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Research Article

Generalized Contractions on Extended *b*-Metric Space Endowed With a Graph

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Abstract. The goal of this study is to obtain fixed point and coincidence point results for interpolative Hardy Rogers and Ćirić-Reich-Rus type contractive type mappings in an extended b-metric space, which is a generalization of b-metric space, based on Errai *et al*. (Some new results of interpolative Hardy-Rogers and Ćirić-Reich-Rus type contraction, **2021** (2021), Article ID 9992783). We present a variety of examples which backs up our findings.

Keywords. Contraction, Fixed point, Extended b-metric space, Graph

Mathematics Subject Classification (2020). 26D07, 37C25

1. Introduction

The fixed point theory has been used in a variety of mathematical disciplines, including graph theory, non-linear analysis, and differential equations ([2, 4, 5, 8, 14, 15, 17, 27]). In 1922, Banach [8] established Banach contraction principle which was followed by various generalizations and improvements in generalized spaces. In 1968, the first improvement related to this theorem was seen by Kannan [22]. After that Rus, Ćirić, Reich, Hardy, and Rogers worked on this famous principle. Introduced the concept of b-metric space in 1993 and developed a number of fixed point results for contractive type mappings in b-metric space. Kamran *et al*. [21] proposed the notion of extended b-metric space in 2017. For more information in this space (see [3, 7, 18, 19, 23–25]). Errai *et al*. [16] recently acquired various Ćirić-Rus-Reich type and interpolative Hardy-Rogers type contraction mapping results. Recently, many authors have been working on combining fixed-point results with visual concepts such as edge preserves, graph preservence, weak graph preservence of the mappings and involved, transitivity of the graph, etc. Jachymski [20] initiated use of the graph concepts in the sector of fixed point and established the new direction in fixed point theory. He generalized the prominent "Banach contraction principle" for mapping in metric spaces by approaching the graph concept. Inspired by the significant work of Jachymski [20], more results in this direction were considered by Beg *et al.* [9], Bojor [10, 11], Samreen and Kamran [26] etc. Later this concept was extended and generalized by a lot of researchers. Motivated by [16] and [20], we generalize the results of Errai *et al.* [16] in an extended b-metric space fit up with a graph. Throughout this paper, let Δ represent the diagonal of the product $\mathbf{\mathring{A}} \times \mathbf{\mathring{A}}$ where $\mathbf{\mathring{A}} \neq \phi$. Let us choose $\widehat{\mathbf{G}}$ as a graph, where the set of vertices are indicated by $V(\widehat{\mathbf{G}})$ coincide with $\mathbf{\mathring{A}}$, edges and loops are contained by $E(\widehat{\mathbf{G}})$. Thus a pair $(V(\widehat{\mathbf{G}}, E(\widehat{\mathbf{G}}) = \widehat{\mathbf{G}}$ represents a graph. Let $\widehat{\mathbf{G}}^{-1}$ represents a change in $\widehat{\mathbf{G}}$ i.e. $E(\widehat{\mathbf{G}}^{-1}) = \{(\mathbf{v},\mathbf{u})|(\mathbf{u},\mathbf{v}) \in E(\widehat{\mathbf{G}})\}$ and $\ddot{\mathbf{G}}$ indicates an undirected graph from $\widehat{\mathbf{G}}$, when the direction of edges are not considered consequently $E(\widehat{\mathbf{G}}) \bigcup E(\widehat{\mathbf{G}}^{-1}) = E(\ddot{\mathbf{G}})$.

2. Definitions and Preliminaries

Definition 2.1 ([12])**.** Consider a non-empty set $\mathbf{\mathring{A}} \neq \phi$ and $s \geq 1$ a given real number. Then a function $d_b : \mathbf{\mathring{A}} \times \mathbf{\mathring{A}} \to [0,\infty)$ is a b-metric space if the following axioms are met for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{\mathring{A}}$.

(b_1) $d_b(\mathbf{u},\mathbf{v}) = 0$ iff $\mathbf{u} = \mathbf{v}$,

(b_2) $d_b(\mathbf{u},\mathbf{v}) = d_b(\mathbf{v},\mathbf{u})$,

(b_3) $d_b(\mathbf{u},\mathbf{w}) \leq s[d_b(\mathbf{u},\mathbf{v}) + d_b(\mathbf{v},\mathbf{w})]$.

Then, the pair $(\mathbf{\mathring{A}}, d_b)$ is b-metric space.

Definition 2.2 ([21])**.** Consider a function $\theta : \mathbf{\mathring{A}} \times \mathbf{\mathring{A}} \to [1,\infty)$ with a non-empty set $\mathbf{\mathring{A}} \neq \phi$. A function $d_\theta : \mathbf{\mathring{A}} \times \mathbf{\mathring{A}} \to [0,\infty)$ is an extended b-metric space if it satisfies the following axioms for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{\mathring{A}}$.

($d_\theta 1$) $d_\theta(\mathbf{u},\mathbf{v}) = 0$ iff $\mathbf{u} = \mathbf{v}$,

($d_\theta 2$) $d_\theta(\mathbf{u},\mathbf{v}) = d_\theta(\mathbf{v},\mathbf{u})$,

($d_\theta 3$) $d_\theta(\mathbf{u},\mathbf{w}) \leq \theta(\mathbf{u},\mathbf{w})[d_\theta(\mathbf{u},\mathbf{v}) + d_\theta(\mathbf{v},\mathbf{w})]$.

So, the pair $(\mathbf{\mathring{A}}, d_\theta)$ is an extended b-metric space.

Example 2.1. Consider $\mathbf{\mathring{A}} = \{-2,1,2\}$ and define the function $\theta : \mathbf{\mathring{A}} \times \mathbf{\mathring{A}} \to [1,\infty)$ as $\theta(\mathbf{u},\mathbf{v}) = |\mathbf{u}| + |\mathbf{v}|$. Let $d_\theta(\mathbf{u},\mathbf{v})$ define as: $d_\theta(1,1) = d_\theta(2,2) = d_\theta(-2,-2) = 0$, $d_\theta(1,2) = d_\theta(2,1) = \frac{1}{2}$ and $d_\theta(-2,1) = d_\theta(1,-2) = d_\theta(2,-2) = d_\theta(-2,2) = \frac{1}{3}$. We observe that $d_\theta(\mathbf{u},\mathbf{v})$ clearly satisfy the first two axioms. So, we have to prove only the last axiom:

$$d_\theta(1,2) = \frac{1}{2} \leq 3\left[\frac{1}{3} + \frac{1}{3}\right] = \theta(1,2)[d_\theta(1,-2) + d_\theta(-2,2)],$$

$$d_\theta(1,-2) = \frac{1}{3} \leq 3\left[\frac{1}{2} + \frac{1}{3}\right] = \theta(1,-2)[d_\theta(1,2) + d_\theta(2,-2)],$$

$$d_\theta(2,-2) = \frac{1}{3} \leq 4\left[\frac{1}{2} + \frac{1}{3}\right] = \theta(2,-2)[d_\theta(2,1) + d_\theta(1,-2)].$$

Definition 2.3 ([12]). Suppose $(\mathbf{\mathring{A}}, d_\theta)$ is an extended b-metric space.

(i) A sequence $\{\mathbf{u_n}\}$ in $\mathbf{\mathring{A}}$ converges to point $\mathbf{u}$ if for any $\epsilon > 0$ there is a number $N = N(\epsilon) \in \mathbb{N}$ with $d_\theta(\mathbf{u_n}, \mathbf{u}) < \epsilon$ for any $n \geq N$ and we write $\lim\limits_{\mathbf{n}\to\infty} \mathbf{u_n} = \mathbf{u}$, where $\mathbf{u}$ is the unique limit of the convergent sequence.

(ii) A sequence $\{\mathbf{u_n}\}$ in $\mathbf{u}$ is said to be a Cauchy sequence for any $\epsilon > 0$ there is $N = N(\epsilon) \in \mathbb{N}$ with $d_\theta(\mathbf{u_n}, \mathbf{u_m}) < \epsilon$ for any $n, m \geq N$.

(iii) An extended b-metric space $(\mathbf{\mathring{A}}, d_\theta)$ is said to be complete if we take every Cauchy sequence in $\mathbf{\mathring{A}}$ which is convergent in $\mathbf{\mathring{A}}$.

Definition 2.4 ([1]). Let $\{\mathbf{u_n}\}$ be a sequence in an extended b-metric space $(\mathbf{\mathring{A}}, d_\theta)$. Then the maps $G, T : \mathbf{\mathring{A}} \to \mathbf{\mathring{A}}$ and a point $\mathbf{u} \in \mathbf{\mathring{A}}$ possesses a property of coincidence point for the pair $\mathbf{G}, \mathbf{T}$ if $\mathbf{Gu} = \mathbf{Tu}$.

Definition 2.5 ([6, 13]). Let Ψ denote the set of all non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$ along with $\sum\limits_{\mathbf{k}=1}^{\infty} \psi^{\mathbf{k}}(\mathbf{t}) < \infty$, for all $\mathbf{t} > 0$ and

(i) $\psi(\mathbf{t}) < \mathbf{t}$ for any $\mathbf{t} > 0$,

(ii) $\psi(0) = 0$.

Definition 2.6. Assume $\mathbf{\mathring{A}} \neq \phi$ and $(\mathbf{\mathring{A}}, d_\theta)$ is an extended b-metric space. Let $\widehat{\mathbf{G}} = (V, E)$ represents the graph where the set of a vertices is $V(\widehat{\mathbf{G}})$ and set of an edge is denoted by $E(\widehat{\mathbf{G}})$ (including loops also). Then

(i) A sequence $\{\mathbf{u_n}\}$ in $\mathbf{\mathring{A}}$ converges to a point $\mathbf{u}$ of $\mathbf{\mathring{A}}$, if for any $\epsilon > 0$ there is a number $N = N(\epsilon) \in \mathbb{N}$ with $d_\theta(\mathbf{u_n}, \mathbf{u}) < \epsilon$ for any $n \geq N$ imply $(\mathbf{u_n}, \mathbf{u}) \in E(\widehat{\mathbf{G}})$ and we write $\lim\limits_{n\to\infty} \mathbf{u_n} = \mathbf{u}$, where $\mathbf{u}$ is the unique limit of the convergent sequence.

