\frac{\vartheta}{2})^{4g-4}}{|r+s\sigma|^{6g-4}}.\end{aligned} \tag{2.16}$$

When σ is real and $\vartheta = (2n+1)\pi$, the sum vanishes. If σ is real and $\vartheta = 2n\pi$, the sum is '$2^{4g-4}C(g)\sum'_{r,\theta}\frac{1}{(r+s\sigma)^{2g}}$.

3. Modular Invariance of Compactifications of Ten-Dimensional Theory

Modular forms arise upon compactification of the string background, and especially, the $SL(d;\mathbb{Z})$, $SO(d,d;\mathbb{Z})$ and $E_{d+1(d+1)}(\mathbb{Z})$ Eisenstein series for theories with S-duality, T-duality and U-duality when the compact dimensions describe the torus T^d. There remains, however, an inherent $SL(2;\mathbb{Z})$ modular invariance arising from the summation over Riemann surfaces, since $SL(2;\mathbb{Z}) \subset Sp(2g;\mathbb{Z})$ and $SL(2;\mathbb{Z}) \subset SL(2;\mathbb{R}) \subset SL(2;\mathbb{C})$, the global conformal group of the sphere. This symmetry occurs in coefficients of both R^4 and similar terms in the superstring effective action in ten dimensions [14].

The partition function of string theory after compactification over T^d at genus one is

$$\begin{aligned}I_d &= 2\pi\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}Z_{d,d}(g,B;\tau),\\ Z_{d,d} &= \tau_2^{\frac{d}{2}}\sum_{m_i,n^i} e^{-\pi(m^+\tau n^i)(g_{ij}+B_{ij})(m^j+\tau n^j)},\end{aligned} \tag{3.1}$$

where g_{ij} are block diagonal elements in an $SL(d;\mathbb{R})$ subgroup and B_{ij} parameterize a nilpotent subgroup in the Iwasawa decomposition of $SO(d,d;\mathbb{R})$, such that

$$\begin{aligned}I_d &= 2\mathcal{E}^{SO(d,d;\mathbb{Z})}_{S;s=1} + 2\mathcal{E}^{SO(d,d;\mathbb{Z})}_{C;s=1},\\ I_1 &= 2(2\zeta(2)R^{-1} + 2\zeta(2)R) = 4\zeta(2)\left(R+\frac{1}{R}\right) = \frac{2\pi^2}{3}\left(R+\frac{1}{R}\right),\\ I_2 &= 2(-\pi\log(4e^{-4\gamma}\tau_2(U)|\eta(U)|^4) - \pi\log(4e^{-4\gamma}\tau_2(T)|\eta(T)|^4)\\ &= -2\pi\log(16e^{-8\gamma}\tau_2(T)\tau_2(U)|\eta(T)\eta(U)|^4)\end{aligned} \tag{3.2}$$

with $\tau_2 = \mathrm{Im}\,\tau$ [18].

Now consider the genus-g amplitude

$$I_d^g = \int_{\mathcal{M}_g} d\mu_g Z_{d,d}^g(g_{ij}, B_{ij}; \tau_{AB}),$$

$$Z_{d,d}^g = \frac{1}{4}\tau_{2AC}\tau_{2BD}\left(\frac{\partial}{\partial\tau_{1AB}}\frac{\partial}{\partial\tau_{1CD}} + \frac{\partial}{\partial\tau_{2AB}}\frac{\partial}{\partial\tau_{2CD}}\right), \tag{3.3}$$

after defining the real and imaginary parts of elements of the period matrix τ_{AB} to be $\tau_{1AB} = \mathrm{Re}\,\tau_{AB}$ and $\tau_{2AB} = \mathrm{Im}\,\tau_{AB}$ [18]. Instead of restricting the period matrices to a fundamental domain of $Sp(2g;\mathbb{Z})$ in the Siegel upper half plane, the sum over m_A^i and n^{iA} can be constrained. Then $u(x_1, x_2, t) = \frac{\partial^2}{\partial x_2^2}\ln\Theta(Ux_1 + Vx_2 + Wt + X|\tau)$ must satisfy the Kadomtsev-Petviashvili equation $u_{x_1x_1} + \frac{2}{3}(u_t - 3uu_{x_2} + \frac{1}{2}u_{x_2x_2x_2})_{x_2}$ when τ is the period matrix of a Riemann surface.

Theorem 3.1. *The solutions to the Kadomtsev-Petviashvili equation form a set invariant under the action of the symplectic modular group at arbitrary genus.*

Proof. Under a modular transformation, $\tau \to (A\tau + B)(C\tau + D)^{-1}$,

$$\Theta(((C\tau + D)^{-1})^T z|(A\tau + B)(C\tau + D)^{-1})) = e^{\frac{2\pi i}{8}}(\det(C\tau + D))^{\frac{1}{2}} e^{\pi i z(C\tau+D)^{-1}C\cdot z}\Theta(z|\tau).$$

If $z = Ux_1 + Vx_2 + Wt + X$,

$$u(x_1, x_2, t) = 2\frac{V^A V^B}{\Theta(z|\tau)}\left[\Theta(z|\tau)\frac{\partial}{\partial z^A}\frac{\partial}{\partial z^B}\Theta(z|\tau) - \frac{\partial}{\partial z^A}\Theta(z|\tau)\frac{\partial}{\partial z^B}\Theta(z|\tau)\right]. \tag{3.4}$$

