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Research Article

A New Special Function with Applications to Quantum Mechanics

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Abstract. In this paper, a special function using the particular Sturm-Liouville Equation is introduced. Sequences of recurrence relations are presented for that special function. Also, its series and definite integral representations are provided. This special function is a radial solution to the Laplace equation in the 5-dimensional hyperspherical coordinate system. It is also a solution to the 4-dimensional Radial Schrödinger equation. All results are verified numerically using different mathematical softwares.

Keywords. Sturm-Liouville differential equation, Bessel Function, Special Function, Laplace equation, Schrödinger equation

Mathematics Subject Classification (2020). 33E20, 33E30, 34A25, 34B24, 35J05, 35J10

1. Introduction

The special functions have huge applications in the field of *Nanotechnology*. Some of the special functions viz. gamma functions are not expressible in the series form. Many of the special functions expressible in the series form are derived from the Bessel's equations (Andrews and Shivamogg [3], and Cicchetti and Faraone [6]) of different orders. In this work, a special function as a solution of Sturm-Liouville differential equation is introduced that is a radial solution to the Laplace equation in the 5-dimensional hyperspherical coordinate system and a solution to 4-dimensional radial Schrödinger equation.

2. Preliminaries

Bessel's equation (Jirari [7]) is obtained from the Sturm-Liouville differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[\lambda w(x)-q(x)]y=0, \quad p(x)>0, w(x)>0 \tag{1}$$

by taking

$$p(x)=x,\ q(x)=-x,\ w(x)=\frac{1}{x},\ \lambda=-n^2.$$

For a real number $n \notin \{-1,-2,-3,\dots\}$, the Bessel function of the first kind of order n is defined by

$$J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\cdot\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}. \tag{2}$$

3. Introduction to Special Functions and Recurrences

If we take

$$p(x)=x^3,\ q(x)=-x^3,\ \lambda=1-n^2,\ w(x)=x; \quad x>0,$$

then the Sturm-Liouville differential equation (1) becomes

$$\frac{d}{dx}\left[x^3\frac{dy}{dx}\right]+[(1-n^2)x+x^3]y=0, \quad x>0. \tag{3}$$

The simplified form of this equation is

$$\frac{d^2y}{dx^2}+\frac{3}{x}\frac{dy}{dx}+\left(1-\frac{n^2-1}{x^2}\right)y=0, \quad x>0. \tag{4}$$

The proposed special function is the solution of the Sturm-Liouville differential equation (4).

The series form of a new special function of order n is obtained by Frobenius' method as follows:

Let

$$y(x)=\sum_{m=0}^{\infty}a_m x^m$$

be the series solution of equation (4). Then $y(x)$ and its derivatives

$$\frac{dy}{dx}=\sum_{m=1}^{\infty}m a_m x^{m-1}$$

and

$$\frac{d^2y}{dx^2}=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}$$

satisfy the differential equation (4). Hence, we have the series solution of differential equation (4) as

$$C_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m n!}{4^m\cdot m!\cdot(m+n)!}x^{2m+n-1}, \quad n=0,1,2,3,\dots.$$

This is the new special function of order n and the equation (4) is the new special equation of order n associated with it. So, the new special equation of order n can be written as

$$C_n''(x)+\frac{3}{x}C_n'(x)+\left(1-\frac{n^2-1}{x^2}\right)C_n(x)=0, \quad x>0. \tag{5}$$

Theorem 3.1. *The general formula for new special function of real order n such that $n \notin \{-1,-2,-3,\dots\}$ is given by*

$$C_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-1},\quad x\neq 0. \tag{6}$$

Proof. Differentiating equation (6) with respect to 'x', we get

$$C_n'(x)=\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n-1)\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-2}, \tag{7}$$

$$C_n''(x)=\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n-1)(2m+n-2)\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-3}$$

$$\begin{aligned}\Rightarrow\quad \frac{d^2C_n}{dx^2}+\frac{3}{x}\frac{dC_n}{dx}+\left(1-\frac{n^2-1}{x^2}\right)C_n&=\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}[(2m+n-1)(2m+n-2)\\&\quad+3(2m+n-1)-(n^2-1)]x^{2m+n-3}\\&\quad+\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-1}\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad \frac{d^2C_n}{dx^2}+\frac{3}{x}\frac{dC_n}{dx}+\left(1-\frac{n^2-1}{x^2}\right)C_n&=\sum_{m=1}^{\infty}\frac{(-1)^m4m(m+n)\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-3}\\&\quad+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\Gamma(n+1)}{4^{m-1}\cdot(m-1)!\cdot\Gamma(m-1+n+1)}x^{2(m-1)+n-1}\\&=0.\end{aligned}$$

Thus, equation (6) is the general solution of the differential equation (5).

