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Research Article

# A Study of Anti-Magic Graphs on Corona Product of Complete Graphs and Complete Bipartite Graphs

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**Abstract.** Graph labeling has a wide range of applications such as coding theory, X-ray crystallography, network design, and circuit design. It can be done by assigning numbers to edges, vertices or to both. An anti-magic labeling of a graph $G$ is a one-to-one correspondence between the edge set $E(G)$ and the set $\{1,2,3,\dots,|E|\}$ such that the vertex sums are pairwise distinct. The vertex sum is the sum of labels assigned to edges incident to a vertex. Corona product of the graphs $H$ and $T$ is the graph $H \odot T$ which is obtained by taking one copy of $H$ and $|V(H)|$ copies of $T$ and making the $i$th vertex of $H$ adjacent to every vertex of the $i$th copy of $T$, $1 \le i \le |V(H)|$. In this study, we prove that the Corona product $K_n \odot K_{m,m}$ generates anti-magic graphs. We also develop a programme using MATLAB to demonstrate this anti-magic property.

**Keywords.** Complete graph, Complete bipartite graph, Corona product, Anti-magic labeling

**Mathematics Subject Classification (2020).** 05C76, 05C78



## 1. Introduction

All graphs in this paper are finite, simple and undirected. Graph theory and its branches have become topics of interest in various fields of Mathematics and other areas of sciences. Graph labeling is an assignment of integers to vertices, edges or to both. In this paper, we have assigned the integers to edges and obtained the labels of vertices by adding the labels of edges incident at each vertex. Suppose $G = (V, E)$ is a graph and $f : E \to \{1, 2, \dots, |E|\}$ is a bijective mapping. For each vertex $x$ of $G$, the vertex sum $\varphi_f(x)$ at $x$ is defined as $\varphi_f(x) = \sum_{e \in E(x)} f(e)$, where $E(x)$ is the set of edges incident on $x$. If $\varphi_f(x) \neq \varphi_f(y)$ for any two distinct vertices $x, y \in V(G)$, then $f$ is called an anti-magic labeling of $G$. Hartsfield and Ringel [5] introduced anti-magic labeling and they put forth the following conjecture.

**Conjecture** ([5])**.** *Every connected graph other than $K_2$ is anti-magic.*

The conjecture remains open and research continues to be carried out concerning it. Liang *et al.* [6] studied the anti-magic labeling of trees. Reddy *et al.* [9] proved that splittance cycles are anti-magic. Other works on anti-magic labeling have been done by Alon *et al.* [1], Wang and Hsiao [10], and Bérczi *et al.* [2]. For exhaustive survey of anti-magic graphs, we refer to Gallian [4].

There are various products of graphs such as lexicographic product, Cartesian product, strong product and corona product among others. In this paper, we have taken an interest in corona product. It was introduced by Frucht and Harary [3]. Corona product of $G$ and $H$ is the graph $G \odot H$, where $G$ has $n$ vertices, $e$ edges, and $H$ has $k$ vertices, $l$ edges. The graph $G \odot H$ is obtained by taking one copy of $G$ and $n$ copies of $H$ and making the $i$th vertex of $G$ adjacent to every vertex of the $i$th copy of $H$, where $1 \leq i \leq n$. Nada *et al.* [8] investigated the cordiality of Corona between cycles and paths. Ma *et al.* [7] proved that Lexicographic product graphs $P_m[P_n]$ are anti-magic. In our study we aim at proving the conjecture. We prove that the corona product of complete graphs ($K_n$) and complete bipartite graphs ($K_{m,m}$) given by $K_n \odot K_{m,m}$ is anti-magic. We also develop a programme using MATLAB to have a better demonstration of this anti-magic property.

## 2. Preliminaries

**Definition 2.1** ([5])**.** The degree of a vertex $v_i$ is the number of edges incident on $v_i$ and is denoted by $d(v_i)$.

**Definition 2.2** ([5])**.** A graph $G$ is bipartite if its vertex set can be partitioned into two subsets such that no two vertices in the same partition are adjacent.

**Definition 2.3** ([5])**.** A complete bipartite graph is a bipartite graph in which each vertex in one set is adjacent to all the vertices in the other set. A complete bipartite graph with $m$ vertices in one set and $n$ vertices in the other set is denoted by $K_{m,n}$.

**Definition 2.4** ([5])**.** A complete graph is a graph in which every two distinct pair of vertices are joined by an edge. A complete graph with $n$ vertices is denoted by $K_n$ and it is $(n-1)$ regular.