(ii) A sequence $\{\mathbf{u_n}\}$ in $\mathbf{\mathring{A}}$ is considered a Cauchy sequence for any $\epsilon > 0$ if $\exists\, N = N(\epsilon) \in \mathbb{N}$ with $d_\theta(\mathbf{u_n}, \mathbf{u_n}) < \epsilon$ for any $n, m \geq N$ imply $(\mathbf{u_n}, \mathbf{u_n}) \in E(\widehat{\mathbf{G}})$.

(iii) If we take any Cauchy sequence that is convergent in $\mathbf{\mathring{A}}$ implies convergent in $E(\widehat{\mathbf{G}})$ then $(\mathbf{\mathring{A}}, d_\theta)$ is a complete extended b-metric space.

Throughout the upcoming two lemmas, we assume that $\theta(u, v) = L \geq 1$ (a finite number).

Lemma 2.1. *Consider the function $\theta : \mathbf{\mathring{A}} \times \mathbf{\mathring{A}} \to [1, \infty)$ in an extended b-metric space $(\mathbf{\mathring{A}}, d_\theta)$. Let $\widehat{\mathbf{G}} = (V, E)$ represents a graph which contains loops. Suppose $\{\mathbf{u_n}\}$, $\{\mathbf{v_n}\}$ are two sequences converging to $\mathbf{u}$, $\mathbf{v}$ respectively with $(\mathbf{u_n}, \mathbf{v_n}) \in E(\widehat{\mathbf{G}})$ and $(\mathbf{u_n}, \mathbf{u})$, $(\mathbf{v_n}, \mathbf{v}) \in E(\widehat{\mathbf{G}})$ then*

(i) $\frac{1}{L^2} d_\theta(\mathbf{u}, \mathbf{v}) \leq \liminf\limits_{n\to\infty} d_\theta(\mathbf{u_n}, \mathbf{v_n}) \leq \limsup\limits_{n\to\infty} d_\theta(\mathbf{u_n}, \mathbf{v_n}) \leq L^2 d_\theta(\mathbf{u}, \mathbf{v})$.

Besides, if $\mathbf{u} = \mathbf{v}$, then $\lim\limits_{n\to\infty} d_\theta(u_n, v_n) = 0$. Further, for any $\mathbf{w}$ in $\mathbf{\mathring{A}}$ which is a loop. Also, we get

(ii) $\frac{1}{L} d_\theta(\mathbf{u}, \mathbf{w}) \leq \liminf\limits_{n\to\infty} d_\theta(\mathbf{u_n}, \mathbf{w}) \leq \limsup\limits_{n\to\infty} d_\theta(\mathbf{u_n}, \mathbf{w}) \leq L d_\theta(\mathbf{u}, \mathbf{v})$.

Proof. By employing the triangle inequality of an extended b-metric space, we observe that

$$d_\theta(\mathbf{u},\mathbf{v}) \le \theta(\mathbf{u},\mathbf{v})[d_\theta(\mathbf{u},\mathbf{u_n}) + d_\theta(\mathbf{u_n},\mathbf{v})],$$

$$d_\theta(\mathbf{u_n},\mathbf{v}) \le \theta(\mathbf{u_n},\mathbf{v})[d_\theta(\mathbf{u_n},\mathbf{v_n}) + d_\theta(\mathbf{v_n},\mathbf{v})]$$

Thus, we obtain

$$d_\theta(\mathbf{u},\mathbf{v}) \le L d_\theta(\mathbf{u},\mathbf{u_n}) + L^2 d_\theta(\mathbf{u_n},\mathbf{v_n}) + L^2 d_\theta(\mathbf{v_n},\mathbf{v}). \tag{2.1}$$

Further,

$$d_\theta(\mathbf{u_n},\mathbf{v_n}) \le \theta(\mathbf{u_n},\mathbf{v_n})[d_\theta(\mathbf{u_n},\mathbf{u}) + d_\theta(\mathbf{u},\mathbf{v_n})],$$

$$d_\theta(\mathbf{u},\mathbf{v_n}) \le \theta(\mathbf{u},\mathbf{v_n})[d_\theta(\mathbf{u},\mathbf{v}) + d_\theta(\mathbf{v},\mathbf{v_n})].$$

This implies

$$d_\theta(\mathbf{u_n},\mathbf{v_n}) \le L d_\theta(\mathbf{u_n},\mathbf{u}) + L^2 d_\theta(\mathbf{u},\mathbf{v}) + L^2 d_\theta(\mathbf{v_n},\mathbf{v}). \tag{2.2}$$

Taking the lower limit as $n \to \infty$ in (2.1) and upper limit as $n \to \infty$ in equation (2.2), we obtain the first required result. Similarly, by using triangle inequality of an extended b-metric space we obtain condition (ii) of Lemma 2.1. □

Lemma 2.2. *Consider a sequence* $\{\mathbf{u_n}\}$ *which is defined on an extended b-metric space* $(\mathbf{Å}, d_\theta)$ *and satisfies the axioms listed below:*

(i) *if* $\{\mathbf{u_n}\}$ *is a convergent sequence in* $(\mathbf{Å}, d_\theta)$*, then* $(\mathbf{u_n},\mathbf{u}) \in E(\widehat{\mathbf{G}})$.

(ii) $d_\theta(\mathbf{u_{n+1}},\mathbf{u_n}) \le \psi(d_\theta(\mathbf{u_n},\mathbf{u_{n-1}})) \le \psi^2(d_\theta(\mathbf{u_{n-1}},\mathbf{u_{n-2}}) \le \dots \le \psi^2(d_\theta(\mathbf{u_1},\mathbf{u_0})$*, where* $\psi \in \Psi$.

Then $\{\mathbf{u_n}\}$ *is a Cauchy sequence in an extended b-metric space.*

Proof. Let $p \in \mathbb{N} - \{0\}$ and employing the triangle inequality of the extended b-metric space $(\mathbf{Å}, d_\theta)$ along with condition (ii) of Lemma 2.2, we obtain

$$\begin{aligned}
d_\theta(\mathbf{u_n},\mathbf{u_{n+p}}) &\le L[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}}) + d_\theta(\mathbf{u_{n+1}},\mathbf{u_{n+p}})] \\
&\le L[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}}) + L d_\theta(\mathbf{u_{n+1}},\mathbf{u_{n+2}}) + L d_\theta(\mathbf{u_{n+2}},\mathbf{u_{n+p}})] \\
&\;\vdots \\
&\le L\psi^n[d_\theta(\mathbf{u_0},\mathbf{u_1}) + L^2\psi^{n+1} d_\theta(\mathbf{u_0},\mathbf{u_1}) + \dots + L^p \psi^{n+p-1} d_\theta(\mathbf{u_0},\mathbf{u_1})] \\
&\le L^p \sum_{k=1}^{n+p-1} \psi^k d_\theta(\mathbf{u_0},\mathbf{u_1}) \\
&\le L^p \sum_{k=n}^{\infty} \psi^k d_\theta(\mathbf{u_0},\mathbf{u_1}).
\end{aligned}$$

Since we observe that $\sum_{k=0}^{\infty} \psi^k(\mathbf{t}) < \infty$, for all $\mathbf{t} > 0 \Rightarrow \lim_{n\to\infty} \sum_{k=n}^{\infty} \psi^k d_\theta(\mathbf{u_0},\mathbf{u_1}) = 0 \Rightarrow$ for any finite integer $p \ge 1$, $\lim_{n\to\infty} d_\theta(\mathbf{u_n},\mathbf{u_{n+p}}) = 0$. Hence $\{\mathbf{u_n}\}$ is a Cauchy sequence in an extended b-metric space $(\mathbf{Å}, d_\theta)$. □

3. Main Results

Now, we present here our main results. We denote the set of all functions i.e.,

$$\Omega = \{\xi(\mathbf{t}) < \mathbf{t},\ \text{for any } \mathbf{t} > 0, \mathbf{t} \in [0,1]\}$$

where ξ is function on $[0,\infty)$.

Definition 3.1. Consider an extended b-metric space $(\mathbf{\mathring{A}}, d_\theta)$ and $\widehat{\mathbf{G}} = (V(\widehat{\mathbf{G}}), E(\widehat{\mathbf{G}}))$ contains loops. A self map A on $\mathbf{\mathring{A}}$ is termed as generalized ξ-interpolative Hardy-Rogers type contraction if it satisfy the following conditions:

(i) There $\exists\ \kappa, \beta, \gamma, \delta \in (0,1)$ such that $\kappa + \beta + \gamma + \delta > 1$ imply $\kappa, \beta, \gamma, \delta \in E(\widehat{\mathbf{G}})$.

(ii) $\theta(\mathbf{u},\mathbf{v}) d_\theta(A\mathbf{u}, A\mathbf{v}) \le \xi\left([d_\theta(\mathbf{u},\mathbf{v})]^\kappa [d_\theta(\mathbf{u}, A\mathbf{v})]^\beta [d_\theta(\mathbf{v}, A(\mathbf{v}))]^\gamma \left[\dfrac{d_\theta(\mathbf{u}, A(\mathbf{v})) + d_\theta(\mathbf{v}, A(\mathbf{u}))}{2}\right]^\delta\right)$

for any $\mathbf{u},\mathbf{v} \in \mathbf{\mathring{A}} - \mathrm{Fix}(A)$ imply $(\mathbf{u},\mathbf{v}) \in E(\widehat{\mathbf{G}})$ where $\mathrm{Fix}(A) = \{\mathbf{u} \in \mathbf{\mathring{A}} : A\mathbf{u} = \mathbf{u}\}$, $\xi \in \mathbf{\Omega}$.