Since

$$\tilde{z} = ((C\tau + D)^{-1})^T Ux_1 + ((C\tau + D)^{-1})^T Vx_2 + ((C\tau + D)^{-1})^T Wt + ((C\tau + D)^{-1})X$$

$$\tilde{u}(x_1, x_2, t) = 2\frac{((C\tau + D)^{-1}V)^A((C\tau + D)^{-1}V)^B}{\Theta(\tilde{z}|\tilde{\tau})^2}\left[\Theta(\tilde{z}|\tilde{\tau})\frac{\partial}{\partial\tilde{z}^A}\frac{\partial}{\partial\tilde{z}^B}\Theta(\tilde{z}|\tilde{\tau}) - \frac{\partial}{\partial\tilde{z}^A}\Theta(\tilde{z}|\tilde{\tau})\frac{\partial}{\partial\tilde{z}^B}\Theta(\tilde{z}|\tilde{\tau})\right] \tag{3.5}$$

and

$$\begin{aligned}&2V^A V^B\frac{1}{\Theta(\tilde{z}|\tilde{\tau})^2}\left[\Theta(\tilde{z}|\tilde{\tau})\frac{\partial}{\partial z^A}\frac{\partial}{\partial z^B}\Theta(\tilde{z}|\tilde{\tau}) - \frac{\partial}{\partial z^A}\Theta(\tilde{z}|\tilde{\tau})\frac{\partial}{\partial z^B}\Theta(\tilde{z}|\tilde{\tau})\right]\\ &\quad = 2V^A V^B[2\pi i((C\tau + D)^{-1}C)_{AB} - 4\pi^2 z_E((C\tau + D)^{-1}C)_{EA}((C\tau + D)^{-1}C)_{FB}z_F].\end{aligned} \tag{3.6}$$

The first term $2\pi i V^A V_B((C\tau_D)^{-1}C)_{AB}$ can be removed through the addition of a constant to u, which would remain a solution to the differential equation. The second term yields a non-zero function of z. If the remainder is denoted by u_r,

$$\begin{aligned}&(u + u_r)_{x_1x_1} + \frac{2}{3}\left((u + u_r)_t - 4(u + u_r)(u + u_r)_{x_2} + \frac{1}{2}(u + u_r)_{x_2x_2x_2}\right)_{x_2}\\ &\quad = (u_r)_{x_1x_1} + \frac{2}{3}(u_r)_{x_2t} - 2(u_r)_{x_2}^2 - 4u_{x_2}(u_r)_{x_2} - 2u(u_r)_{x_2x_2}\end{aligned} \tag{3.7}$$

since u satisfies the Kadomtsev-Petviashvili equation. The second derivatives u_r in this equation

$$\begin{aligned}(u_r)_{x_1x_1} &= U^A U^B\frac{\partial}{\partial z^A}\frac{\partial}{\partial z^B}u_r,\\ (u_r)_{x_2t} &= V^A W^B\frac{\partial}{\partial z^A}\frac{\partial}{\partial z^B}u_r\end{aligned} \tag{3.8}$$

are constants and the equation has the form

$$c_1(\tau)z\frac{du}{dz}+c_2(\tau)u=c_3(\tau)z^2+c_4(\tau) \tag{3.9}$$

which can be solved by another remainder term

$$\begin{aligned} u_{r_1}&=c_5(\tau)z^2+c_6(\tau),\\ c_5(\tau)&=\frac{c_3(\tau)}{2c_1(\tau)+c_2(\tau)},\\ c_6(\tau)&=\frac{c_4(\tau)}{c_2(\tau)}. \end{aligned} \tag{3.10}$$

The sequence $u+u_r+u_{r_1}$ continued to an infinite series $u+u_r+u_{r_1}+\dots$ with cancellations of the constant and quadratic terms. The coefficients $c_1(\tau)$ and $c_3(\tau)$ derived from

$$\begin{aligned} -4u_{x_2}(u_r)_{x_2}&=-4V^AV^B\frac{\partial u}{\partial z^A}\frac{\partial u_r}{\partial z_B}\\ &=64\pi^2V^AV^BV^GV^H\frac{\partial u}{\partial z^A}((C\tau+D)^{-1}C)_{BG}((C\tau_+D)^{-1}C)_{FH}z_F,\\ 2(u_r)^2_{x_2}&=2((-8\pi^2)^2)V^AV^B\frac{\partial u_r}{\partial z^A}\frac{\partial u_r}{\partial z^B}\\ &=512\pi^4V^AV^BV^GV^HV^IV^J((C\tau+D)^{-1}C)_{AG}((C\tau+D)^{-1}C)_{FH}z_F\\ &\quad\cdot((C\tau+D)^{-1}C)_{BI}((C\tau+D)^{-1}C)_{KJ}z_K, \end{aligned} \tag{3.11}$$

respectively. Similarly, $c_2(\tau)$ and $c_4(\tau)$ are derived from

$$\begin{aligned} -2(u_r)_{x_2x_2}&=-2V^AV^B\frac{\partial^2u_r}{\partial z^A\partial z^B}\\ &=32\pi^2V^AV^BV^GV^H((C\tau+D)^{-1}C)_{AG}((C\tau+D)^{-1}C)_{BH},\\ -u_{x_1x_1}-\frac{2}{3}u_{x_2t}&=-U^AU^B\frac{\partial^2u_r}{\partial z^A\partial z^B}-\frac{2}{3}V^AW^B\frac{\partial^2u_r}{\partial z^A\partial z^B}\\ &=16\pi^2U^AU^BV^GV^H((C\tau+D)^{-1}C)_{AG}((C\tau+D)^{-1}C)_{BH}\\ &\quad+\frac{32}{3}\pi^2V^AW^BV^GV^H((C\tau+D)^{-1}C)_{AG}((C\tau+D)^{-1}C)_{BH}. \end{aligned} \tag{3.12}$$

