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Research Article

Some Results on Fixed Sets and Kernel Sets of Permuting n-(f,g)-derivation of Lattices

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Abstract. In this paper, we introduce a multivariate fixed set, a partial fixed set, a multivariate kernel set and a partial kernel set of permuting n-(f,g)-derivation in lattices, and investigate some related properties. We also give some conditions under which these sets are sublattices and ideals.

Keywords. Lattice, Derivation, (f,g)-derivation, Permuting n-(f,g)-derivation, Fixed set, Kernel set

Mathematics Subject Classification (2020). 06A12, 17A36, 06B35, 06B99

1. Introduction

The notion of derivations in lattices have been studied by Szász [9], Ferrari [4], and Xin *et al*. [11]. They studied some properties of derivations, and characterized modular and distributive lattices by some special derivations. The concept of derivation in lattices has been generalized in several ways by various authors (see [1–3, 5, 10, 12]). In [8], Öztürk *et al*. introduced the permuting tri-derivations in lattices. Yazarli and Ozturk [13] generalized the permuting tri-derivations to permuting tri-f-derivations. Xin [12] introduced the fixed set of derivations in lattices and proved that the fixed set of a derivation is an ideal in lattices. Furthermore, by using the fixed sets of isotone derivations, he established characterizations of a chain, a distributive lattice, a modular lattice and a relatively pseudo-complemented lattice, respectively.

Recently, Leerawat and Chotchaya [6] generalized the permuting tri-f-derivations to the permuting n-(f,g)-derivation, where n is a positive integer, and investigated some related properties. Moreover, they introduced the concept of trace of permuting n-(f,g)-derivation of a lattice and discussed some related properties. But the relations among derivations, sublattices, ideals, fixed sets and kernel sets were not investigated in that paper. In this paper, we introduce some kind of fixed sets and kernel sets of permuting n-(f,g)-derivations in lattices and investigate the structure fixed sets, kernel sets, sublattices and ideals in lattices. This paper is a continuation to the paper [6].

2. Preliminaries

First, we will give some basic definitions and some results used throughout the entire paper. Details and proofs can be found in Lidl and Pilz [7].

Definition 2.1 ([7]). A lattice $(L,\wedge,\vee)$ is a nonempty set L with two binary operation "$\wedge$" and "$\vee$" (read "meet" and "join", respectively) on L which satisfy the following conditions for all $x,y,z\in L$:

(i) $x\wedge x=x$, $\quad x\vee x=x$.

(ii) $x\wedge y=y\wedge x$, $\quad x\vee y=y\vee x$.

(iii) $x\wedge(y\wedge z)=(x\wedge y)\wedge z$, $\quad x\vee(y\vee z)=(x\vee y)\vee z$.

(iv) $(x\wedge y)\vee x=x$, $\quad (x\vee y)\wedge x=x$.

In what follows, we denote by L a lattice $(L,\wedge,\vee)$, unless otherwise specified.

Definition 2.2 ([7]). A nonempty subset S of a lattice L is called sublattice of L if S is a lattice with respect to the restriction of $\wedge$ and $\vee$ of L onto S.

Lemma 2.3 ([7]). *Let L be a lattice. Define the binary operation "$\leqslant$" by $x\leqslant y$ if and only if $x\wedge y=x$. Then $(L,\leqslant)$ is a poset and for any $x,y\in L$, $x\wedge y$ is the greatest lower bound of $\{x,y\}$ (or $\inf\{x,y\}$) and $x\vee y$ is the least upper bound of $\{x,y\}$ (or $\sup\{x,y\}$).*

Definition 2.4 ([7]). A poset $(L,\leqslant)$ is a lattice ordered if and only if for every pair x,y of elements of L both the $\sup\{x,y\}$ and the $\inf\{x,y\}$ exist.

Theorem 2.5 ([7]). (i) *Let $(L,\leqslant)$ be a lattice ordered set. If we define $x\wedge y=\inf\{x,y\}$ and $x\vee y=\sup\{x,y\}$, then $(L,\wedge,\vee)$ is a lattice.*

(ii) *Let $(L,\wedge,\vee)$ be a lattice. If we define $x\leqslant y$ if and only if $x\wedge y=x$ (or $x\leqslant y$ if and only if $x\vee y=y$) then $(L,\leqslant)$ is a lattice ordered set.*

It can be verified that Theorem 2.5 yields a one-to-one relationship between lattice ordered sets and lattices. Therefore, we shall use the term lattice for both concepts.

Theorem 2.6 ([7]). (i) *Every ordered set is lattice ordered.*

(ii) *In a lattice ordered set* $(L, \leqslant)$ *the following statements are equivalent for all* $x, y \in L$*:*

(a) $x \leqslant y$;

(b) $\sup\{x, y\} = y$;

(c) $\inf\{x, y\} = x$.