Theorem 3.1. *Consider an extended b-metric space $(\mathbf{\mathring{A}}, d_\theta)$ which is complete and $\widehat{\mathbf{G}} = (V(\widehat{\mathbf{G}}), E(\widehat{\mathbf{G}}))$ represents a graph containing loops also. Choose a generalized ξ-interpolative Hardy-Rogers type contraction map A on $\mathbf{\mathring{A}}$. Assume that $\exists\ \mathbf{u} \in \mathbf{\mathring{A}}$ with $d_\theta(\mathbf{u}, A\mathbf{u}) < 1$ imply A, possesses a fixed point in $\mathbf{\mathring{A}}$ and $(\mathbf{u},\mathbf{u}) \in E(\widehat{\mathbf{G}})$.*

Proof. Choose a sequence $\{\mathbf{u_n}\}$ defined by $\mathbf{u_0} = u$ and $\mathbf{u_{n+1}} = A\mathbf{u_n}$ for any integer $\mathbf{n}$. If there exists $\mathbf{n_0}$ with $\mathbf{u_{n_0}} = \mathbf{u_{n_0+1}}$ then $\mathbf{u_{n_0}}$ is a fixed point of A. Suppose $\mathbf{u_{n+1}} = A(\mathbf{u_n})$, for all $\mathbf{n} \ge 0$ in of Definition 3.1 we obtain

$$\begin{aligned} d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}}) &\le \theta(\mathbf{u_n}, \mathbf{u_{n+1}}) d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}}) \\ &\le \xi\left([d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n})]^\kappa [d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n})]^\beta [d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}})]^\gamma \left[\frac{d_\theta(\mathbf{u_{n-1}}, \mathbf{u_{n+1}}) + d_\theta(\mathbf{u_n}, \mathbf{u_n})}{2}\right]^\delta\right) \\ &\le \xi\left([d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n})]^{\kappa+\beta} [d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}})]^\gamma \left[\frac{d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n}) + d_\theta(\mathbf{u_n}, \mathbf{u_n})}{2}\right]^\delta\right). \end{aligned} \tag{3.1}$$

By employing the condition $\xi(\mathbf{t}) < t$, for all $\mathbf{t} > 0$, we have

$$d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}}) < \left([d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n})]^{\kappa+\beta} [d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}})]^\gamma \left[\frac{d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n}) + d_\theta(\mathbf{u_n}, \mathbf{u_n})}{2}\right]^\delta\right). \tag{3.2}$$

If $d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n}) < d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}})$ for some $\mathbf{n} \ge 1$, imply

$$\frac{d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n}) + d_\theta(\mathbf{u_n}, \mathbf{u_n})}{2} \le d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}}). \tag{3.3}$$

From (3.2), we get

$$d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}}) < [d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n})]^{\kappa+\beta} [d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}})]^{\gamma+\delta}. \tag{3.4}$$

So,

$$d_\theta(\mathbf{u_n}, \mathbf{u_{n+1}})^{1-\gamma-\delta} < [d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n})]^{\kappa+\beta}. \tag{3.5}$$

This implies

$$d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n})^{1-\gamma-\delta} < [d_\theta(\mathbf{u_{n-1}}, \mathbf{u_n}]^{\kappa+\beta}. \tag{3.6}$$

From (3.6) we arrive at contradiction i.e. $1-\gamma-\delta < \kappa+\beta$ and $d_\theta(\mathbf{u_{n-1}},\mathbf{u_n}) < 1$. Thus, $d_\theta(\mathbf{u_{n-1}},\mathbf{u_n}) \geq 1$, $d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})$ for all $\mathbf{n} \geq 1$ and $\frac{d_\theta(\mathbf{u_{n-1}},\mathbf{u_n})+d_\theta(\mathbf{u_n},\mathbf{u_n})}{2} \leq d_\theta(\mathbf{u_{n-1}},\mathbf{u_n})$. Using (3.2) we obtain

$$d_\theta(\mathbf{u_{n-1}},\mathbf{u_n})^{1-\delta} < [d_\theta(\mathbf{u_{n-1}},\mathbf{u_n})]^{\kappa+\beta+\gamma}, \quad \text{for all } \mathbf{n} \geq 1. \tag{3.7}$$

Also, $d_\theta(\mathbf{u_0},\mathbf{u_1}) < 1$ implies there exists a number $k \in (0,1)$ such that $d_\theta(\mathbf{u_0},\mathbf{u_1}) \leq 1$ and $k = \frac{d_\theta(\mathbf{u_0},\mathbf{u_1})+1}{2}$.

From (3.7), we get $d_\theta(\mathbf{u_1},\mathbf{u_2}) \leq d_\theta(\mathbf{u_0},\mathbf{u_1})^{\frac{\kappa+\beta+\gamma}{1-\delta}} \leq k^{\frac{\kappa+\beta+\gamma}{1-\delta}}$. Suppose there exists a real number $\wp^{\mathbf{n}}$, for all $\mathbf{n} \geq 0$. From (3.7) we obtain

$$d_\theta(\mathbf{u_{n+1}},\mathbf{u_{n+2}})^{1-\delta} < [d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})]^{\kappa+\beta+\gamma} \leq k^{(\kappa+\beta+\gamma)\wp(\mathbf{n})}. \tag{3.8}$$

Thus, we obtain

$$d_\theta(\mathbf{u_{n+1}},\mathbf{u_{n+2}}) < k^{(\kappa+\beta+\gamma)\wp(\mathbf{n+1})}. \tag{3.9}$$

Let $\wp(\mathbf{n}+\mathbf{1}) = \left(\frac{\kappa+\beta+\gamma}{1-\delta}\right)\wp(\mathbf{n})$, for all $\mathbf{n} \geq 1$ such that $\wp(0) = 1$.

Also, $\frac{\kappa+\beta+\gamma}{1-\delta} > 1$ with $\lim\limits_{\mathbf{n}\to\infty} \wp(\mathbf{n}) = \infty$. This implies

$$\sum_{\mathbf{n}=0}^{\infty} d_\theta(\mathbf{u_n},\mathbf{u_{n+1}}) \leq \sum_{\mathbf{n}=0}^{\infty} k^{\wp(\mathbf{n})}. \tag{3.10}$$

Thus $\{\mathbf{u_n}\}$ represents a cauchy sequence in $(\mathbf{Å}, d_\theta)$ converging to $\mathbf{u}$ in $\mathbf{Å} \Rightarrow (\mathbf{u},\mathbf{u}) \in E(\widehat{\mathbf{G}})$. Let us suppose that $A\mathbf{u} \neq \mathbf{u}$, then by (ii) of Definition 3.1

$$\begin{aligned}
&\theta(\mathbf{u_{n+1}},\mathbf{u_{n+1}})d_\theta(\mathbf{u_{n+1}},A\mathbf{u}) \\
&\quad \leq \xi\left([d_\theta(\mathbf{u_n},\mathbf{u})]^\kappa[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})]^\gamma[d_\theta(\mathbf{u},A\mathbf{u}]^\gamma\left[\frac{d_\theta(\mathbf{u_n},A\mathbf{u})+d_\theta(\mathbf{u},\mathbf{u_n})}{2}\right]^\delta\right) \\
&\quad < \left([d_\theta(\mathbf{u_n},\mathbf{u})]^\kappa[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})]^\beta[d_\theta(\mathbf{u},A\mathbf{u}]^\gamma\left[\frac{d_\theta(\mathbf{u_n},A\mathbf{u})+d_\theta(\mathbf{u},\mathbf{u_{n+1}})}{2}\right]^\delta\right).
\end{aligned} \tag{3.11}$$

Letting $\mathbf{n}\to\infty$, we obtain $d_\theta(\mathbf{u},A\mathbf{u})<0$. Thus $d_\theta(\mathbf{u},A\mathbf{u})=0$, which is contradiction so $A\mathbf{u}=\mathbf{u}$. □

Example 3.1. Let $\mathbf{Å} = [0,3]$ be a non-empty set endowed with an extended b-metric space $d_\theta : \mathbf{Å}\times\mathbf{Å} \to [0,\infty)$ defined by

$$d_\theta(\mathbf{u},\mathbf{v}) = \begin{cases} 0, & \text{if } \mathbf{u}=\mathbf{v} \\ 2, & \text{if } \mathbf{u},\mathbf{v} \in [0,1] \text{ and } \mathbf{u} \neq \mathbf{v} \\ 3, & \text{otherwise} \end{cases}$$

where $\theta : \mathbf{Å}\times\mathbf{Å} \to [1,\infty)$ defined as $\theta(\mathbf{u},\mathbf{v}) = \mathbf{u}+\mathbf{v}$ along with Graph, i.e., $[0,1] = V(\mathbf{Å})E(\widehat{\mathbf{G}}) = \{(\mathbf{u},\mathbf{v})| \in \mathbf{Å}\times\mathbf{Å}\}$. Consider the map, $A : \mathbf{Å} \to \mathbf{Å}$ defined as

$$A(\mathbf{u}) = \begin{cases} \frac{1}{3}, & \mathbf{u} \in [0,1] \\ \frac{\mathbf{u}}{3}, & \mathbf{u} \in (1,5) \end{cases}$$

and the function $\xi(\mathbf{t}) = \frac{2}{7\mathbf{u}^2}$, for all $\mathbf{t} \in [0,\infty)$. Choose $\kappa = 0.7$, $\beta = 0.6$, $\gamma = 0.8$, $\delta = 0.5$ and $d_\theta(A\mathbf{u},A\mathbf{v}) \leq 2$, $\theta(\mathbf{u},\mathbf{v}) \leq 6$, for all $\mathbf{u},\mathbf{v} \in \mathbf{Å}$.