Then,

$$c_5(\tau)\sim\frac{512\pi^4|V|^6((C\tau+D)^{-1}C)^4}{160\pi^2|V|^4((C\tau+D)^{-1}C)^2}\sim\frac{16}{5}\pi^2|V|^2((C\tau+D)^{-1}C)^2$$

and

$$c_6(\tau)\sim\frac{16\pi^2|U|^2|V|^2+\frac{32}{3}\pi^2|V|^3|W|}{32\pi^2|V|^2}\sim\frac{|U|^2+\frac{2}{3}|V||W|}{2|V|^2}.$$

Consequently u_1 does not satisfy the Kadomtsev-Petviashvili equation and u cannot equal u_1.

If V^A is chosen such that

$$2V^AV^B\Big[2\pi i((C\tau+D)^{-1}C)_{AB}-4\pi^2z_E((C\tau+D)^{-1}C)_{EA}((C\tau+D)^{-1}C)_{FB}z_F\Big]=0, \tag{3.13}$$

or $V^A((C\tau+D)^{-1}C)_{AB}=0$, the extra term will have no effect on the solution, since $u_r=0$. This condition reduces the number of independent components of V^A to $g-1$. It follows that

$$(V(C\tau+D)^{-1})^AC_{AB}=\tilde{V}^AC_{AB}=0. \tag{3.14}$$

Only a single component of $\tilde{V}^A$ is determined, and the vector is not zero. The map from $\{V^A\}$ to $\{\tilde{V}^A\}$ is now an isomorphism of $g-1$ complex dimensional spaces. Then it follows that $\tilde{u}$ will be a solution to the Kadomtsev-Petviashvili equation representing the period matrix $(A\tau+B)(C\tau+D)^{-1}$ of another Riemann surface in Teichmüller space. □

The Kadomtsev-Petviashvili equation is equivalent to the condition for a Plücker embedding of the universal Grassmannian in the exterior power of the space of one-particle states in a free-fermion theory on a Riemann surface [28]. Since the Teichmüller space of any uniformizing Fuchsian group can be embedded in the universal Grassmannian [30], it follows that the period matrices related by a symplectic modular transformation $\tilde{\tau}=(A\tau+B)(C\tau+D)^{-1}$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g;\mathbb{Z})$ correspond to theta functions such that $\frac{\partial^2}{\partial x_2^2}\log\Theta(\tilde{U}x_1+\tilde{V}x_2+\tilde{W}t+\tilde{X}|\tilde{\tau})$ is a solution to the differential equation.

The preservation of $Sp(2g;\mathbb{Z})$ invariance in the covering space, the domain of the differential equation, yields the proof of $\Delta_{SO(d,d;\mathbb{Z})}I_d^g = \frac{dg(g+1-d)}{4}I_g^d$ [18]. There are other symmetries which yield equalities between Eisenstein series and restrict the number of different types of coefficients, and an example arises in the compactification over T^d. The Laplacian $\Box_d$ has eigenfunctions in the vector and spinor representations of the group $SO(d,d)$ $\Box_d\mathcal{E}_{V;s}^{SO(d,d;\mathbb{Z})} = \frac{s(s-d+1)}{2}\mathcal{E}_{V;s}^{SO(d,d;\mathbb{Z})}$, $\Box_d\mathcal{E}_{S;s=1}^{SO(d,d;\mathbb{Z})} = \frac{d(2-d)}{8}\mathcal{E}_{S;s=1}^{SO(d,d;\mathbb{Z})}$ and $\Box_d\mathcal{E}_{C;s=1}^{SO(d,d;\mathbb{Z})} = \frac{d(2-d)}{8}\mathcal{E}_{C;s=1}^{SO(d,d;\mathbb{Z})}$, and the nonperturbative amplitude for this compactification has been shown to be equal to $I_d = 2\frac{\Gamma\left(\frac{d}{2}-1\right)}{\pi^{\frac{d}{2}-2}}\mathcal{E}_{V;s=\frac{d}{2}-1}^{SO(d,d;\mathbb{Z})} = 2\mathcal{E}_{S;s=1}^{SO(d,d;\mathbb{Z})} = 2\mathcal{E}_{C;s=1}^{SO(d,d;\mathbb{Z})}$. When $d=4$, $I_4 = 2\mathcal{E}_{V;s=1}^{SO(4,4;\mathbb{Z})} = 2\mathcal{E}_{S;s=1}^{SO(4,4;\mathbb{Z})} = 2\mathcal{E}_{C;s=1}^{SO(4,4;\mathbb{Z})}$ [18]. This equality reflects the existence of a triality automorphism in $SO(4,4)$. The triality in $SO(8)$ relates the vector and spinor and conjugate spinor representations of the isometry group of the parallelizable seven-sphere. The other seven-dimensional parallelizable manifold is the quadric admitting the transitive action of $SO(4,4)$. The triality automorphism follows from the S_3 permutation symmetry of the root diagram of D_4 and the weights of the $so(4,4)$ root system [13]. Neither the paralellizability or the triality would be present for the other groups $SO(m,n)$, $m+n=8$, $m\neq 0,4,8$, $n\neq 0,4,8$.