Definition 2.7 ([7]). If a lattice L contains a least (greatest) element with respect to $\leqslant$ then this uniquely determined element is called the zero element (one element), denoted by 0 (by 1).

Definition 2.8 ([7]). A lattice L is called distributive if the identity (i) or (ii) holds for all $x, y, z \in L$:

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

In any lattice, the conditions (i) and (ii) are equivalent.

Definition 2.9 ([7]). A nonempty subset I of a lattice L is called ideal of L if the following conditions holds:

(i) If $x, y \in L$ such that $x \leqslant y$ and $y \in I$ then $x \in I$.

(ii) If $x, y \in I$ then $x \vee y \in I$.

Definition 2.10 ([7]). Let L and M be two lattices and $f : L \to M$ be a function.

(i) f is called a join-homomorphism if $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$.

(ii) f is called a meet-homomorphism if $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in L$.

(iii) f is called a lattice-homomorphism if f is both a join-homomorphism and a meet-homomorphism.

(iv) f is called an order-preserving if $x \leqslant y$ implies $f(x) \leqslant f(y)$ for all $x, y \in L$.

From now on, let L denote a lattice and $f, g : L \to L$ be functions. Let n be a fixed positive integer and L^n denote $L \times L \times \cdots \times L$ (n terms). We collect some definitions and some results from [6], which are essential for developing the proofs of our main results.

Definition 2.11 ([6]). A mapping $D : L^n \to L$ is said to be permuting if the relation

$$D(x_1, x_2, \ldots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$$

holds for all $x_i \in L$ and for every permutation $\pi \in S_n$, where S_n is the permutation group on $\{1, 2, \ldots, n\}$.

Definition 2.12 ([6]). A mapping $D : L^n \to L$ is called an n-join-homomorphism of L if D satisfies the following conditions:

$$D(x_1 \vee y, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n) \vee D(y, x_2, \ldots, x_n)$$

$$D(x_1,x_2 \vee y,\dots,x_n)=D(x_1,x_2,\dots,x_n)\vee D(x_1,y,\dots,x_n)$$

$$\vdots$$

$$D(x_1,x_2,\dots,x_n \vee y)=D(x_1,x_2,\dots,x_n)\vee D(x_1,x_2,\dots,y)$$

for all $x_1,x_2,\dots,x_n,y \in L$.

Definition 2.13 ([6]). A mapping $D : L^n \to L$ is called an n-(f,g)-derivation of L if D is an n-join-homomorphism of L and satisfies the following conditions:

$$D(x_1 \wedge y,x_2,\dots,x_n)=(D(x_1,x_2,\dots,x_n)\wedge f(y))\vee(g(x_1)\wedge D(y,x_2,\dots,x_n))$$

$$D(x_1,x_2 \wedge y,\dots,x_n)=(D(x_1,x_2,\dots,x_n)\wedge f(y))\vee(g(x_2)\wedge D(x_1,y,\dots,x_n))$$

$$\vdots$$

$$D(x_1,x_2,\dots,x_n \wedge y)=(D(x_1,x_2,\dots,x_n)\wedge f(y))\vee(g(x_n)\wedge D(x_1,x_2,\dots,y))$$

for all $x_1,x_2,\dots,x_n,y \in L$.

Definition 2.14 ([6]). A mapping $D : L^n \to L$ is called a permuting n-(f,g)-derivation of L if D is a permuting and satisfies the following conditions:

$$D(x_1 \vee y,x_2,\dots,x_n)=D(x_1,x_2,\dots,x_n)\vee D(y,x_2,\dots,x_n) \quad \text{and}$$

$$D(x_1 \wedge y,x_2,\dots,x_n)=(D(x_1,x_2,\dots,x_n)\wedge f(y))\vee(g(x_1)\wedge D(y,x_2,\dots,x_n))$$

for all $x_1,x_2,\dots,x_n,y \in L$.

Theorem 2.15 ([6]). *Let D be an n-(f,g)-derivation of L. Then*

$$D(x_1,x_2,\dots,x_n)\leqslant f(x_1)\vee g(x_1)$$

$$D(x_1,x_2,\dots,x_n)\leqslant f(x_2)\vee g(x_2)$$

$$\vdots$$

$$D(x_1,x_2,\dots,x_n)\leqslant f(x_n)\vee g(x_n)$$

for all $x_1,x_2,\dots,x_n \in L$.