The following cases are under consideration:

Case I: If $\mathbf{u},\mathbf{v}$ in $[0,1]$ or $\mathbf{u}=\mathbf{v}$, for all $\mathbf{u},\mathbf{v}$ in $\mathring{\mathbf{A}}$, we obtain $\xi(\mathbf{t})$ is non negative, for all $\mathbf{t}\in[0,\infty)$ also $d_\theta(\mathbf{u},\mathbf{v})=0$, for all $\mathbf{u},\mathbf{v}\in[0,1]$ or $\mathbf{u}=\mathbf{v}$, for all $\mathbf{u},\mathbf{v}$ in $[0,3]$. Clearly, *Cases* (i) and (ii) of Definition 3.1 are fulfilled.

Case II: If $\mathbf{u},\mathbf{v}$ in $(1,3]$ and $\mathbf{v}\neq\mathbf{u}$, we get

$$\begin{aligned}\theta(\mathbf{u},\mathbf{v})d_\theta(A\mathbf{u},A\mathbf{v}) &\le \xi\left([d_\theta(\mathbf{u},\mathbf{v})]^\kappa[d_\theta(\mathbf{u},A\mathbf{u})]^\beta[d_\theta(\mathbf{v},A\mathbf{v})]^\gamma\left[\frac{d_\theta(\mathbf{u},A\mathbf{v})+d_\theta(\mathbf{v},A\mathbf{u})}{2}\right]^\delta\right)\\ &=\xi(2^{\kappa+\beta+\gamma+\delta})=\frac{2}{7}2^{5.2}\ge 12.\end{aligned}\tag{3.12}$$

Case III: If $\mathbf{u}\in[0,1]$ and $\mathbf{v}\in(1,3]$, $\mathbf{u}\neq\frac{1}{3}$, we get

$$\begin{aligned}\theta(\mathbf{u},\mathbf{v})d_\theta(A\mathbf{u},A\mathbf{v}) &\le \xi\left([d_\theta(\mathbf{u},\mathbf{v})]^\kappa[d_\theta(\mathbf{u},A\mathbf{u})]^\beta[d_\theta(\mathbf{v},A\mathbf{v})]^\gamma\left[\frac{d_\theta(\mathbf{u},A\mathbf{v})+d_\theta(\mathbf{v},A\mathbf{u})}{2}\right]^\delta\right)\\ &=\xi(3^{\kappa+\gamma}.2^{\beta-\delta}.3^{\kappa+\gamma}.5^\delta)=\frac{2}{7}.3^3.2^{0.2}.5\ge 12.\end{aligned}\tag{3.13}$$

Case IV: If $\mathbf{u}$ in $(1,3]$ and $\mathbf{v}$ in $[0,1]$, $\mathbf{v}\neq\frac{1}{3}$, we get

$$\begin{aligned}\theta(\mathbf{u},\mathbf{v})d_\theta(A\mathbf{u},A\mathbf{v}) &\le \xi\left([d_\theta(\mathbf{u},\mathbf{v})]^\kappa[d_\theta(\mathbf{u},A\mathbf{u})]^\beta[d_\theta(\mathbf{v},A\mathbf{v})]^\gamma\left[\frac{d_\theta(\mathbf{u},A\mathbf{v})+d_\theta(\mathbf{v},A\mathbf{u})}{2}\right]^\delta\right)\\ &=\xi(3^{\kappa+\gamma}.2^{\beta-\delta}.5^\delta)=\frac{2}{7}.3^{2.6}.2^{0.6}.5\ge 12.\end{aligned}\tag{3.14}$$

Hence in all the cases, the map A and distance function d_θ satisfies generalized ξ-interpolative Hardy-Rogers type contraction, for all $\mathbf{u},\mathbf{v}\in\mathring{\mathbf{A}}-\frac{1}{3}$. Consequently, all the conditions of Theorem 3.1 are satisfied so A possesses a fixed point say $\mathbf{u}=\frac{1}{3}\approx 0.33$ as shown Figure 1.

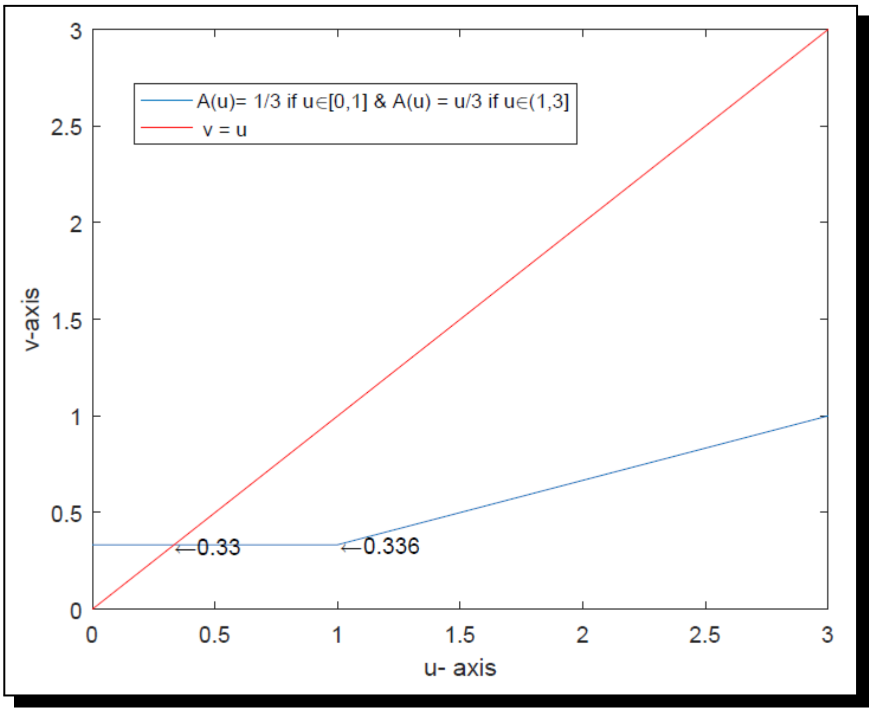

Figure 1. Graph of a fixed point for A

If we take $\xi(\mathbf{u}) = \kappa\mathbf{u}$ such that $\kappa \in (0,1)$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.1. *Consider an extended b-metric space $(\mathring{\mathbf{A}}, d_\theta)$ which is complete and $\widehat{\mathbf{G}} = (V(\widehat{\mathbf{G}}), E(\widehat{\mathbf{G}}))$ represents a graph containing loops also. Consider a map A on $\mathring{\mathbf{A}}$ satisfied the following conditions:*

(i) *There exists $\kappa,\beta,\gamma,\delta \in (0,1)$ such that $\kappa+\beta+\gamma+\delta > 1$ imply $\kappa,\beta,\gamma,\delta \in E(\widehat{\mathbf{G}})$.*

(ii) $\theta(\mathbf{u},\mathbf{v})d_\theta(A\mathbf{u},A\mathbf{v}) \le \xi\left([d_\theta(\mathbf{u},\mathbf{v})]^\kappa [d_\theta(\mathbf{u},A\mathbf{v})]^\beta [d_\theta(\mathbf{v},A(\mathbf{v}))]^\gamma \left[\dfrac{d_\theta(\mathbf{u},A(\mathbf{v}))+d_\theta(\mathbf{v},A(\mathbf{u}))}{2}\right]^\delta\right)$

for any $\mathbf{u},\mathbf{v} \in \mathring{\mathbf{A}} - \mathrm{Fix}(A)$ imply $(\mathbf{u},\mathbf{v}) \in E(\widehat{\mathbf{G}})$ where $\mathrm{Fix}(A) = \{\mathbf{u} \in \mathring{\mathbf{A}} : A\mathbf{u} = \mathbf{u}\}$, $\xi \in \mathbf{\Omega}$.

Definition 3.2. Let $(\mathring{\mathbf{A}}, d_\theta)$ be an extended b-metric space that represents a graph $\widehat{\mathbf{G}} = (V(\widehat{\mathbf{G}}), E(\widehat{\mathbf{G}}))$ containing loops. For these self maps A, B on $\mathring{\mathbf{A}}$ are termed as generalized b-interpolative Hardy-Rogers type contraction if it satisfies the following conditions:

(i) There exists $\kappa,\beta,\gamma \in (0,1)$ such that $\kappa+\beta+\gamma > 1$ imply $\kappa,\beta,\gamma \in E(\widehat{\mathbf{G}})$.

(ii) $\theta(\mathbf{u},\mathbf{u})d_\theta(A\mathbf{u},A\mathbf{v})$

$$\le \xi\left([d_\theta(B\mathbf{u},B\mathbf{v})]^\kappa [d_\theta(B\mathbf{u},A\mathbf{u})]^\beta [d_\theta(B\mathbf{v},A\mathbf{v})]^\gamma \left[\frac{d_\theta(B\mathbf{u},A\mathbf{v})+d_\theta(B\mathbf{v},A\mathbf{u})}{2\theta}\right]^{1-\kappa-\beta-\gamma}\right)$$

for any $\mathbf{u},\mathbf{v} \in \mathring{\mathbf{A}}$ implies $(\mathbf{u},\mathbf{v}) \in E(\mathbf{G})$ such that $A\mathbf{u} \ne B\mathbf{u}$, $A\mathbf{v} \ne B\mathbf{v}$, and $B\mathbf{u} \ne B\mathbf{v}$ and $\psi \in \Psi$.