The Eisenstein series that do arise in the expressions for the amplitudes are included in the basis for modular forms of arbitrary weight. The subset of forms that occur and the role of the spanning set in the series expansion of amplitudes and coefficients in the effective action will be described.

4. The Dimensions of Spaces of Modular Forms and Transcendental Numbers

The decomposition of modular forms of weight k, $M_k=\langle G_k\rangle + S_k$, where G_k is the Eisenstein series $G_k(\tau)=\frac{(k-1)!}{2(2\pi i)^k}\sum_{m,n}{}'\frac{1}{(m\tau+n)^k}$ and $\mathcal{S}_k$ is the space of cusp forms with no constant term majorized by $Im(\tau)^{-\frac{k}{2}}$ as $\mathrm{Im}\,\tau$ tends to ∞. The Eisenstein series equals

$$G_k(\tau)=\frac{(-1)^{\frac{k}{2}}(k-1)!}{(2\pi)^k}\zeta(k)+\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}r^{k-1}e^{2\pi irm\tau}$$

$$= -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{4.1}$$

where B_k is the kth Bernoulli number, $\sigma_{k-1}(n) = \sum_{r|n} r^{k-1}$ and $q = e^{2\pi i\tau}$, and it may be proven that G_4 and G_6 form a basis for the modular forms of even weight k for all $k \in 2\mathbb{Z}$, $k > 0$ [31].

A formula for the dimension of the space of cusp forms of even weight will be given.

Lemma 4.1. *The dimension of the space of cusp forms of weight k, $k \in 2$ is* $\left\lfloor \frac{\lfloor \frac{k}{6} \rfloor}{2} + \left[\frac{k}{4}\right] \right\rfloor + \delta_{k,2} - \left\{\left[\frac{k}{4}\right]\right\}$.

Proof. Consider the spanning set $\langle G_4^a G_6^b | 4a+6b = k\rangle$. The constant term in the sum $\sum_{\substack{a,b \\ 4a+6b=k}} c_{ab} G_4^a G_6^b$ is

$$\sum_{\substack{a,b=0 \\ 4a+6b=k}} c_{ab} \left(\frac{1}{2}\zeta(-3)\right)^a \left(\frac{1}{2}\zeta(-5)\right)^b = \sum_{\substack{a,b=0 \\ 4a+6b=k}} (-1)^b c_{ab} \frac{1}{2^{a+b}} \left(\frac{1}{240}\right)^a \left(\frac{1}{504}\right)^b. \tag{4.2}$$

The integers c_{ab} must be algebraic because the Fourier coefficients of cusp forms are algebraic [31].

The number of solutions to $4a+6b = k$ is the number of solutions to $2a+3b = \ell$ where $\ell = \frac{k}{2}$, $0 \le a \le \frac{\ell}{2}$ and $0 \le b \le \frac{\ell}{3}$. There is only one independent exponent since $a = \frac{\ell - 3b}{2}$. If ℓ is odd, $\lfloor \frac{\ell}{3} \rfloor$ is odd, the number of values of b between 0 and $\frac{\ell}{3}$ is $\frac{\lfloor \frac{\ell}{3} \rfloor + 1}{2}$ if $\lfloor \frac{\ell}{3} \rfloor$ is odd, and $\frac{\lfloor \frac{\ell}{3} \rfloor}{2}$ when $\lfloor \frac{\ell}{3} \rfloor$ is even. If ℓ is even, the number of values of b is $1 + \lfloor \frac{\lfloor \frac{\ell}{3} \rfloor}{2} \rfloor$ because there is an extra solution with $b = 0$ and $a = \frac{\ell}{2}$. Representing this additional number as $1 - \left\{\left[\frac{\ell}{2}\right]\right\}$. Then the number of integer solutions to $2a+3b = \ell$ or $4a+6b = k$ is

$$\left\lfloor \frac{\lfloor \frac{\ell}{3} \rfloor}{2} + \left[\frac{\ell}{2}\right] \right\rfloor + \left(1 - \left\{\left[\frac{\ell}{2}\right]\right\}\right) = \left\lfloor \frac{\lfloor \frac{k}{6} \rfloor}{2} + \left[\frac{k}{4}\right] \right\rfloor + \left(1 - \left\{\left[\frac{\ell}{2}\right]\right\}\right) \tag{4.3}$$

The relation (4.2) would be solved for one coefficient, reducing the dimension by 1, except for $k = 2$, when there are no modular forms of weight 2 and no relation amongst elements of the spanning set. It follows that the dimension of the space of cusp forms of weight k is

$$\left\lfloor \frac{\lfloor \frac{k}{6} \rfloor}{2} + \left[\frac{k}{4}\right] \right\rfloor + \left(1 - \left\{\left[\frac{\ell}{2}\right]\right\}\right) + \delta_{2\ell,2} - 1 = \left\lfloor \frac{\lfloor \frac{k}{6} \rfloor}{2} + \left[\frac{k}{4}\right] \right\rfloor + \delta_{k,2} - \left\{\left[\frac{k}{4}\right]\right\}. \tag{4.4}$$