## 3. Main Results

Hartsfield and Ringel [5] proved that complete graphs are anti-magic. In this paper we prove that the corona product of the complete graph $\{K_n, n \geq 2\}$ with the complete bipartite graph $\{K_{m,m}, m \geq 1\}$ is anti- magic.

**Lemma 3.1.** *Corona product $G \odot H$ of connected graphs $G$ and $H$ is a connected graph.*

*Proof.* A graph is connected if there is at least a path between every pair of distinct vertices. Since graph $G$ and graph $H$ are each connected, then there is at least a path between every pair of distinct vertices in graph $G$ as well as in graph $H$. On performance of Corona product, every $i$th vertex of graph $G$ is made to be adjacent to every vertex of the $i$th copy of graph $H$. It then follows that there is at least a path between every pair of distinct vertices in $G \odot H$ since every $i$th vertex of graph $G$ is connected to every vertex of the $i$th copy of graph $H$. Hence $G \odot H$ is a connected graph. In Figure 1 we have graph $G$, Figure 2 is graph $H$ and Figure 3 shows $G \odot H$.

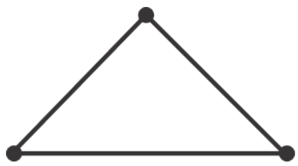

**Figure 1.** Graph $G$

**Figure 2.** Graph $H$

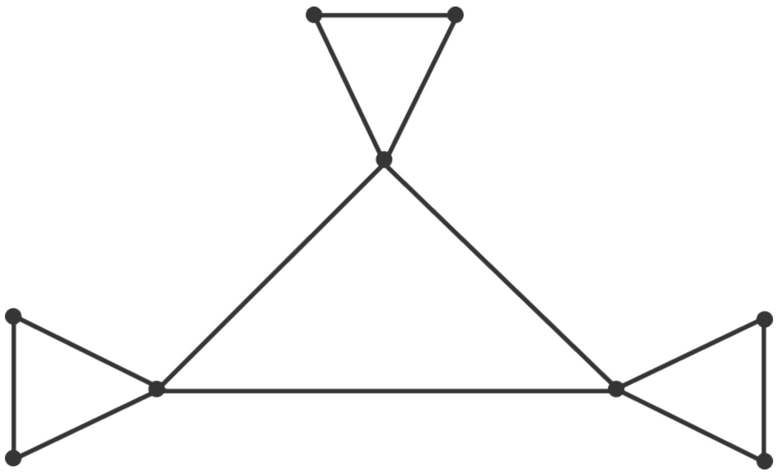

**Figure 3.** Graph $G \odot H$

It then follows that the graph $K_n \odot K_{m,m}$ is connected. □

**Theorem 3.1.** *Corona product $K_n \odot K_{m,m}$ is not commutative.*

*Proof.* By definition $K_n \odot K_{m,m}$ has $n+2nm$ vertices and $\left[n\left(m^2+2m\right)+\frac{n(n-1)}{2}\right]$ edges. This is as a result of centering $K_n$ and connecting the $i$th vertex to every vertex of $i$th copy of $K_{m,m}$. On the other hand $K_{m,m} \odot K_n$ has $2m(n+1)$ vertices and $[mn(n+1)+m^2]$ edges. This is done by centering $K_{m,m}$ and connecting the $i$th vertex to every vertex of $i$th copy of $K_n$. The vertex sets and the edge sets are not the same for any $n$ and $m$, then $K_n \odot K_{m,m} \neq K_{m,m} \odot K_n$. Thus $K_n \odot K_{m,m}$ is not commutative. $\square$

**Theorem 3.2.** *Corona product $K_n \odot K_{m,m}$ is not associative.*

*Proof.* Let $G$ be a graph with $g$ vertices and $t$ edges. The corona product $G \odot (K_n \odot K_{m,m})$ has $[g+gn(1+2m)]$ vertices while corona product $(G \odot K_n) \odot K_{m,m}$ has $g(1+n)(1+2m)$ vertices. The vertex sets are not the same for any $n$ and $m$. On the other hand, corona product $G \odot (K_n \odot K_{m,m})$ has $\left\{t+g\left[n(m^2+4m+1)+\frac{n(n-1)}{2}\right]\right\}$ edges while corona product $(G \odot K_n) \bigodot K_{m,m}$ has $\left[(g+gn)\left(m^2+2m\right)+t+\frac{gn(n+1)}{2}\right]$ edges. The edge sets are not the same for any $n$ and $m$. Thus $G \odot (K_n \odot K_{m,m}) \neq (G \odot K_n) \odot K_{m,m}$. Hence $K_n \odot K_{m,m}$ is not associative. $\square$

**Theorem 3.3.** *Corona product of complete graph and complete bipartite graph $K_n \odot K_{m,m}$ $(n \geq 2,\ m \geq 1)$ is anti-magic.*

*Proof.* A complete graph has $n$ vertices and $\frac{n(n-1)}{2}$ edges. Let us arrange the vertices as per increasing order of vertex labels $v_i$, $1 \leq i \leq n$. Let us label the edges as $\varphi(e_i) = i$, $1 \leq i \leq \frac{n(n-1)}{2}$.