Theorem 3.2. *Consider an extended b-metric space $(\mathring{\mathbf{A}}, d_\theta)$ which is complete that represents a graph $\widehat{\mathbf{G}} = (V(\widehat{\mathbf{G}}), E(\widehat{\mathbf{G}}))$ containing loops. Consider a generalized b-interpolative Hardy-Rogers type contraction map A on $\mathring{\mathbf{A}}$ with*

(i) *$A\mathring{\mathbf{A}} \subseteq B\mathring{\mathbf{A}}$.*

(ii) *$B\mathring{\mathbf{A}}$ is closed.*

Then A and B possess a coincidence point.

Proof. Consider a point $\mathbf{u} \in \mathring{\mathbf{A}}$, since $A\mathring{\mathbf{A}} \subseteq B\mathring{\mathbf{A}}$, we define a sequence $\{\mathbf{u_n}\}$ such that

$$\mathbf{u_0} = \mathbf{u} \text{ and } B\mathbf{u_n} = A\mathbf{u_n}, \quad \text{for all integer } \mathbf{n}. \tag{3.15}$$

Consider a sequence $\{\mathbf{u_n}\}$ defined by $\mathbf{u_0} = \mathbf{u}$ and $B\mathbf{u_{n+1}} = A\mathbf{u_n}$ for any integer $\mathbf{n}$. If there exists $\mathbf{n} \in \{0,1,2,\dots\}$ with $A\mathbf{u_n} = B\mathbf{u_n}$ then $\mathbf{u_n}$ is a coincidence point. Let us suppose that $A\mathbf{u_n} \ne B\mathbf{u_n}$, for all $\mathbf{n}$. From (ii) of Definition 3.2 we obtain

$$\begin{aligned} d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n}) &\le \theta(\mathbf{u_{n+1}},\mathbf{u_n})d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n}) \\ &\le \psi\Bigg([d_\theta(B\mathbf{u_{n+1}},B\mathbf{u_n})]^\kappa [d_\theta B(\mathbf{u_{n+1}},A\mathbf{u_{n+1}})]^\beta [d_\theta B(\mathbf{u_n},A\mathbf{u_n})]^\gamma \\ &\quad \cdot \left[\frac{d_\theta(B\mathbf{u_{n+1}},A\mathbf{u_n})+d_\theta B(\mathbf{u_n},A\mathbf{u_{n+1}})}{2}\right]^{1-\kappa-\beta-\gamma}\Bigg) \\ &= \psi\Big([d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})]^\kappa [d_\theta(A\mathbf{u_n},A\mathbf{u_{n+1}})]^\beta [d_\theta(A\mathbf{u_{n-1}},A\mathbf{u_n})]^\gamma \end{aligned}$$

$$\cdot\left[\frac{d_\theta(A\mathbf{u_n},A\mathbf{u_n})+d_\theta(A\mathbf{u_{n-1}},\mathbf{u_{n+1}})}{2}\right]^{1-\kappa-\beta-\gamma}\Bigg)$$

$$\le \psi\Bigg([d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})]^{\kappa+\gamma}[d_\theta(A\mathbf{u_n},\mathbf{u_{n+1}})]^{\beta}$$

$$\cdot\left[\frac{d_\theta(A\mathbf{u_{n-1}},A\mathbf{u_n})+d_\theta(A\mathbf{u_n},A\mathbf{u_{n+1}})}{2}\right]^{1-\kappa-\beta-\gamma}\Bigg). \tag{3.16}$$

By employing the condition $\psi(\mathbf{t})<t$ for any $\mathbf{t}>0$, we have

$$d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})\le\Bigg([d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})]^{\kappa+\gamma}[d_\theta(A\mathbf{u_n},\mathbf{u_{n+1}})]^{\beta}$$

$$\cdot\left[\frac{d_\theta(A\mathbf{u_{n-1}},A\mathbf{u_n})+d_\theta(A\mathbf{u_n},A\mathbf{u_{n+1}})}{2}\right]^{1-\kappa-\beta-\gamma}\Bigg). \tag{3.17}$$

If $d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})<d_\theta(A\mathbf{u_n},A\mathbf{u_{n+1}})$ for some $\mathbf{n}\ge 1$. Then

$$\frac{d_\theta(A\mathbf{u_{n-1}},A\mathbf{u_n})+d_\theta(A\mathbf{u_n},A\mathbf{u_{n+1}})}{2}\le d_\theta(A\mathbf{u_n},A\mathbf{u_{n+1}}). \tag{3.18}$$

Thus, from (3.17) we obtain

$$d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})\le[d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})]^{\kappa+\gamma}[d_\theta(A\mathbf{u_n},A\mathbf{u_{n+1}})]^{1-\kappa-\gamma}. \tag{3.19}$$

Further,

$$d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})^{\kappa+\gamma}\le[d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})]^{\kappa+\gamma}. \tag{3.20}$$

Thus, we obtain

$$d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})\le[d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})] \tag{3.21}$$

which is contradiction. Thus

$$d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})\le[d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})],\quad \text{for all } \mathbf{n}\ge 1. \tag{3.22}$$

Thus, the positive sequence $\{d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})\}$ is monotone decreasing sequence so there exists $h\ge 0$ such that $\lim_{\mathbf{n}\to\infty} d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})=h$. By using (3.17), (3.18) and (3.22) we conclude

$$d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})\le[d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})]^{\kappa+\gamma}[d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})]^{\beta}[d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})]^{1-\kappa-\gamma}$$

$$=d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}}). \tag{3.23}$$

Thus from (3.17) and with non-decreasing character of ψ, we obtain

$$d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})\le\psi d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}}), \tag{3.24}$$

$$d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})\le\psi d_\theta(A\mathbf{u_n},A\mathbf{u_{n-1}})\le\psi^2 d_\theta(A\mathbf{u_{n-1}},A\mathbf{u_{n-2}})\le\ldots\le\psi^n d_\theta(A\mathbf{u_1},A\mathbf{u_0}). \tag{3.25}$$

As $\mathbf{n}\to\infty$ in (3.26) and with using the fact $\lim_{\mathbf{n}\to\infty}\psi^{\mathbf{n}}(\mathbf{t})=0$, we conclude that

$$\lim_{\mathbf{n}\to\infty} d_\theta(A\mathbf{u_{n+1}},A\mathbf{u_n})=0. \tag{3.26}$$

Thus by Lemma 2.2 $\{A\mathbf{u_n}\}$ is Cauchy sequence in an extended b-metric space and consequently $B(\mathbf{u_n})$ is also a is Cauchy sequence in same so there exists $\mathbf{t}\in\mathbf{\mathring{A}}$ imply $(\mathbf{t},\mathbf{t})\in E(\widehat{\mathbf{G}})$ such that

$$\lim_{\mathbf{n}\to\infty} d_\theta(A\mathbf{u_n},\mathbf{t})=\lim_{\mathbf{n}\to\infty} d_\theta(B\mathbf{u_{n+1}},\mathbf{t})=0. \tag{3.27}$$

As $\mathbf{t}\in B(\mathbf{\mathring{A}})$, there exists $\mathbf{v}\in\mathbf{\mathring{A}}$ such that $\mathbf{t}=B\mathbf{v}$. Suppose $\mathbf{v}$ is coincidence point of A and B.

Further, we claim that $B\mathbf{v}=A\mathbf{v}$, we have

$$\begin{aligned}
d_\theta(A\mathbf{u_n},A\mathbf{v}) &\le \theta(A\mathbf{u_n},A\mathbf{v})d_\theta(A\mathbf{u_n},A\mathbf{v})\\
&\le \psi\Bigg([d_\theta(B\mathbf{u_n},B\mathbf{v})]^\kappa[d_\theta(B\mathbf{u_n},A\mathbf{u_n})]^\beta[d_\theta(B\mathbf{v},A\mathbf{v})]^\gamma\\
&\quad\cdot\left[\frac{d_\theta(B\mathbf{u_n},A\mathbf{v})+d_\theta(B\mathbf{v},A\mathbf{u_n})}{2\theta}\right]^{1-\kappa-\beta-\gamma}\Bigg)\\
&< \Bigg([d_\theta(B\mathbf{u_n},B\mathbf{v})]^\kappa[d_\theta(B\mathbf{u_n},A\mathbf{u_n})]^\beta[d_\theta(B\mathbf{v},A\mathbf{v})]^\gamma\\
&\quad\cdot\left[\frac{d_\theta(B\mathbf{u_n},A\mathbf{v})+d_\theta(B\mathbf{v},A\mathbf{u_n})}{2\theta}\right]^{1-\kappa-\beta-\gamma}\Bigg).
\end{aligned}\tag{3.28}$$

Thus

$$\frac{1}{\theta}d_\theta(\mathbf{t},A\mathbf{v})\le\left([\theta d_\theta(\mathbf{t},B\mathbf{v})]^\kappa[\theta^2 d_\theta B(\mathbf{t},\mathbf{t})]^\beta[d_\theta(B\mathbf{v},A\mathbf{v})]^\gamma\left[\frac{d_\theta(\mathbf{t},A\mathbf{v})+d_\theta B(\mathbf{v},\mathbf{t})}{2}\right]^{1-\kappa-\beta-\gamma}\right)=0.$$

which is a contradiction. Thus $A\mathbf{v}=\mathbf{t}=B\mathbf{v}$, i.e., $\mathbf{t}$ is the coincidence point of A and B. □

Example 3.2. Consider a graph $\widehat{\mathbf{G}}=(V(\widehat{\mathbf{G}}),E(\widehat{\mathbf{G}}))$ which contains loops. Consider a set $\mathbf{\mathring{A}}=[0,\infty)$ and $d_\theta:\mathbf{\mathring{A}}\times\mathbf{\mathring{A}}\to[0,\infty)$ defined by

$$d_\theta(\mathbf{u},\mathbf{v})=\begin{cases}(\mathbf{u}+\mathbf{v})^2, & \text{if } \mathbf{u}\ne\mathbf{v},\\ 0, & \text{if } \mathbf{u}=\mathbf{v}.\end{cases}$$

Then $(\mathbf{\mathring{A}},d_\theta)$ is complete extended b-metric space. Define the self maps A and B on $\mathbf{\mathring{A}}$ as $B(\mathbf{u})=\mathbf{u}^2$, for all $\mathbf{u}\in\mathbf{\mathring{A}}$ and

$$A(\mathbf{u})=\begin{cases}1, & \mathbf{u}\in[0,2],\\ e^{-\mathbf{u}} & \mathbf{u}\in(2,\infty).\end{cases}$$

A is b-interpolative Hardy-Rogers type contraction map with $\kappa=0.3$, $\beta=0.6$, $\gamma=0.4$, $\psi(\mathbf{t})=\frac{3}{5\mathbf{t}^2}$, for all $\mathbf{t}\in[0,\infty)$. Now, we discuss the following cases:

Case I: If $\mathbf{u},\mathbf{v}$ in $[0,2]$ or $\mathbf{u}=\mathbf{v}$ for all $\mathbf{u}\in[0,\infty)$, this is trivial.