□

It can be verified that $\dim \mathcal{S}_k = \dim \mathcal{M}_{k-12}$ for $k \ge 16$ because

$$\dim \mathcal{M}_k = \left\lfloor \frac{\lfloor \frac{k}{6} \rfloor}{2} + \left[\frac{k}{4}\right] \right\rfloor - \left\{\left[\frac{k}{4}\right]\right\} + 1$$

and

$$\begin{aligned} \dim \mathcal{S}_{k+12} &= \left\lfloor \frac{\lfloor \frac{k+12}{6} \rfloor}{2} + \left[\frac{k+12}{4}\right] \right\rfloor + \delta_{k+12,2} - \left\{\left[\frac{k}{4}\right]\right\} \\ &= \left\lfloor \frac{\lfloor \frac{k}{6} \rfloor}{2} + \left[\frac{k}{4}\right] \right\rfloor + \frac{\frac{12}{6}}{2} - \left\{\left[\frac{k}{4}\right]\right\} - \left\{\left[\frac{12}{4}\right]\right\} = \dim \mathcal{M}_k \end{aligned} \tag{4.5}$$

while $\dim \mathcal{S}_{12} = \dim M_0 = 1$ and $\dim \mathcal{S}_{14} = \dim M_2 = 0$.

The product of the constant term of $G_k(\tau)$ with $\frac{1}{2}(1+(-1)^k)$ equals $\frac{1}{2}\zeta(1-k)$ by the functional relation for the Riemann zeta function $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta((k)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-k)$.

Lemma 4.2. *The relation*

$$\frac{(-1)^{\frac{k}{2}}}{(2\pi)^k}\frac{1}{2}(1+(-1)^k))\Gamma(k)\zeta(k)=\frac{1}{2}\zeta(1-k) \tag{4.6}$$

is valid for all integers k. The factor $\frac{1}{2}(1+(-1)^k)$ is required since $\zeta(s)$ has trivial zeros at all negative integer zeros.

Proof. Substituting $s=k$ in the functional relation for the Riemann zeta function gives

$$\pi^{\frac{1}{2}-k}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{1-k}{2}\right)\zeta(k)=\zeta(1-k). \tag{4.7}$$

The doubling relation for the gamma function $\Gamma(2z)=(2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$ at negative arguments is

$$\Gamma(-s)=(2\pi)^{-\frac{1}{2}}2^{-s-\frac{1}{2}}\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right). \tag{4.8}$$

Similarly, the identity $\Gamma(z)\Gamma(-z)=-\frac{\pi}{z\sin(\pi z)}$ at $z=-\frac{k}{2}$ is

$$\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}\right)=-\frac{\pi}{\frac{k}{2}\sin\left(\frac{k\pi}{2}\right)} \tag{4.9}$$

generally. Nevertheless, it will be useful to transpose factors when the denominator vanishes and write the identity in the form

$$\frac{1}{2}(-1)^{\frac{k-1}{2}}(1-(-1)^k)\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k}{2}\right)=-\frac{2\pi}{k},\quad k\in\mathbb{Z}^*. \tag{4.10}$$

Therefore,

$$\frac{1}{2}(1-(-1)^k)\Gamma\left(-\frac{k-1}{2}\right)=-\frac{2\pi}{k}\Gamma\left(\frac{k}{2}\right)^{-1},\quad k\in\mathbb{Z}^* \tag{4.11}$$

and

$$-\frac{2\pi}{k}\Gamma\left(\frac{k}{2}\right)^{-1}\Gamma\left(\frac{1-k}{2}\right)=\frac{1}{2}(-1)^{\frac{k}{2}-1}(1-(-1)^k)2^{k+\frac{1}{2}}(2\pi)^{\frac{1}{2}}\Gamma(-k). \tag{4.12}$$

Since $\Gamma(-k)\Gamma(k)=-\frac{\pi}{k\sin(k\pi)}$ and $\sin\,k\pi=2\sin\frac{k\pi}{2}\cos\frac{k\pi}{2}=2\frac{1}{2}(-1)^{\frac{k-1}{2}}(1-(-1)^k)\frac{1}{2}(1+(-1)^k)(-1)^{\frac{k}{2}}=\frac{1}{2}(-1)^{k-\frac{1}{2}}(1-(-1)^{2k})$ for $k\in\mathbb{Z}^*$,

$$\Gamma(-k)^{-1}=-\frac{\pi}{k}\frac{1}{2}(-1)^{k-\frac{1}{2}}(1-(-1)^{2k})\Gamma(k),\quad k\in\mathbb{Z}^*. \tag{4.13}$$

Then

$$\frac{(-1)^{\frac{k}{2}}}{(2\pi)^k}\frac{1}{2}(1+(-1)^k)\Gamma(k)\zeta(k)=\frac{1}{2}\zeta(1-k),\quad k\in\mathbb{Z}^*. \tag{4.14}$$

If $k=2n+1$ for integer n, $1+(-1)^k=1-(-1)^{2n}=0$, which is consistent with $\zeta(-2n)=0$. Let $k=\epsilon$ and consider the limit

$$\lim_{\epsilon\to 0}\frac{(-1)^{\frac{\epsilon}{2}}}{(2\pi)^{\frac{\epsilon}{2}}}\frac{1}{2}(1+(-1)^{\frac{\epsilon}{2}})\Gamma(\epsilon)\zeta(\epsilon)=\frac{1}{2}\lim_{\epsilon\to 0}\zeta(1-\epsilon),$$

$$\lim_{\epsilon\to 0}\Gamma(\epsilon)\zeta(0)=\frac{1}{2}\lim_{\epsilon\to 0}\zeta(1-\epsilon)$$
$$\lim_{\epsilon\to 0}\left[\frac{1}{\epsilon}+\dots\right]\left(-\frac{1}{2}\right)=\frac{1}{2}\lim_{\epsilon\to 0}\left[\frac{1}{1-\epsilon-1}+\dots\right]. \tag{4.15}$$

It follows that the identity is valid for $k=0$ and, therefore, for all integers. □

There are no non-zero modular forms of negative weight [31]. The modular forms of weight k can be analytically continued to negative values of k.