*Labeling technique*

For the complete bipartite graph $K_{m,m}$, let us denote the vertices by $\{u_{1i}, u_{2i}, u_{3i}, \dots, u_{(2m)i}\}$, $1 \leq i \leq n$. We now define the edge labels by the function $\psi$. For the original edges of complete bipartite graph $K_{m,m}$, we label the edges as follows; For $i=1$, we have

$$\psi(u_{11}, u_{(m+1)1}) = 1,\ \psi(u_{11}, u_{(m+2)1}) = 2, \dots, \psi(u_{11}, u_{(2m)1}) = m$$

$$\psi(u_{21}, u_{(m+1)1}) = m+1,\ \psi(u_{21}, u_{(m+2)1}) = m+2, \dots, \psi(u_{21}, u_{(2m)1}) = 2m$$

$$\vdots$$

$$\psi(u_{m1}, u_{(m+1)1}) = m(m-1)+1,\ \psi(u_{m1}, u_{(m+2)1}) = m(m-1)+2, \dots, \psi(u_{m1}, u_{(2m)1}) = m^2$$

Since the number of new edges added to each vertex of the complete graph is given by $m(m+2)$, then we have:

For $i = 2,3,4,\dots,n$, the edge labels $\psi(u_{1i}, u_{(m+1)i}), \dots, \psi(u_{1i}, u_{(2m)i}), \psi(u_{2i}, u_{(m+1)i}), \dots,$ $\psi(u_{2i}, u_{(2m)i}), \dots, \psi(u_{(m)i}, u_{(2m)i})$ are obtained by taking the corresponding edge labels for $i=1$, and adding them to $(i-1)m(m+2)$.

Hence the edge labels are given by:

$$\psi(u_{li}, u_{ji}) = m(l-1) - m + j + (i-1)(m^2+2m), \quad 1 \leq i \leq n, 1 \leq l \leq m, m+1 \leq j \leq 2m. \tag{3.1}$$

For the edges joining $K_n$ to $K_{m,m,}$, we have

For $i = 1$, we have

$$\psi(u_{11}, v_1) = m^2 + 1,$$
$$\psi(u_{21}, v_1) = m^2 + 2,$$
$$\psi(u_{31}, v_1) = m^2 + 3,$$
$$\vdots$$
$$\psi(u_{(2m)1}, v_1) = m^2 + 2m$$

For $i = 2, 3, 4, \dots, n$, the edge labels $\psi(u_{1i}, v_i), \psi(u_{2i}, v_i), \dots, \psi(u_{(2m)i}, v_i)$ are obtained by taking the corresponding edge labels for $i = 1$, and adding them to $(i-1)(m^2 + 2m)$.
Hence the edge labels are given by:

$$\psi(u_{li}, v_i) = m^2 + l + (i-1)(m^2 + 2m), \quad 1 \le i \le n, 1 \le l \le 2m. \tag{3.2}$$

Since the total number of new edges is $(m^2 + 2m)$, the edges of $K_n$ will have labels given as:

$$\psi(e_i) = n(m^2 + 2m) + i, \quad 1 \le i \le \frac{n(n-1)}{2}. \tag{3.3}$$

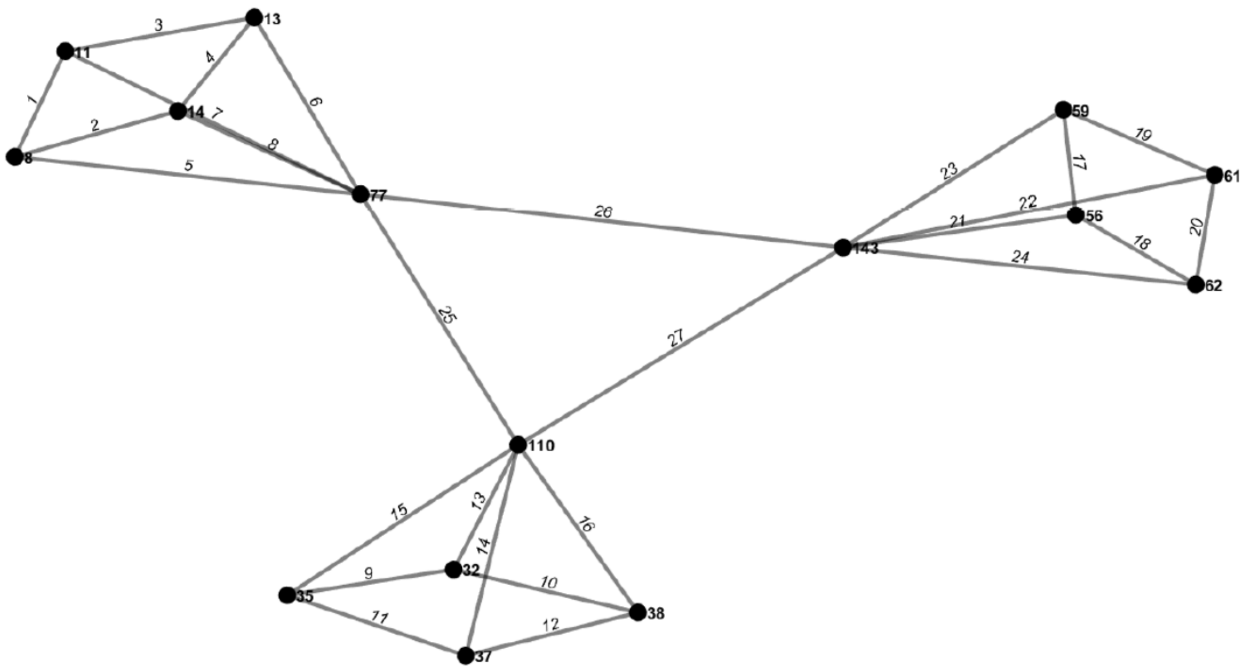


**Figure 4.** Graph $K_3 \odot K_{2,2}$

Figure 4 shows the corona product $K_3 \odot K_{2,2}$. □

The following is the programme developed using MATLAB to show the anti-magic properties of $K_n \odot K_{m,m}$. This programme has been very useful in verifying the anti-magic property for large values of $n$ and $m$.

**Listing 1.** Matlab code for generalising anti-magic labeling of $K_n \odot K_{m,m}$

```
clear
n=3;
m=2;
%%
% |_Plotting complete graph_|
Mat2 = ones(n);
Mat2 = Mat2 - diag(diag(Mat2));
Kn = graph(Mat2);
% plot(Kn)
for a=1:0.5*n*(n-1)
    c(a)=(m^2+2*m)*n+a;
end
Wf = c;
Kn.Edges.Weight =Wf';
% plot(Kn,"EdgeLabel",Kn.Edges.Weight)
Whw = Kn.Edges.Weight;
Whw = Whw';
% |_Plotting complete bipartite graph_|
Mat = ones(2*m);
Mat = Mat - diag(diag(Mat));
Km = graph(Mat);
% plot(Km)
for ap=1:m
    Km =rmedge(Km,1:m,ap);
end
for bp=m+1:2*m
    Km=rmedge(Km,m+1:2*m,bp);
end
% plot(Km)
e1 = Km.Edges;
e1 = table2array(e1);
s1 = [e1(:,1)];
s1 = s1';
t1 = [e1(:,2)];
t1 = t1';
s11 = ones(length(s1));
t11 = ones(length(s1));
for i = 1:n
    s2 = (n+((i-1)*(2*m)).*[s11(1,:)]);
    s = s2 + s1;
    t2 = (n+((i-1)*(2*m)).*[t11(1,:)]);
    t = t2 + t1;
    smin = min(s);
    tmax = max(t);
    weight1 = ones(length(t1));
    weight1 = [weight1(1,:)];
    weight2 = ones(length(t));
```

```
    weight2 = [weight2(1,:)];
    Kn = addedge(Kn,s,t,weight2);
end
% |_Joining the complete graph to the complete bipartite graphs_|
for  i=1:n
    for l=1:m
        for j=m+1:2*m
            if j>l
               w4(i,j,l)=m*(l-1)-m+j+(i-1)*(m^2+2*m);
            end
        end
    end
end
ww =pagectranspose(w4);
ww = nonzeros(w4);
Wkm = ww(:);
Wkm = Wkm';
Wkm = sort(Wkm);
Whwkm = [Whw,Wkm];
Kn.Edges.Weight = Whwkm';
for l= 1:2*m
    for i= 1:n
        Wcon(i,l) = m^2+l+(m^2+2*m)*(i-1);
    end
end
Wcon = sort(Wcon(:));
Wcon = Wcon';
for jj = 1:n
    s2 = (n+((jj-1)*((2*m)).*[s11(1,:)]));
    sjj = s2 + s1;
    t2 = (n+((jj-1)*((2*m)).*[t11(1,:)]));
    tjj = t2 + t1;
    swh = ones(2*m);
    swh = swh(1,:);
    smin2 = min(sjj);
    tmax2 = max(tjj);
    strt = 1+(jj-1)*((2*m));
    tmt = jj*((2*m));
    Wconnect = Wcon(strt:tmt);
    Kn = addedge(Kn,jj.*swh,[smin2:tmax2],Wconnect);
end
for i = 1:n*(2*m+1)
    Sum(i) = sum(Kn.Edges.Weight(outedges(Kn,i)));
end
Kn.Nodes.Size = Sum';
qq = Kn.Edges.Weight;
plot(Kn, "EdgeLabel",qq,"NodeLabel",Sum,...
    "NodeColor","black","EdgeColor","black",...
    "NodeFontWeight","bold","MarkerSize",8,"LineWidth",2)
```

```
% |_Test uniqueness of each node size_|
vldt = Kn.Nodes.Size;
vldt = sort(vldt);
UnqSize = unique(vldt);
Distinct_Nodes = histc(vldt, UnqSize);
```