Case II: If $\mathbf{u}$ in $[0,2]-1$ and $\mathbf{v}$ in $(2,\infty)$ then

$$\begin{aligned}
d_\theta(A\mathbf{u},A\mathbf{v}) &\le \theta(A\mathbf{u},A\mathbf{v})d_\theta(A\mathbf{u},A\mathbf{v})\\
&\le \psi\Bigg([d_\theta(B\mathbf{u},B\mathbf{v})]^\kappa[d_\theta(B\mathbf{u},A\mathbf{u})]^\beta[d_\theta(B\mathbf{v},A\mathbf{v})]^\gamma\\
&\quad\cdot\left[\frac{d_\theta(B\mathbf{u},A\mathbf{v})+d_\theta(B\mathbf{v},A\mathbf{u})}{2\theta}\right]^{1-\kappa-\beta-\gamma}\Bigg)\\
&= \left((\mathbf{u}^2+\mathbf{v}^2)^{2\kappa}(\mathbf{u}^2+1)^{2\beta}(\mathbf{v}^2+e^{-\mathbf{v}})^{2\gamma}\left[\frac{(\mathbf{u}^2+e^{-\mathbf{u}})^2+(\mathbf{v}^2+1)^2}{4}\right]^{1-\kappa-\beta-\gamma}\right)\\
&\ge \psi\left(4^{2\kappa}1^{2\beta}4^{2\gamma}\left[\frac{5^2}{4}\right]^{1-\kappa-\beta-\gamma}\right)\\
&= \psi(4^{1.7}5^{-0.6})
\end{aligned}$$

$$= 3.4^{3.4}5^{-2.2} \geq (1+e^{-2})^2, \tag{3.29}$$

where $d_\theta(A\mathbf{u}, A\mathbf{v}) = (1+e^{-\mathbf{v}})^2 \leq (1+e^{-2})^2$.

Thus

$$\theta(Au, Av)d_\theta(A\mathbf{u}, A\mathbf{v}) \leq \psi\Bigg([d_\theta(B\mathbf{u}, B\mathbf{v})]^\kappa [d_\theta(B\mathbf{u}, A\mathbf{u})]^\beta [d_\theta(B\mathbf{v}, A\mathbf{v})]^\gamma \cdot \left[\frac{d_\theta(B\mathbf{u}, A\mathbf{v}) + d_\theta(B\mathbf{v}, A\mathbf{u})}{2\theta}\right]^{1-\kappa-\beta-\gamma}\Bigg). \tag{3.30}$$

Case III: If $\mathbf{u} \in (2,\infty)$ and $\mathbf{v} \in [0,2]-1$, we obtain

$$d_\theta(A\mathbf{u}, A\mathbf{v}) = (1+e^{-\mathbf{v}})^2 \leq (1+e^{-2})^2, \tag{3.31}$$

$$\begin{aligned}
d_\theta(A\mathbf{u}, A\mathbf{v}) &\leq \theta(A\mathbf{u}, A\mathbf{v})d_\theta(A\mathbf{u}, A\mathbf{v}) \\
&\leq \psi\Bigg([d_\theta(B\mathbf{u}, B\mathbf{v})]^\kappa [d_\theta(B\mathbf{u}, A\mathbf{u})]^\beta [d_\theta(B\mathbf{v}, A\mathbf{v})]^\gamma \\
&\qquad \left[\frac{d_\theta(B\mathbf{u}, A\mathbf{v}) + d_\theta(B\mathbf{v}, A\mathbf{u})}{2\theta}\right]^{1-\kappa-\beta-\gamma}\Bigg) \\
&= \Bigg((\mathbf{u}^2+\mathbf{v}^2)^{2\kappa}(\mathbf{u}^2+e^{-\mathbf{u}})^{2\beta}(v^2+1)^{2\gamma}\left[\frac{(\mathbf{u}^2+1)^2+(\mathbf{v}^2+e^{-\mathbf{u}})^2)}{4}\right]^{1-\kappa-\beta-\gamma}\Bigg) \\
&\geq \psi\Bigg(4^{2\kappa}4^{2\beta}1^{2\gamma}\left[\frac{5^2}{4}\right]^{1-\kappa-\beta-\gamma}\Bigg) \\
&= 3.4^{4.2}5^{-2.2} \geq (1+e^{-2})^2.
\end{aligned} \tag{3.32}$$

Thus

$$\theta(A\mathbf{u}, A\mathbf{v})d_\theta(A\mathbf{u}, A\mathbf{v}) \leq \psi\Bigg([d_\theta(B\mathbf{u}, B\mathbf{v})]^\kappa [d_\theta(B\mathbf{u}, A\mathbf{u})]^\beta [d_\theta(B\mathbf{v}, A\mathbf{v})]^\gamma \cdot \left[\frac{d_\theta(B\mathbf{u}, A\mathbf{v}) + d_\theta(B\mathbf{v}, A\mathbf{u})}{2\theta}\right]^{1-\kappa-\beta-\gamma}\Bigg). \tag{3.33}$$

Case IV: If $\mathbf{u},\mathbf{v}$ in $(2,\infty)$ and $\mathbf{u} \neq \mathbf{v}$, we get

$$d_\theta(A\mathbf{u}, A\mathbf{v}) = (e^{-\mathbf{u}}+e^{-\mathbf{v}})^2 \leq 2e^{-4}, \tag{3.34}$$

$$\begin{aligned}
d_\theta(A\mathbf{u}, A\mathbf{v}) &\leq \theta(A\mathbf{u}, A\mathbf{v})d_\theta(A\mathbf{u}, A\mathbf{v}) \\
&\leq \psi\Bigg([d_\theta(B\mathbf{u}, B\mathbf{v})]^\alpha [d_\theta(B\mathbf{u}, A\mathbf{u})]^\beta [d_\theta(B\mathbf{v}, A\mathbf{v})]^\gamma \\
&\qquad \cdot \left[\frac{d_\theta(B\mathbf{u}, A\mathbf{v}) + d_\theta(B\mathbf{v}, A\mathbf{u})}{2\theta}\right]^{1-\kappa-\beta-\gamma}\Bigg) \\
&= \Bigg((\mathbf{u}^2+\mathbf{v}^2)^{2\alpha}(\mathbf{u}^2+e^{-\mathbf{u}})^{2\beta}(\mathbf{v}^2+e^{-\mathbf{v}})^{2\gamma}\left[\frac{(\mathbf{u}^2+e^{-\mathbf{v}})^2+(\mathbf{v}^2+e^{-\mathbf{u}})^2)}{4}\right]^{1-\kappa-\beta-\gamma}\Bigg) \\
&\geq \psi\Bigg(8^{2\kappa}4^{2\beta}4^{2\gamma}\left[\frac{4^2+4^2}{4}\right]^{1-\kappa-\beta-\gamma}\Bigg) \\
&= \frac{3}{5}2^{9.8} \geq 2e^{-4}.
\end{aligned} \tag{3.35}$$

This implies

$$\theta(A\mathbf{u},A\mathbf{v})d_\theta(A\mathbf{u},A\mathbf{v}) \le \psi\Bigg([d_\theta(B\mathbf{u},B\mathbf{v})]^\kappa[d_\theta(B\mathbf{u},A\mathbf{u})]^\beta[d_\theta(B\mathbf{v},A\mathbf{v})]^\gamma \cdot \left[\frac{d_\theta(B\mathbf{u},A\mathbf{v})+d_\theta(B\mathbf{v},A\mathbf{u})}{2\theta}\right]^{1-\kappa-\beta-\gamma}\Bigg) \tag{3.36}$$

for all $\mathbf{u},\mathbf{v} \in \mathring{\mathbf{A}}-1$, A and B met Definition 3.2. Further 1 is the coincidence point of A and B.

Definition 3.3. Let $(\mathring{\mathbf{A}},d_\theta)$ be an extended b-metric space and $\widehat{\mathbf{G}}=(V(\widehat{\mathbf{G}}),E(\widehat{\mathbf{G}}))$ represents a graph which contains loops. For a self map A on $\mathring{\mathbf{A}}$ is said to be generalized interpolative weakly contractive mapping type Ćirić-Reich-Rus, if it satisfied the following conditions:

(i) There exists $\kappa,\beta \in (0,1)$ such that imply $\kappa,\beta \in E(\mathbf{G})$.