Theorem 2.16 ([6]). *Let D be a permuting n-(f,g)-derivation of L. If $g(x)\leqslant f(x)$ for all $x \in L$ Then*

$$\begin{aligned}D(x_1,x_2,\dots,x_n)\wedge D(y,x_2,\dots,x_n) &\leqslant D(x_1 \wedge y,x_2,\dots,x_n)\\ &\leqslant D(x_1,x_2,\dots,x_n)\vee D(y,x_2,\dots,x_n)\end{aligned}$$

for all $x_1,x_2,\dots,x_n,y \in L$.

Let $D : L^n \to L$ be a mapping. We recall the following notations. For simplicity, we denote from now on $D(x^{(n-k)},y^{(k)})$ by $D(\underbrace{x,x,\dots,x}_{n\text{-}k \text{ copies}},\underbrace{y,y,\dots,y}_{k \text{ copies}})$, where $k=1,2,3,\dots,n-1$, and $x,y \in L$.

A mapping $d : L \to L$ is called a trace of D if $d(x)=D(x,x,\dots,x)$ for all $x \in L$.

Theorem 2.17 ([6]). *Let D be a permuting n-(f,g)-derivation of L and d be a trace of D. Then*

$$d(x \vee y) = d(x) \vee d(y) \vee [D(x^{(n-1)}, y) \vee D(x^{(n-2)}, y^{(2)}) \vee \ldots \vee D(x, y^{(n-1)})],$$

for all $x, y \in L$.

Theorem 2.18 ([6]). *Let L be a distributive lattice. Let D be a permuting n-(f,g)-derivation on L and d be a trace of D. Then*

$$d(x\wedge y) = (d(x)\wedge f(y))\vee(g(x)\wedge d(y))\vee[(g(x)\wedge f(y))\wedge[D(x^{(n-1)}, y)\vee D(x^{(n-2)}, y^{(2)})\vee\cdots\vee D(x, y^{(n-1)})]],$$

for all $x, y \in L$.

Corollary 2.19 ([6]). *Let L be a distributive lattice and D be a permuting n-(f,g)-derivation on L with a trace d. Then for all $x, y \in L$,*

(i) $d(x) \vee d(y) \leqslant d(x \vee y)$.

(ii) $g(x) \wedge f(y) \wedge [D(x^{(n-1)}, y) \vee D(x^{(n-2)}, y^{(2)}) \vee \cdots \vee D(x, y^{(n-1)})] \leqslant d(x \wedge y)$.

(iii) $g(x) \wedge d(y) \leqslant d(x \wedge y)$.

(iv) $d(x) \wedge f(y) \leqslant d(x \wedge y)$.

3. Fixed Sets and Kernel Sets of Permuting n-(f,g)-derivation of Lattice

In this section, let $D : L^n \to L$ denote a permuting n-(f,g)-derivation on L. We introduce some kind of fixed sets and kernel sets of D on L and investigate some related properties.

Definition 3.1. Let $D : L^n \to L$ be a permuting n-(f,g)-derivation on L.

(i) An element $a \in L$ is called a *multivariate fixed point* of (D, f, g), if $D(a, a, \ldots, a) = f(a) = g(a)$.

(ii) An element $a \in L$ is called a *partial fixed point* of (D, f, g), if $D(a, a_2, \ldots, a_n) = f(a) = g(a)$ for some $a_2, a_3, \ldots, a_n \in L$.

(iii) Define a set $\mathrm{Fix}(L)$ by

$$\mathrm{Fix}(L) = \{x \in L \mid D(x, x, \ldots, x) = f(x) = g(x)\}.$$

Then $\mathrm{Fix}(L)$ is called a *multivariate fixed set* of (D, f, g).

(iv) For $a_2, a_3, a_4, \ldots, a_n \in L$, define a set $\mathrm{Fix}(L, a_2, \ldots, a_n)$ by

$$\mathrm{Fix}(L, a_2, \ldots, a_n) = \{x \in L \mid D(x, a_2, \ldots, a_n) = f(x) = g(x)\}.$$

Then $\mathrm{Fix}(L, a_2, \ldots, a_n)$ is called a *partial fixed set* of (D, f, g).

Theorem 3.2. *Let f and g be lattice-homomorphisms and $g(x) \leqslant f(x)$ for all $x \in L$. If* $\mathrm{Fix}(L, a_2, \ldots, a_n) \neq \emptyset$ *for some $a_2, a_3, \ldots, a_n$. Then* $\mathrm{Fix}(L, a_2, \ldots, a_n)$ *is a sublattice of L.*

Proof. Assume that $\mathrm{Fix}(L,a_2,\ldots,a_n)\neq\emptyset$ for some $a_2,a_3,\ldots,a_n$.

Let $x,y\in\mathrm{Fix}(L,a_2,\ldots,a_n)$ then $D(x,a_2,\ldots,a_n)=f(x)=g(x)$ and $D(y,a_2,\ldots,a_n)=f(y)=g(y)$.