Theorem 4.1. *The constant terms in the analytically continued forms of weight $-k$ must be transcendental if $\zeta(5)$ and $\zeta(7)$ are algebraically independent.*

Proof. The Eisenstein series form a basis for the modular forms of $PSL(2,\mathbb{Z})$ of weight k with a non-zero constant term. When k is analytically continued to negative values, the constant terms in these forms are required to satisfy an algebraic relation such that the combination equals zero for equality with a cusp form with the analytically continued non-constant terms being rational, since the coefficients of the basis of Hecke forms are algebraic numbers [31]. There must be no solution to these algebraic relations for any set of analytically continued powers $G_4^a G_6^b$, beginning with the analytic continuation of a sum of two terms through the substitution of k by $-k$. Since the constant term of $G_k(\tau)$ is $\frac{1}{2}\zeta(1-k)$ for even k, analytic continuation from k to $-k$ would give $\frac{1}{2}\zeta(1+k)$. The values of the zeta functions $\zeta(2n)$ are known to be periods and include $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$, $\zeta(6)=\frac{\pi^6}{945},\dots$, and, while there is no closed analytic formula for $\zeta(2n+1)$ when n is a positive integer, there exist identities of the form $\zeta(2n+1)=q_{2n+1}\pi^{2n+1}-r_{2n+1,-}S_-(2n+1)-r_{2n+1,+}S_+(2n+1)$, q_{2n+1}, $r_{2n+1,-}$, $r_{2n+1,+}\in\mathbb{Q}$, with $S_\pm(2n+1)=\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}(e^{2\pi k}\pm 1)}$ [1]. The infinite series renders $\frac{1}{2}\zeta(2n+1)$ to be transcendental, and analytic continuation of the sum $\sum_{a,b}c_{a,b}G_4^aG_6^b$ to $\lim_{k_1\to -4,k_2\to -6}\sum_{a,b}c_{a,b}G_{k_1}^aG_{k_2}^b$ gives the constant term

$$\sum_{a,b}\frac{1}{4^{a+b}}c_{a,b}\zeta(5)^a\zeta(7)^b. \tag{4.16}$$

Even if the dimension of the modular forms is not set equal to zero for negative weights $-k$, $k\in\mathbb{Z}^+$, the transcendence of the constant terms would follow from algebraic independence of $\{\zeta(2n+1),\ n\in\mathbb{Z}^+\}$ [2].

Consequently, the analytic continuation of the relation must be transcendental, and each of the constant terms would be transcendental numbers since the dimension of the space of modular forms of weight $-k$ is zero, given the algebraic independence of each of series over $\mathbb{A}[q]$ and consequently the constant terms. The analytically continued series with transcendental coefficients are not modular forms. □

The Eisenstein series for theories invariant under the U-duality group $E_{d+1(d+1)}(\mathbb{Z})$ have been found to represent string multiplets, particle multiplets and multipets of higher-dimensional states, with an example being the R^4 coupling for a toroidal compactification of a

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Type II theory, f_{R^4}, that is conjectured to be

$$\frac{V_{d+1}}{\ell_M^2}\mathcal{E}_{d+1(d+1)}(\mathbb{Z})_{\text{strings};s=\frac{3}{2}},$$

equal to the sum of terms

$$2\zeta(2s)\frac{V_d}{g_s^2}+\left(\frac{V_d}{g_s^2}\right)^{\frac{3}{2}-s}\mathcal{E}^{SO(d,d;\mathbb{Z})}_{S;s=\frac{1}{2}}+\left(\frac{V_d}{g_s^2}\right)^{\frac{3-2s}{4}}\frac{2\pi^s}{\Gamma(s)}\sum_{m\neq 0}\sum_{m^i,m_i}\delta(m_im^i)\left[\frac{m^2V_d}{(\tilde{m}_i)^2+V_d^2(m_i)^2}\right]^{\frac{2s-1}{4}}$$
$$\cdot K_{s-\frac{1}{2}}\left(-\frac{2\pi|m|}{g_s}\sqrt{(\tilde{m}^i)^2+V_d^2(m_i)^2}\right)+\dots \tag{4.17}$$

and

$$\left(\frac{V_d}{g_s^2}\right)^{\frac{3-2s}{4}}\frac{2\pi^s}{\Gamma(s)}\int_0^1 d\theta\sum_{m_i}\left[\frac{m^2g_s^2V_d}{V_d^2\bar{m}^2+g_s^2(\tilde{m}^i)^2+g_s^2V_d^2(m_i)^2}\right]^{\frac{2s-1}{4}}$$
$$\cdot K_{s-\frac{1}{2}}\left(-\frac{2\pi|m+\theta\bar{m}|}{g_s^2}\sqrt{V_d^2\bar{m}^2+g_s^2(\tilde{m}^i)^2+g_s^2V_d^2(m_i)^2}\right)e^{2\pi i\theta m^im_i}, \tag{4.18}$$

with the large-volume limits of the Bessel functions producing nonperturbative effects of $\mathcal{O}(e^{-\frac{1}{g_s}})$ and $\mathcal{O}(e^{-\frac{1}{g_s^2}})$ [18]. These terms result from string and point particle instantons. The $\mathcal{O}(e^{-\frac{1}{g_s}})$ exponential has been demonstrated to have physical effects that are unique to string theory [26]. Both terms represent an integral component of the nonperturbative effects when the elementary particle interactions are described by gauge field theories.