(ii) $\theta(\mathbf{u},\mathbf{v})\zeta(d_\theta(A\mathbf{u},A\mathbf{v})) \le \zeta(S(\mathbf{u},\mathbf{v}))-\phi(S(\mathbf{u},\mathbf{v}))$

for any $\mathbf{u},\mathbf{v} \in \mathring{\mathbf{A}}-\text{Fix}(A)$ imply $(\mathbf{u},\mathbf{v}) \in E(\mathbf{G})$, where $\text{Fix}(A)=\{\mathbf{u}\in\mathring{\mathbf{A}}: A\mathbf{u}=\mathbf{u}\}$.

$S(\mathbf{u},\mathbf{v})=[d_\theta(\mathbf{u},\mathbf{v})]^\kappa[d_\theta(\mathbf{u},A\mathbf{u})]^\beta[d_\theta(\mathbf{v},A\mathbf{v})]^{1-\kappa-\beta}$ and ϕ,ζ are the functions on $[0,\infty)$ which are lower semi continuous and continuous monotone nondecreasing function respectively and $\phi(\mathbf{t})$, $\zeta(\mathbf{t})=0 \Leftrightarrow \mathbf{t}=0$.

Theorem 3.3. *Consider an extended b-metric space $(\mathring{\mathbf{A}},d_\theta)$ which is complete that represents graph $\widehat{\mathbf{G}}=(V(\widehat{\mathbf{G}}),E(\widehat{\mathbf{G}}))$ containing loops. If a self map A on $\mathring{\mathbf{A}}$ satisfied generalized interpolative weakly contractive mapping type Ćirić-Reich-Rus, then A has a fixed point.*

Proof. Choose a point $\mathbf{u_0}$ along with a sequence $\{\mathbf{u_n}\}$ given by $\mathbf{u_0}=\mathbf{u}$ and $\mathbf{u_{n+1}}=A\mathbf{u_n}$ for $\mathbf{n}$ in $\mathbb{N}\bigcup\{0\}$. If there exists $\mathbf{n_0}$ with $\mathbf{u_{n_0}}=\mathbf{u_{n_0+1}}$ then $\mathbf{u_{n_0}}$ is a fixed point of A. Replaces $\mathbf{u_n}=\mathbf{u}$ and $\mathbf{v}=\mathbf{u_{n-1}}$ in (ii) of Definition 3.3, we obtain

$$\zeta(d_\theta(\mathbf{u_{n+1}},\mathbf{u_n})) \le \theta(\mathbf{u_{n+1}},\mathbf{u_n})\zeta(d_\theta(\mathbf{u_{n+1}},\mathbf{u_n})). \tag{3.37}$$

Now

$$\zeta([d_\theta(\mathbf{u_n},\mathbf{u_{n-1}})]^\kappa[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})]^\beta[d_\theta(\mathbf{u_{n-1}},\mathbf{u_n})]^{1-\kappa-\beta}) - \phi([d_\theta(\mathbf{u_n},\mathbf{u_{n-1}})]^\kappa[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})]^\beta[d_\theta(\mathbf{u_{n-1}},\mathbf{u_n})]^{1-\kappa-\beta}).$$

This implies

$$\theta(\mathbf{u_{n+1}},\mathbf{u_n})\zeta(d_\theta(\mathbf{u_{n+1}},\mathbf{u_n})) \le \zeta([d_\theta(\mathbf{u_n},\mathbf{u_{n-1}})]^{1-\kappa}[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})]^\beta) - \phi([d_\theta(\mathbf{u_n},\mathbf{u_{n-1}})]^{1-\kappa}[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})]^\beta). \tag{3.38}$$

Using the axiom of ζ and (3.37), we obtain

$$d_\theta(\mathbf{u_n},\mathbf{u_{n-1}}) \le ([d_\theta(\mathbf{u_n},\mathbf{u_{n-1}})]^{1-\kappa}[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})]^\beta). \tag{3.39}$$

This implies

$$[d_\theta(\mathbf{u_{n+1}},\mathbf{u_n})]^{1-\kappa} \le [d_\theta(\mathbf{u_n},\mathbf{u_{n-1}})]^{1-\kappa}. \tag{3.40}$$

So, we can write

$$[d_\theta(\mathbf{u_{n+1}},\mathbf{u_n})] \le [d_\theta(\mathbf{u_n},\mathbf{u_{n-1}})]^{1-\kappa}, \quad \text{for all } \mathbf{n} \ge 1. \tag{3.41}$$

Thus the positive sequence $\{d_\theta(\mathbf{u_{n+1}},\mathbf{u_n})\}$ is a decreasing sequence consequently there exists $h \ge 0$ such that $\lim\limits_{\mathbf{n}\to\infty} d_\theta(\mathbf{u_{n+1}},\mathbf{u_n}) = h$. As $n \to \infty$ in (3.38) and using (3.37), we get

$$\zeta(h) \le \zeta(h) - \phi(h). \tag{3.42}$$

We conclude that $h = 0$ consequently

$$\lim_{\mathbf{n}\to\infty} d_\theta(\mathbf{u_n},\mathbf{u_{n-1}}) = 0. \tag{3.43}$$

We observe that $\{\mathbf{u_n}\}$ is a Cauchy sequence. If not then there exists a real number $\epsilon \ge 0$ for all $i \in \mathbb{N}$, and $\mathbf{n_i}$, $\mathbf{m_i} \ge \mathbf{i}$ such that

$$d_\theta(\mathbf{u_{m_i}},\mathbf{u_{n_i}}) \ge \epsilon \quad \text{and} \quad d_\theta(\mathbf{u_{m_i-1}},\mathbf{u_{m_i}}) < \epsilon. \tag{3.44}$$

Substituting $\mathbf{u} = \mathbf{u_{n_{i-1}}}$ and $\mathbf{v} = \mathbf{u_{m_{i-1}}}$ imply $(\mathbf{u},\mathbf{v}) \in inE(\widehat{\mathbf{G}})$ (3.35) and using (ii) of Definition 3.3 and using (3.42) we write

$$\theta(\mathbf{u},\mathbf{v})\zeta(\epsilon) \le \theta(\mathbf{u},\mathbf{v})\zeta(d_\theta(A\mathbf{u},A\mathbf{v}) \le \zeta(S(\mathbf{u},\mathbf{v})) - \phi(S(\mathbf{u},\mathbf{v})), \tag{3.45}$$

$$S(\mathbf{u_{m_i-1}},\mathbf{u_{n_i-1}})[d_\theta(\mathbf{u_{m_i-1}},\mathbf{u_{n_i-1}})]^\kappa[d_\theta(\mathbf{u_{m_i-1}},\mathbf{u_{m_i}})]^\beta[d_\theta(\mathbf{u_{n_i-1}},\mathbf{u_{n_i}})]^{1-\kappa-\beta} \tag{3.46}$$

and

$$d_\theta(\mathbf{u_{m_i-1}},\mathbf{u_{n_i-1}}) \le d_\theta(\mathbf{u_{m_i-1}},\mathbf{u_{m_i}}) + d_\theta(\mathbf{u_{n_i}},\mathbf{u_{n_i-1}}) \le \epsilon + d_\theta(\mathbf{u_{n_i-1}},\mathbf{u_{n_i-1}}) \tag{3.47}$$

As $\mathbf{n} \to \infty$ and using (3.43), we obtain

$$\lim_{\mathbf{n}\to\infty} S(\mathbf{u_{m_i-1}},\mathbf{u_{n_i-1}}) = 0. \tag{3.48}$$

This implies

$$\theta(\mathbf{u},\mathbf{v})\zeta(\epsilon) \le \zeta(0) - \phi(0) = 0. \tag{3.49}$$

This leads to a contradiction since $\theta(\mathbf{u},\mathbf{v})\zeta(\epsilon) > 0$. So $\{\mathbf{u_n}\}$ is a cauchy sequence along with $(\mathbf{\mathring{A}}, d_\theta)$ so there exists $\mathbf{t} \in \mathbf{\mathring{A}}$ imply $(\mathbf{t},\mathbf{t})$ in $E(\widehat{\mathbf{G}})$ such that $\lim\limits_{\mathbf{n}\to\infty} d_\theta(\mathbf{u_n},\mathbf{t}) = 0$ and let us assume that $A\mathbf{t} \ne \mathbf{t}$, we obtain

$$\theta(\mathbf{u_n},\mathbf{u})\zeta(d_\theta(\mathbf{u_{n+1}},A\mathbf{t})) \le \zeta(S(\mathbf{u_n},\mathbf{t})) - \phi(S(\mathbf{u_n},\mathbf{t}), \quad \text{for all } \mathbf{n} \tag{3.50}$$

where

$$S(\mathbf{u_n},\mathbf{t}) = [d_\theta(\mathbf{u_n},\mathbf{t})]^\kappa[d_\theta(\mathbf{u_n},\mathbf{u_{n+1}})]^\beta[(d_\theta)(\mathbf{t},A\mathbf{t})]^{1-\kappa-\beta}. \tag{3.51}$$

Using (3.43), we get

$$S(\mathbf{u_n},\mathbf{t}) = 0. \tag{3.52}$$

As $\mathbf{n} \to \infty$ in (3.50), we get

$$\theta(\mathbf{u_n},\mathbf{v})\zeta(d_\theta(\mathbf{n}+\mathbf{1},A\mathbf{t})) \le \zeta(S(\mathbf{n},\mathbf{t})) - \phi(S(\mathbf{n},\mathbf{t})) = 0. \tag{3.53}$$