So $f(x\wedge y)=f(x)\wedge f(y)=g(x)\wedge g(y)=g(x\wedge y)$ and $f(x\vee y)=f(x)\vee f(y)=g(x)\vee g(y)=g(x\vee y)$.

By Theorem 2.16, we get $f(x\wedge y)=f(x)\wedge f(y)=D(x,a_2,\ldots,a_n)\wedge D(y,a_2,\ldots,a_n)\leqslant D(x\wedge y,a_2,\ldots,a_n)$. Therefore $f(x\wedge y)\leqslant D(x\wedge y,a_2,\ldots,a_n)$.

By Theorem 2.15, we get $D(x\wedge y,a_2,\ldots,a_n)\leqslant f(x\wedge y)\vee g(x\wedge y)=f(x\wedge y)$.

Hence $D(x\wedge y,a_2,\ldots,a_n)=f(x\wedge y)=g(x\wedge y)$.

So $x\wedge y\in\mathrm{Fix}(L,a_2,\ldots,a_n)$.

Consider $D(x\vee y,a_2,\ldots,a_n)=D(x,a_2,\ldots,a_n)\vee D(y,a_2,\ldots,a_n)=f(x)\vee f(y)=f(x\vee y)=g(x\vee y)$.

Hence $D(x\vee y,a_2,\ldots,a_n)=f(x\vee y)=g(x\vee y)$.

So $x\vee y\in\mathrm{Fix}(L,a_2,\ldots,a_n)$.

Therefore $\mathrm{Fix}(L,a_2,\ldots,a_n)$ is a sublattice of L. □

Theorem 3.3. *Let L be a lattice with a greatest element 1. Assume that $f(x)=g(x)$ for all $x\in L$ and $f(1)=1$. If $D(1,a_2,\ldots,a_n)=1$ for some $a_2,a_3,\ldots,a_n\in L$ then $D(x,a_2,\ldots,a_n)=f(x)$ for all $x\in L$ and $L=\mathrm{Fix}(L,a_2,\ldots,a_n)$.*

Proof. Let $D(1,a_2,\ldots,a_n)=1$ for some $a_2,a_3,\ldots,a_n\in L$. Let $x\in L$. Then

$$\begin{aligned}D(x,a_2,\ldots,a_n)&=D(x\wedge 1,a_2,\ldots,a_n)\\&=[D(x,a_2,\ldots,a_n)\wedge f(1)]\vee[g(x)\wedge D(1,a_2,\ldots,a_n)]\\&=[D(x,a_2,\ldots,a_n)\wedge 1]\vee[f(x)\wedge 1]\\&=D(x,a_2,\ldots,a_n)\vee f(x).\end{aligned}$$

So $D(x,a_2,\ldots,a_n)=D(x,a_2,\ldots,a_n)\vee f(x)$, this implies $f(x)\leqslant D(x,a_2,\ldots,a_n)$.

$$\begin{aligned}D(x,a_2,\ldots,a_n)&=D(x\wedge x,a_2,\ldots,a_n)\\&=[D(x,a_2,\ldots,a_n)\wedge f(x)]\vee[g(x)\wedge D(x,a_2,\vee,a_n)]\\&=[D(x,a_2,\ldots,a_n)\wedge f(x)]\vee[f(x)\wedge D(x,a_2,\ldots,a_n)]\\&=D(x,a_2,\ldots,a_n)\wedge f(x).\end{aligned}$$

So $D(x,a_2,\ldots,a_n)=D(x,a_2,\ldots,a_n)\wedge f(x)$, this implies

$$D(x,a_2,\ldots,a_n)\leqslant f(x).$$

Hence $D(x,a_2,\ldots,a_n)=f(x)$ for all $x\in L$.

Next, we will show that $L=\mathrm{Fix}(L,a_2,\ldots,a_n)$.

Clearly, $\mathrm{Fix}(L,a_2,\ldots,a_n)\subseteq L$.

Let $x\in L$. Then $D(x,a_2,\ldots,a_n)=f(x)=g(x)$. That is $x\in\mathrm{Fix}(L,a_2,\ldots,a_n)$.