The sufficiency of the modular form $\mathcal{E}^{SL(2;\mathbb{Z})}_{s=\frac{3}{2}}$ for the R^4 coupling in Type IIB superstring theory [22] has prompted consideration of the contribution of the cusp forms. There are holomorphic cusp forms of weight $k+\frac{1}{2}$ of congruence subgroups of the modular group that are constructed from cusp forms of integral weight [27]. Cusp forms are characterized by Fourier coefficients which increase as $a(n)=\mathcal{O}(n^{\frac{k-1}{2}+\epsilon})$ while modular forms have coefficients $a(n)=\mathcal{O}(n^{k-1})$ [6]. Consequently, the coefficient $a(2n)$ would be multiplying an expansion parameter $\frac{1}{g_s^2}$ rather than $\frac{1}{g_s}$, and it is necessary only to select cusp forms such that $a(2n-1)=0$ or $a(2n)=0$ for all $n\in\mathbb{Z}$. The arithmetic density of nonvanishing coefficients can tend to zero as $n\to\infty$ for lacunary complex multiplication curves and half of the prime coefficients vanish if it is a modular form of weight greater than or equal to 2 with a character defined over an imaginary quadratic field [25]. If $a(2)=0$, for example, $a(2(2k+1))=a(2)a(2\ell+1)=0$, while $a(4\ell)=a(4)a(\ell)=a(2)^2-2^{k-1}a(1)a(\ell)=-2^{k-1}a(\ell)$ for a Hecke eigenform of weight k. A cusp form with these coefficients then would equal $\sum_{\ell=0}^{\infty}a(2\ell+1)q^{2\ell+1}+\sum_{\ell=1}^{\infty}a(4\ell)q^{4\ell}$. The coefficients would not be equal to those of a modular form with coupling g_s^2. Instead, the coefficients $a(2\ell+1)$ are $\mathcal{O}\big(2^{\frac{k-1}{2}+\epsilon}\big(\ell+\frac{1}{2}\big)^{\frac{k-1}{2}+\epsilon}\big)$, which characterize a cusp form with coupling g_s^2. The asymptotics of this series then would give rise to an exponential term $\mathcal{O}(e^{-\frac{1}{g_s^2}})$ that can represent a set of special instanton effects in point particle field theories. It may be noted that the initial series with $a(2)=0$ also would introduce another exponential term $\mathcal{O}(e^{-\frac{1}{g_s^4}})$. Another choice would be a series with a quadratic form in the exponent of q, which is derived from complex multiplication and equals a theta series when the terms are proportional to q^{n^2}. When $a(m)=0$ for $m\neq n^2$, and $a(n^2)=\mathcal{O}\left(n^{k-1+2\epsilon}\right)$, which matches the coefficients in the modular forms of $E_{d(d)}(\mathbb{Z})$ except that the non-zero terms are multiplied by $g_s^{n^2}$. Theta series have been found to occur in instanton effects in string theory [1].

The results on the expansion of the forms and the exponential terms have been extended to the group $E_{d+1(d+1)}(\mathbb{Z})$ [12]. A class of parabolic Eisenstein series representing non-perturbative corrections in string theory are found to include sets of vanishing coefficients, and the $\mathcal{O}(e^{-\frac{1}{g_s}})$ asymptotics of the series for toroidal compactifications to lower dimensions reflect the contribution of instantons describing the winding of nonperturbative supersymmetric states about the circular directions [11]. Setting g_s proportional to g_{YM}^2 yields the exponential term $\mathcal{O}(e^{-\frac{1}{g_{YM}^2}})$, which is characteristic of instantons in Yang-Mills theories. The $\mathcal{O}(e^{-\frac{1}{g_s^2}})$ and $\mathcal{O}(e^{-\frac{1}{g_s^4}})$ terms similarly would be equivalent to instanton contributions in Yang-Mills theory weighted by $\mathcal{O}(e^{-\frac{1}{g_{YM}^4}})$ and $\mathcal{O}(e^{-\frac{1}{g_{YM}^8}})$. It is known that a duality between Type II worldsheet and heterotic string space-time instantons interchanges $\mathcal{O}(e^{-\frac{1}{g_{s,II}}})$ and $\mathcal{O}(e^{-\frac{1}{g_{s,het}^2}})$ effects [29]. Since a similar phenomenon occurs for the duality between Type I and heterotic string theory, the $\mathcal{O}(e^{-\frac{1}{g_{s,II}}})$ term can be attributed to the open string coupling, which has a magnitude different from the $\mathcal{O}(e^{-\frac{1}{g_{YM}^2}})$ effect in point particle field theories, since $g_{s,open} \propto g_{YM}$. By the relation $g_{YM}^2 = g_{het}^2(2\alpha')^3$ [7], the nonperturbative effect in space-time is restored under duality. These terms then could be derived from a duality transformation of the modular form with a $\mathcal{O}(e^{-\frac{1}{g_{s,II}}})$ term, determined by the asymptotics of the solution to the equations for the scattering amplitude, as an alternative to the $\mathcal{O}(e^{-\frac{1}{g_s^2}})$ exponential arising from a series with every other coefficient nonvanishing.