This leads us to a contradiction so $A\mathbf{t} = \mathbf{t}$. □

Example 3.3. Let $\widehat{\mathbf{G}} = (V(\widehat{\mathbf{G}}), E(\widehat{\mathbf{G}}))$ represents a graph, where $V(\widehat{\mathbf{G}}) = [0,5]$ and $E(\widehat{\mathbf{G}}) = [(\mathbf{u},\mathbf{v}) : \mathbf{u},\mathbf{v} \in [0,5]]$ which contains loops. Consider a set $\mathbf{\mathring{A}} = [0,5]$ and a function $d_\theta : \mathbf{\mathring{A}} \times \mathbf{\mathring{A}} \to [0,\infty)$ defined as:

$$d_\theta(\mathbf{u},\mathbf{v}) = \begin{cases} 0, & \text{if } \mathbf{u} = \mathbf{v}, \\ 5, & \text{if } \mathbf{u},\mathbf{v} \in [0,1], \\ 3, & \text{otherwise} \end{cases}$$

and $\theta : \mathbf{\mathring{A}} \times \mathbf{\mathring{A}} \to [1,\infty) = 1 + \log\left(1 + \frac{\mathbf{u}+\mathbf{v}}{50}\right)$. Then $(\mathbf{\mathring{A}}, d_\theta)$ is a complete extended b-metric space. Define the self map A on $\mathbf{\mathring{A}}$ as:

$$A(\mathbf{u}) = \begin{cases} 0, & \mathbf{u} \in [0,1), \\ 3, & \mathbf{u} \in [1,5). \end{cases}$$

Consider $\mathbf{t}^2 = \zeta(\mathbf{t})$ and $\frac{\mathbf{t}}{3} = \phi(\mathbf{t})$ be two functions defined, for all $\mathbf{t} \in [0,\infty)$, takes $\kappa = 0.4$, $\beta = 0.3$, then the following cases arises:

Case I: If $\mathbf{u} = \mathbf{v}$ or $\mathbf{u},\mathbf{v} \in (0,1)$ or $\mathbf{u},\mathbf{v} \in [1,5] - 3$ imply $(\mathbf{u},\mathbf{v}) \in E(\widehat{\mathbf{G}})$ such that $\mathbf{u} \neq \mathbf{u}$, this is true.

Case II: If $\mathbf{u}$ in $(0,1)$ and $\mathbf{v}$ in $[1,5] - 3$ imply $(\mathbf{u},\mathbf{v}) \in E(\widehat{\mathbf{G}})$ we obtain

$$\zeta(d_\theta(A\mathbf{u}, A\mathbf{v})) = \zeta(d_\theta(0,3)) = \zeta(3) = 9, \tag{3.54}$$

$$\theta(\mathbf{u},\mathbf{v}) \leq 1.079, \tag{3.55}$$

$$\begin{aligned}&\zeta([d_\theta(\mathbf{u},\mathbf{v})]^\kappa [d_\theta(\mathbf{u},A\mathbf{u})]^\beta [d_\theta(\mathbf{v},A\mathbf{v})]^{1-\kappa-\beta})) - \phi([d_\theta(\mathbf{u},\mathbf{v})]^\kappa [d_\theta(\mathbf{u},A\mathbf{u})]^\beta [d_\theta(\mathbf{v},A\mathbf{v})]^{1-\kappa-\beta}) \\ &\quad = \left(\frac{5}{3}\right)^\kappa \left[9.\left(\frac{5}{3}\right)^\kappa - 1\right] \geq 9 = \zeta(3) = \zeta(d_\theta(0,3)).\end{aligned} \tag{3.56}$$

Case III: If $\mathbf{v} \in (0,1)$ and $\mathbf{u} \in [1,5] - 3$ imply $(\mathbf{u},\mathbf{v}) \in E(\widehat{\mathbf{G}})$ we observe

$$\zeta(d_\theta(A\mathbf{u}, A\mathbf{v}) = \zeta(d_\theta(3,0)) = 9, \tag{3.57}$$

$$\begin{aligned}&\zeta([d_\theta(\mathbf{u},\mathbf{v})]^\kappa [d_\theta(\mathbf{u},A\mathbf{u})]^\beta [d_\theta(\mathbf{v},A\mathbf{v})]^{1-\kappa-\beta})) - \phi([d_\theta(\mathbf{u},\mathbf{v})]^\kappa [d_\theta(\mathbf{u},A\mathbf{u})]^\beta [d_\theta(\mathbf{v},A\mathbf{v})]^{1-\kappa-\beta}) \\ &\quad = \left(\frac{3}{5}\right)^{\kappa+\beta} \left[25.\left(\frac{3}{5}\right)^{\kappa+\beta} - \frac{5}{3}\right] \geq 9 = \zeta(3) = \zeta(d_\theta(A\mathbf{u}, A\mathbf{v})).\end{aligned} \tag{3.58}$$

Hence all the cases:

$$\theta(\mathbf{u},\mathbf{v})\zeta(d_\theta(A\mathbf{u},A\mathbf{v})) \leq \zeta(S(\mathbf{u},\mathbf{v})) - \phi(S(\mathbf{u},\mathbf{v})), \tag{3.59}$$

$$\text{for any } \mathbf{u},\mathbf{v} \in \mathbf{\mathring{A}} - [0,5] \text{ imply } (\mathbf{u},\mathbf{v}) \in E(\mathbf{G}). \tag{3.60}$$

Thus, we observe A has two fixed points, 0 and 3.

Corollary 3.2. *Let $\widehat{\mathbf{G}} = (V(\widehat{\mathbf{G}}), E(\widehat{\mathbf{G}}))$ be a graph containing loops and $(\mathbf{\mathring{A}}, d_\theta)$ is complete extended b-metric space. Consider a map A on $\mathbf{\mathring{A}}$ that meets the following requirements:*

(i) *For κ,β in $(0,1)$ imply $\kappa,\beta \in E(\widehat{\mathbf{G}})$.*

(ii) $\theta(\mathbf{u},\mathbf{v}) d_\theta(A\mathbf{u},A\mathbf{v})$
$\leq [d_\theta(\mathbf{u},\mathbf{v})]^\kappa [d_\theta(\mathbf{u},A\mathbf{u})]^\beta [d_\theta(\mathbf{v},A\mathbf{v}]^{1-\kappa-\beta} - \phi([d_\theta(\mathbf{u},\mathbf{v})]^\kappa [d_\theta(\mathbf{u},A\mathbf{u})]^\beta [d_\theta(\mathbf{v},A\mathbf{v}]^{1-\kappa-\beta}$,
for all $\mathbf{v},\mathbf{u} \in \mathbf{\mathring{A}}$ imply $(\mathbf{u},\mathbf{v})$ in $E(\widehat{\mathbf{G}})$ and $A\mathbf{u} \neq \mathbf{u}$, $A\mathbf{v} \neq \mathbf{v}$, where ϕ is same as Theorem 3.3 then A possesses the fixed point.

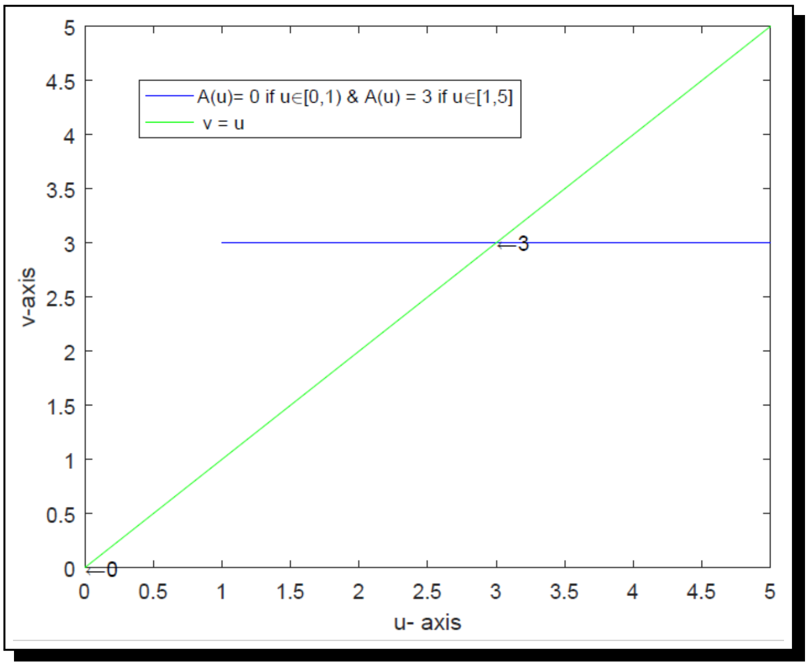

Figure 2. Graph of a fixed point for A

Using the $\phi(\mathbf{t}) = (1-\mu)\mathbf{t}$ for $\kappa, \beta \in (0,1)$ in Corollary 3.2 we get the following corollary:

Corollary 3.3. *Let $\widehat{\mathbf{G}} = (V(\widehat{\mathbf{G}}), E(\widehat{\mathbf{G}}))$ be a graph containing loops and $(\mathring{\mathbf{A}}, d_\theta)$ is complete extended b-metric space. Consider a map A on $\mathring{\mathbf{A}}$ that meets the following requirements:*

(i) *For $\kappa, \beta \in (0,1)$ imply $\kappa, \beta \in E(\widehat{\mathbf{G}})$.*

(ii) $\theta(\mathbf{u},\mathbf{v}) d_\theta(A\mathbf{u}, A\mathbf{v}) \le \mu[d_\theta(\mathbf{u},\mathbf{v})]^\kappa [d_\theta(\mathbf{u}, A\mathbf{u})]^\beta [d_\theta(\mathbf{v}, A\mathbf{v})]^{1-\kappa-\beta}$,
for all $\mathbf{u},\mathbf{v} \in \mathring{\mathbf{A}}$ imply $(\mathbf{u},\mathbf{v}) \in E(\widehat{\mathbf{G}})$ and $A\mathbf{u} \ne \mathbf{u}$, $A\mathbf{v} \ne \mathbf{v}$.

4. Conclusion

In this paper, we generalized the concepts of Errai *et al.* [16] in the framework of complete extended b-metric space endowed with a graph. We also showed a variety of cases are connected to our findings. Our findings are critical in the existing literature on fixed point theory.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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