So, $L\subseteq\mathrm{Fix}(L,a_2,\ldots,a_n)$. Therefore $L=\mathrm{Fix}(L,a_2,\ldots,a_n)$. □

Theorem 3.4. *Let f be order−preserving and $f(x) = g(x)$ for all $x \in L$. If $x, y \in L$ such that $x \leqslant y$ and $y \leqslant \mathrm{Fix}(L, a_2, \dots, a_n)$ for some $a_2, a_3, \dots, a_n \in L$ then $x \in \mathrm{Fix}(L, a_2, \dots, a_n)$.*

Proof. Let $x, y \in L$ be such that $x \leqslant y$ and $y \in \mathrm{Fix}(L, a_2, \dots, a_n)$ for some $a_2, a_3, \dots, a_n \in L$. Then $x = x \wedge y$ and $D(y, a_2, \dots, a_n) = f(y) = g(y)$. Since f is order−preserving and $x \leqslant y$, $f(x) \leqslant f(y)$. By Theorem 2.15, we have $D(x, a_2, \dots, a_n) \leqslant f(x) \vee g(x) = f(x)$. So, $D(x, a_2, \dots, a_n) \leqslant f(x) \leqslant f(y)$. Consider

$$\begin{aligned} D(x, a_2, \dots, a_n) &= D(x \wedge y, a_2, \dots, a_n) \\ &= [D(x, a_2, \dots, a_n) \wedge f(y)] \vee [g(x) \wedge D(y, a_2, \dots, a_n)] \\ &= [D(x, a_2, \dots, a_n) \wedge f(y)] \vee [f(x) \wedge f(y)] \\ &= D(x, a_2, \dots, a_n) \vee f(x) \\ &= f(x) = g(x). \end{aligned}$$

Hence $D(x, a_2, \dots, a_n) = f(x) = g(x)$. That is $x \in \mathrm{Fix}(L, a_2, \dots, a_n)$. □

Theorem 3.5. *Let f be a lattice−homomorphisms and order-preserving. If $f(x) = g(x)$ for all $x \in L$ and there exist $a_2, a_3, \dots, a_n \in L$ such that $\mathrm{Fix}(L, a_2, \dots, a_n) \neq \emptyset$ then $\mathrm{Fix}(L, a_2, \dots, a_n)$ is an ideal of L.*

Proof. Assume that $f(x) = g(x)$ for all $x \in L$ and there exist $a_2, a_3, \dots, a_n \in L$ such that $\mathrm{Fix}(L, a_2, \dots, a_n) \neq \emptyset$.

Let $x, y \in L$ be such that $x \leqslant y$ and $y \in \mathrm{Fix}(L, a_2, \dots, a_n)$.

By Theorem 3.4, we have $x \in \mathrm{Fix}(L, a_2, \dots, a_n)$.

Next, let $x, y \in \mathrm{Fix}(L, a_2, \dots, a_n)$. Then $D(x, a_2, \dots, a_n) = f(x) = g(x)$ and $D(y, a_2, \dots, a_n) = f(y) = g(y)$. Then

$$\begin{aligned} D(x \vee y, a_2, \dots, a_n) &= D(x, a_2, \dots, a_n) \vee D(y, a_2, \dots, a_n) \\ &= f(x) \vee f(y) \\ &= f(x \vee y). \end{aligned}$$

Hence $D(x \vee y, a_2, \dots, a_n) = f(x \vee y) = g(x \vee y)$, so $x \vee y \in \mathrm{Fix}(L, a_2, \dots, a_n)$.

Therefore, $\mathrm{Fix}(L, a_2, \dots, a_n)$ is an ideal of L. This completes the proof. □

Theorem 3.6. *Let f be order−preserving and $f(x) = g(x)$ for all $x \in L$. If there exist $y, a_2, a_3, \dots, a_n \in L$ such that $y \in \mathrm{Fix}(L, a_2, \dots, a_n)$ then $S_y = \{x \in L \mid x \leqslant y\}$ is a sublattice of $\mathrm{Fix}(L, a_2, \dots, a_n)$. Moreover, S_y is an ideal of $\mathrm{Fix}(L, a_2, \dots, a_n)$.*

Proof. Assume that there exist $y, a_2, a_3, \dots, a_n \in L$ such that $y \in \mathrm{Fix}(L, a_2, \dots, a_n)$. Clearly, $S_y = \{x \in L \mid x \leqslant y\} \neq \emptyset$. Now, we will show that $S_y \subseteq \mathrm{Fix}(L, a_2, \dots, a_n)$.

Let $a \in S_y$ then $a \leqslant y$. By Theorem 3.4, and $y \in \mathrm{Fix}(L, a_2, \dots, a_n)$ we get

$$a \in \mathrm{Fix}(L, a_2, \dots, a_n).$$

Therefore $S_y \subseteq \mathrm{Fix}(L, a_2, \dots, a_n)$.

Next, let $a, b \in S_y$ then $a \leqslant y$ and $b \leqslant y$. Hence $a \wedge y = a$ and $b \wedge y = b$.

Therefore, $a \wedge b = (a \wedge y) \wedge (b \wedge y) = (a \wedge b) \wedge y$, and so $a \wedge b \leqslant y$.