## 4. Conclusion

We have proved that the corona product of $K_n \odot K_{m,m}$ produces a graph which is anti-magic in this article. We have also developed a programme using MATLAB to demonstrate this property. This programme is useful in obtaining the anti-magic property for large values of $n$ and $m$ which would otherwise be very tiresome if done manually. Further study can be extended to corona product of complete graph $K_n$ with complete bipartite graph $K_{n,m}$. It would also be interesting to find out if corona product of $K_{m,m} \odot K_n$ is anti-magic since corona product of $K_n \odot K_{m,m}$ is not commutative.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

## References

**[1]** N. Alon, G. Kaplan, A. Lev, Y. Roditty and R. Yuster, Dense graphs are antimagic, *Journal of Graph Theory* **47**(4) (2004), 297 – 309, DOI: 10.1002/jgt.20027.

**[2]** K. Bérczi, A. Bernáth and M. Vizer, Regular graphs are antimagic, *The Electronic Journal of Combinatorics 22*(3) (2015), 1 – 14, DOI: 10.37236/5465.

**[3]** R. Frucht and F. Harary, On the corona of two graphs, *Aequationes Mathematicae* **4**(3) (1970), 322 – 325, DOI: 10.1007/BF01844162.

**[4]** J. A. Gallian, A dynamic survey of graph labeling (Graph labelling), *The Electronic Journal of Combinatorics* **DS6** (2022), 1 – 623, DOI: 10.37236/27.

**[5]** N. Hartsfield and G. Ringel, *Pearls in Graph Theory: A Comprehensive Introduction*, Dover Publications Inc., USA (2003), DOI: 10.2307/2324291.

**[6]** Y.-C. Liang, T.-L. Wong and X. Zhu, Anti-magic labeling of trees, *Discrete Mathematics* **331** (2014), 9 – 14, DOI: 10.1016/j.disc.2014.04.021.

**[7]** W. Ma, G. Dong, Y. Lu and N. Wang, Lexicographic product graphs $P_m$ $[P_n]$ are antimagic, *AKCE International Journal of Graphs and Combinatorics* **15**(3) (2018), 271 – 283, DOI: 10.1016/j.akcej.2017.10.005.

**[8]** S. Nada, A. Elrokh, E. A. Elsakhawi and D. E. Sabra, The corona between cycles and paths, *Journal of the Egyptian Mathematical Society* **25**(2) (2017), 111 – 118, DOI: 10.1016/j.joems.2016.08.004.

**[9]** K. V. Reddy, A. M. Reddy and K. Rajyalakshmi, Splittance of cycles are anti-magic, *Advances in Mathematics: Scientific Journal* **9**(9) (2020), 7165 – 7170, DOI: 10.37418/amsj.9.9.66.

**[10]** T.-M. Wang and C.-C. Hsiao, On anti-magic labeling for graph products, *Discrete Mathematics* **308**(16) (2008), 3624 – 3633, DOI: 10.1016/J.DISC.2007.07.027.