The dimension of space of cusp forms is $\dim(S_k) = \dim(M_{k-12}) = \dim(M_k) - 1$ for even integral weights k. Holomorphic half-integral cusp forms Θ of weight $\ell + \frac{1}{2}$ can be constructed from cusp forms $\varphi \in S_{2\ell}(\Gamma(N))$ through the transform for congruence subgroups of level N through $\Theta(z,\varphi,L,v) = \sum_x v(x).\ \int_{C(x,\Gamma(N))} \varphi(w) x(1,-w)^{\ell-1} e^{\frac{z}{2}(x,x)} dw$, where $x(u,v) = x_1u^2x_2uv + x_3v^2$, L is a $\Gamma(N)$-invariant lattice, L^* is the dual, v is a function on L^*/L satisfying $v(\rho(\gamma)\cdot x) = \xi^{-1}(\gamma)v(x)$, ξ is a character of Γ, $(x,y) = {}^T yQx = M(x_2y_2 - 2x_1y_3 - 2x_3y_1)$, $8D|N$, where $D = \det(\lambda_i,\lambda_j)$ and $\{\lambda_1,\lambda_2,\lambda_3\}$ is a basis of L, $C(x,\Gamma(N))$ is a geodesic in the upper half plane from $\frac{x_2+\sqrt{d_x}}{2x_3}$ to $\frac{x_2-\sqrt{d_x}}{2x_3}$ for $d_x = x_2^2 - 4x_1x_2 = m^2$, $m \in \mathbb{Z}$ or w to $\gamma_x w$, $\gamma_x = \begin{pmatrix} t_x - x_2u & 2x_1u \\ -2x_3u & t_x + x_2u \end{pmatrix} \in \Gamma(N)$, with (t_x,u_x) being the least half-integer solution to $t^2 - u^2d_x = 1$ and $gcd(x_1,x_2,x_3) = 1$, and the sum is evaluated over Γ-equivalence classes of L with positive discriminant [27]. Since $\dim(M_{\ell+\frac{1}{2}}) = \dim(M_{2\ell})$ [31],which equals $\left\lfloor \frac{\lfloor \frac{\ell}{3} \rfloor}{2} + \left[\frac{\ell}{2}\right]\right\rfloor - \left\{\left[\frac{\ell}{2}\right]\right\} + 1$, the space of cusp forms is a subspace of positive codimension

$$\dim(M_{\ell+\frac{1}{2}}) - \dim(S_{\ell+\frac{1}{2}}) = \dim(M_{2\ell})) - \dim(S_{2\ell}) = 1 \tag{4.19}$$

in the space of modular forms of weight $k + \frac{1}{2}$, $k \in \mathbb{Z}$. The nonperturbative effects resulting from cusp forms would be expected to occur only in a limiting set of the modular forms of a given weight.

The space of cusp forms S_k is a Hilbert space with respect to the Peterson inner product

$$\langle f,g\rangle = \int\int_{\mathbb{H}^2/\Gamma} f(z)\bar{g(z)}y^{2k-2}\,dx\,dy, \tag{4.20}$$

where $\Gamma = SL(2;\mathbb{Z})$ [31]. The integration region may be chosen to be a fundamental domain of the principal congruence subgroup $\Gamma(N)$ for $S_k(N)$, when the number of equivalence class of parabolic cusps is $t(1)=1$, $t(2)=3$ and $t(N)=\frac{1}{2N}[\Gamma:\Gamma(N)]=\frac{N^2}{2}\prod_{p|N}\left(1-\frac{1}{p^2}\right)$ [15].

5. Conclusion

The sum over the genus of the diagrams representing the scattering of four external particles in superstring theory has yielded an effective action with $SL(2;\mathbb{Z})$-invariant couplings given by Eisenstein series. When the theory is compactified over a torus, amplitudes at each genus also have been found to equal Eisenstein series of $SO(d,d;\mathbb{Z})$. The formula at genus $g\geq 3$ requires the Laplacian for $Sp(2g.\mathbb{Z})$ of the integral of the partition function for the $SO(d,d;\mathbb{Z})$ lattice that can be evaluated only if the restriction to the space of Riemann surfaces is invariant under the symplectic modular group. After transferring the restriction of the fundamental domain on the period matrix to the sum over the the lattice, it is proven that the condition on the theta function of the period matrix may be satisfied for all matrices related by a symplectic modular transformation, which would follow also from the Plücker embedding of Teichmüller space into the universal Grassmannian.

The Eisenstein series have been found to correspond to various multiplets of U-duality groups which have expansions with both $\mathcal{O}(e^{-\frac{1}{g_s}})$ and $\mathcal{O}(e^{-\frac{1}{g_s^2}})$ terms in the expansion. The latter terms must represents nonperturbative effects in a point particle theory. It can be traced to the presence of a form with coefficients increasing as $a(n)=\mathcal{O}(n^{\frac{k-1}{2}}+\epsilon)$ or $a(2n)=\mathcal{O}(2^{\frac{k-1}{2}+\epsilon}n^{\frac{k-1}{2}+\epsilon})$. For the modular group $SL(2;\mathbb{Z})$, these coefficients define a cusp form. The generalization of these series to forms invariant under $E_{d+1(d+1)}(\mathbb{Z})$ would be sufficient to derive the $\mathcal{O}(e^{-\frac{1}{g_s^2}})$ term in the superstring amplitudes.

Competing Interests

The author declares that there are no competing interests.

Authors' Contributions

This article was written entirely by the only author, who read and approved the final manuscript.

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