This implies $a \wedge b \in S_y$. Similarly, $a \vee b \in S_y$. Hence S_y is a sublattice of $\mathrm{Fix}(L, a_2, \dots, a_n)$. We now show that S_y is an ideal of $\mathrm{Fix}(L, a_2, \dots, a_n)$. Let $a, b \in \mathrm{Fix}(L, a_2, \dots, a_n)$ be such that $a \leqslant b$ and $b \in S_y$. Hence $a \leqslant b \leqslant y$. Thus $a \in S_y$. Therefore S_y is an ideal of $\mathrm{Fix}(L, a_2, \dots, a_n)$. □

Theorem 3.7. *Let L be a distributive lattice. Let D be a permuting n-(f,g)-derivation on L and d be a trace of D. If f and g are lattice−homomorphisms and $\mathrm{Fix}(L) \neq \emptyset$. Then $\mathrm{Fix}(L)$ is a sublattice of L.*

Proof. Let $x, y \in \mathrm{Fix}(L)$, then $d(x) = f(x) = g(x)$ and $d(y) = f(y) = g(y)$. By Theorem 2.18, we have

$$\begin{aligned} d(x \wedge y) &= (d(x) \wedge f(y)) \vee (g(x) \wedge d(y)) \\ &\quad \vee [(g(x) \wedge f(y)) \wedge [D(x^{(n-1)}, y) \vee D(x^{(n-2)}, y^{(2)}) \vee \cdots \vee D(x, y^{(n-1)})]] \\ &= (f(x) \wedge f(y)) \vee (f(x) \wedge f(y)) \\ &\quad \vee [(f(x) \wedge f(y)) \wedge [D(x^{(n-1)}, y) \vee D(x^{(n-2)}, y^{(2)}) \vee \cdots \vee D(x, y^{(n-1)})]] \\ &= (f(x) \wedge f(y)) \vee (f(x) \wedge f(y)) \\ &= f(x) \wedge f(y) = f(x \wedge y). \end{aligned}$$

Clearly, $f(x \wedge y) = g(x \wedge y)$.

Hence $x \wedge y \in \mathrm{Fix}(L)$.

Next, we prove that $x \vee y \in \mathrm{Fix}(L)$. By Theorem 2.15, we have

$$D(x^{(n-1)}, y) \leqslant f(x) \vee g(x) = f(x) \vee f(x) = f(x).$$

Similarly,

$$D(x^{(n-2)}, y^{(2)}) \leqslant f(x), \dots, D(x, y^{(n-1)}) \leqslant f(x).$$

Therefore

$$D(x^{(n-1)}, y) \vee D(x^{(n-2)}, y^{(2)}) \vee \cdots \vee D(x, y^{(n-1)}) \leqslant f(x).$$

By Theorem 2.17, we have

$$\begin{aligned} d(x \vee y) &= d(x) \vee d(y) \vee [D(x^{(n-1)}), y) \vee D(x^{(n-2)}), y^{(2)}) \vee \ldots \vee D(x, y^{(n-1)})] \\ &= f(x) \vee f(y) \vee [D(x^{(n-1)}), y) \vee D(x^{(n-2)}), y^{(2)}) \vee \ldots \vee D(x, y^{(n-1)})] \\ &= f(x) \vee f(y) = f(x \vee y). \end{aligned}$$

Clearly, $f(x \vee y) = g(x \vee y)$.

Hence $x \vee y \in \mathrm{Fix}(L)$. Therefore $\mathrm{Fix}(L)$ is a sublattice of L. □

Theorem 3.8. *Let L be a distributive lattice. Let D be a permuting n-(f,g)-derivation on L and d be a trace of D. Assume that $g(x) \leqslant f(x)$ for all $x \in L$ and f is order−preserving. If $x, y \in L$ such that $x \leqslant y$ and $y \in \mathrm{Fix}(L)$ then $g(x) \leqslant d(x) \leqslant f(x)$.*

Proof. Let $x, y \in L$ such that $x \leqslant y$ and $y \in \mathrm{Fix}(L)$, then $d(y) = f(y) = g(y)$.
By Theorem 2.15, we get $d(x) = D(x,x,\dots,x) \leqslant f(x) \vee g(x) = f(x)$.
Since f is order−preserving and $x \leqslant y$, $f(x) \leqslant f(y)$.
Hence $d(x) \leqslant f(x) \leqslant f(y)$, and $g(x) \leqslant f(x) \leqslant f(y)$.
By Theorem 2.18, we get

$$\begin{aligned}
d(x) &= d(x \wedge y) \\
&= (d(x) \wedge f(y)) \vee (g(x) \wedge d(y)) \vee [(g(x) \wedge f(y)) \wedge [D(x^{(n-1)}, y) \vee D(x^{(n-2)}, y^{(2)}) \vee \cdots \vee D(x, y^{(n-1)})]] \\
&= (d(x) \wedge f(y)) \vee (g(x) \wedge f(y)) \vee [(g(x) \wedge f(y)) \wedge [D(x^{(n-1)}, y) \vee D(x^{(n-2)}, y^{(2)}) \vee \cdots \vee D(x, y^{(n-1)})]] \\
&= (d(x) \wedge f(y)) \vee (g(x) \wedge f(y)) \\
&= d(x) \vee g(x).
\end{aligned}$$

So $d(x) = d(x) \vee g(x)$. Therefore $g(x) \leqslant d(x)$. □

By Theorem 3.7 and Theorem 3.8, we obtain the following corollary.

Corollary 3.9. *Let L be a distributive lattice. Let D be a permuting n-(f,g)-derivation on L and d be a trace of D. Assume that f is a lattice−homomorphisms and order-preserving. If $g(x) = f(x)$ for all $x \in L$ and $\mathrm{Fix}(L) \neq \emptyset$ then $\mathrm{Fix}(L)$ is an ideal of L.*

Definition 3.10. Let L be a lattice with a least element 0. Let D be a permuting n-(f,g)-derivation on L and d be a trace of D.

(i) Define a set $\mathrm{Ker}(d)$ by $\mathrm{Ker}(d) = \{x \in L \mid d(x) = 0\}$. Then $\mathrm{Ker}(d)$ is called a *multiplicative kernel set* of d.

(ii) For $a_2, a_3, \dots, a_n \in L$. Define a set $\mathrm{Ker}(D, a_2, \dots, a_n)$ by

$$\mathrm{Ker}(D, a_2, \dots, a_n) = \{x \in L \mid D(x, a_2, \dots, a_n) = 0\}.$$

Then $\mathrm{Ker}(D, a_2, \dots, a_n)$ is called a *partial kernel set* of (D, f, g).

Theorem 3.11. *Let L be a distributive lattice with a least element 0. Let D be a permuting n-(f,g)-derivation on L. If $x, y \in L$ such that $x \leqslant y$ and $y \in \mathrm{Ker}(d)$ then $x \in \mathrm{Ker}(d)$.*

Proof. Let $x, y \in L$ be such that $x \leqslant y$ and $y \in \mathrm{Ker}(d)$. Then $d(y) = 0$.
By Corollary 2.19(i) we have $d(x) \vee d(y) \leqslant d(x \vee y)$.
Since $x \leqslant y$ and $d(y) = 0$, $d(x) = d(x) \vee 0 \leqslant d(x \vee y) = d(y) = 0$.
Hence $d(x) \leqslant 0$. Therefore $d(x) = 0$, since 0 is the least element.
It follows that $x \in \mathrm{Ker}(d)$. □

Theorem 3.12. *Let L be a lattice with a least element 0. Let D be a permuting n-(f,g)-derivation on L. If $\operatorname{Ker}(D,a_2,\ldots,a_n) \neq \emptyset$ for some $a_2,a_3,\ldots,a_n \in L$ then $\operatorname{Ker}(D,a_2,\ldots,a_n)$ is a sublattice of L.*

Proof. Assume that $\operatorname{Ker}(D,a_2,\ldots,a_n) \neq \emptyset$ for some $a_2,a_3,\ldots,a_n \in L$.
Let $K = \operatorname{Ker}(D,a_2,\ldots,a_n)$.
Let $x,y \in K$ then $D(x,a_2,\ldots,a_n) = 0$ and $D(y,a_2,\ldots,a_n) = 0$.
Then

$$\begin{aligned} D(x \wedge y,a_2,\ldots,a_n) &= [D(x,a_2,\ldots,a_n) \wedge f(y)] \vee [g(x) \wedge D(y,a_2,\ldots,a_n)] \\ &= [0 \wedge f(y)] \vee [g(x) \wedge 0] \\ &= 0 \vee 0 = 0, \\ D(x \vee y,a_2,\ldots,a_n) &= D(x,a_2,\ldots,a_n) \vee D(y,a_2,\ldots,a_n) \\ &= 0 \vee 0 = 0. \end{aligned}$$

Therefore $D(x \wedge y,a_2,\ldots,a_n) = 0$, and $D(x \vee y,a_2,\ldots,a_n) = 0$.
That is $x \wedge y \in K$, and $x \vee y \in K$.
Hence K is a sublattice of L. □

Theorem 3.13. *Let L be a lattice with a least element 0. Let D be a permuting n-(f,g)-derivation on L. If $x,y \in L$ such that $x \leqslant y$ and $y \in \operatorname{Ker}(D,a_2,\ldots,a_n)$ for some $a_2,a_3,\ldots,a_n \in L$ then $x \in \operatorname{Ker}(D,a_2,\ldots,a_n)$.*

Proof. Let $K = \operatorname{Ker}(D,a_2,\ldots,a_n)$.
Assume that $x,y \in L$ such that $x \leqslant y$ and $y \in K$.
Then $D(y,a_2,\ldots,a_n) = 0$. Since $x \leqslant y, x \vee y = y$.
Then

$$\begin{aligned} D(x,a_2,\ldots,a_n) &= D(x,a_2,\ldots,a_n) \vee 0 \\ &= D(x,a_2,\ldots,a_n) \vee D(y,a_2,\ldots,a_n) \\ &= D(x \vee y,a_2,\ldots,a_n) \\ &= D(y,a_2,\ldots,a_n) \\ &= 0. \end{aligned}$$

Hence $D(x,a_2,\ldots,a_n) = 0$, and so $x \in K$. □

By Theorem 3.12 and Theorem 3.13, we obtain the following corollary.

Corollary 3.14. *Let L be a lattice with a least element 0. Let D be a permuting n-(f,g)-derivation on L. If $\operatorname{Ker}(D,a_2,\ldots,a_n) \neq \emptyset$ for some $a_2,a_3,\ldots,a_n \in L$. Then $\operatorname{Ker}(D,a_2,\ldots,a_n)$ is an ideal of L.*

Theorem 3.15. *Let L be a lattice with a least element 0. Let D be a permuting n-(f,g)-derivation on L. Suppose that there exists $y \in \mathrm{Ker}(D,a_2,\dots,a_n)$ for some $a_2,a_3,\dots,a_n \in L$. Then the set $S_y = \{x \in L \mid x \leqslant y\}$ is a sublattice of $\mathrm{Ker}(D,a_2,\dots,a_n)$. Moreover, S_y is an ideal of $\mathrm{Ker}(D,a_2,\dots,a_n)$.*

Proof. The proof is similar to the proof of Theorem 3.6. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] Y. Ceven and M. A. Ozturk, On f-derivations of lattices, *Bulletin of the Korean Mathematical Society* **45**(4) (2008), 701 – 707, DOI: 10.4134/BKMS.2008.45.4.701.

[2] Y. Çeven, Symmetric bi-derivations of lattices, *Quaestiones Mathematicae* **32** (2009), 241 – 245, DOI: 10.2989/QM.2009.32.2.6.799.

[3] M. A. Chaudhry and Z. Ullah, On generalized (α,β)-derivations on lattices, *Quaestiones Mathematicae* **34**(4) (2011), 417 – 424, DOI: 10.2989/16073606.2011.640439.

[4] L. Ferrari, On derivations of lattices, *Pure Mathematics and Applications* **12** (2001), 365 – 382, URL: http://web.math.unifi.it/users/ferrari/derlat.pdf.

[5] S. Harmaitree and U. Leerawat, On f-derivations in lattices, *Far East Journal of Mathematical Sciences* **51**(1) (2011), 27 – 40, URL: http://www.pphmj.com/abstract/5779.html.

[6] U. Leerawat and P. Chotchaya, On permuting n-(f,g)-derivations of lattices, *International Journal of Mathematics and Computer Science* **17**(1) (2022), 485 – 497, URL: http://ijmcs.future-in-tech.net/17.1/R-Leerawat-Chotchaya.pdf.

[7] R. Lidl and G. Pilz, *Applied Abstract Algebra*, Springer-Verlag, Inc., New York, USA (1984), DOI: 10.1007/978-1-4615-6465-2.

[8] M. A. Öztürk, H. Yazarl and K. H. Kim, Permuting tri-derivations in lattices, *Quaestiones Mathematicae* **32** (2009), 415 – 425, DOI: 10.2989/QM.2009.32.3.10.911.

[9] G. Szász, Derivations of lattices, *Acta Scientiarum Mathematicarum* (Szeged) **37** (1975), 149 – 154.

[10] J. Wang, Y. Jun, X. Xin and Y. Zou, On derivations of bounded hyperlattices, *Journal of Mathematical Research with Applications* **36** (2016), 151 – 161, DOI: 10.3770/j.issn:2095-2651.2016.02.003.

[11] X. L. Xin, T. Y. Liu and J. H. Lu, On derivations of lattices, *Information Sciences* **178**(2) (2008), 307 – 316, DOI: 10.1016/j.ins.2007.08.018.

[12] X. L. Xin, The fixed set of a derivation in lattices, *Fixed Point Theory Applications* (A Springer Open Journal) **218** (2012), DOI: 10.1186/16871812-2012-218.

[13] H. Yazarli and M. A. Ozturk, Permuting tri-f-derivations in lattices, *Communications of the Korean Mathematical Society* **26**(1) (2011), 13 – 21, DOI: 10.4134/CKMS.2011.26.1.013.