This proves that the general formula for new special function of real order n; $n\notin\{-1,-2,-3,\dots\}$ is

$$C_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-1}.$$

□

Theorem 3.2. *The new special functions satisfy the following differential recurrence relations:*

$$C_n'(x)+\frac{n+1}{x}C_n(x)=2nC_{n-1}(x), \tag{8}$$

$$C_n'(x)-\frac{n-1}{x}C_n(x)=-\frac{1}{2(n+1)}C_{n+1}(x). \tag{9}$$

Proof. By using the formulae in equations (6) and (7), we have

$$\begin{aligned}C_n'(x)+\frac{n+1}{x}C_n(x)&=\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n-1)\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-2}\\&\quad+\sum_{m=0}^{\infty}\frac{(-1)^m(n+1)\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-2}\\&=\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n-1+n+1)\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-2}\\&=\sum_{m=0}^{\infty}\frac{(-1)^m\cdot 2(m+n)\cdot n\Gamma(n)}{4^m\cdot m!\cdot(m+n)\Gamma(m+n)}x^{2m+n-2}\end{aligned}$$

$$\Rightarrow \qquad C_n'(x) + \frac{n+1}{x}C_n(x) = 2nC_{n-1}(x).$$

This proves the recurrence relation (8).

Similarly, by using the formulae in equations (6) and (7), we have

$$\begin{aligned}
C_n'(x) - \frac{n-1}{x}C_n(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m(2m+n-1)\Gamma(n+1)}{4^m \cdot m! \cdot \Gamma(m+n+1)} x^{2m+n-2} \\
&\quad + \sum_{m=0}^{\infty} \frac{(-1)^m(-n+1)\Gamma(n+1)}{4^m \cdot m! \cdot \Gamma(m+n+1)} x^{2m+n-2} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m(2m+n-1-n+1)\Gamma(n+1)}{4^m \cdot m! \cdot \Gamma(m+n+1)} x^{2m+n-2} \\
&= 2\sum_{m=1}^{\infty} \frac{(-1)^m \cdot (m)\Gamma(n+1)}{4^m \cdot m! \cdot \Gamma(m+n+1)} x^{2m+n-2} \\
&= \frac{1}{2}\sum_{m=0}^{\infty} \frac{(-1)^{m+1} \cdot \Gamma(n+1)}{4^m \cdot (m)! \cdot \Gamma(m+n+2)} x^{2m+2+n-2} \\
&= \frac{-1}{2(n+1)}\sum_{m=0}^{\infty} \frac{(-1)^{m} \cdot \Gamma(n+2)}{4^m \cdot (m)! \cdot \Gamma(m+n+2)} x^{2m+(n+1)-1} \\
&= \frac{-1}{2(n+1)} C_{n+1}(x).
\end{aligned}$$

This proves the recurrence relation (9). □

Corollary 3.2.1. *The new special functions satisfy the following recurrence relation:*

$$2nC_{n-1}(x) + \frac{1}{2(n+1)}C_{n+1}(x) = \frac{2n}{x}C_n(x).$$

Proof. If we subtract equation (9) from equation (8), we get

$$\left(\frac{n+1}{x} + \frac{n-1}{x}\right)C_n(x) = 2nC_{n-1}(x) + \frac{1}{2(n+1)}C_{n+1}(x)$$

$$\Rightarrow \qquad \frac{2n}{x}C_n(x) = 2nC_{n-1}(x) + \frac{1}{2(n+1)}C_{n+1}(x).$$ □

Corollary 3.2.2. *The new special functions satisfy the following differential recurrence relation:*

$$C_n'(x) = (n-1)C_{n-1}(x) - \frac{1}{4n}C_{n+1}(x).$$

Proof. From equations (8) and (9), we get

$$(n-1)C_n'(x) + \frac{(n-1)(n+1)}{x}C_n(x) = 2n(n-1)C_{n-1}(x), \tag{10}$$

$$(n+1)C_n'(x) - \frac{(n-1)(n+1)}{x}C_n(x) = -\frac{1}{2}C_{n+1}(x). \tag{11}$$

Add equations (10) and (11), we get

$$2nC_n'(x) = 2n(n-1)C_{n-1}(x) - \frac{1}{2}C_{n+1}(x)$$

$$\Rightarrow \qquad C_n'(x) = (n-1)C_{n-1}(x) - \frac{1}{4n}C_{n+1}(x).$$ □

Remark 3.1. Equations (8) and (9) are equivalent to equation (5) because equation (8) can be expressed as

$$C_n(x) = \frac{1}{2(n+1)}C'_{n+1}(x) + \frac{n+2}{2x(n+1)}C_{n+1}(x) \tag{12}$$

and by substituting this equation into equation (9), we get

$$\frac{1}{2(n+1)}C''_{n+1}(x) + \frac{n+2}{2x(n+1)}C'_{n+1}(x) - \frac{n+2}{2x^2(n+1)}C_{n+1}(x) - \frac{n-1}{2x(n+1)}C'_{n+1}(x)$$
$$- \frac{(n+2)(n-1)}{2x^2(n+1)}C_{n+1}(x) = -\frac{1}{2(n+1)}C_{n+1}(x)$$
$$\Rightarrow \quad \frac{1}{2(n+1)}C''_{n+1}(x) + \frac{3}{2x(n+1)}C'_{n+1}(x) - \frac{n(n+2)}{2x^2(n+1)}C_{n+1}(x) = -\frac{1}{2(n+1)}C_{n+1}(x)$$
$$\Rightarrow \quad C''_{n+1}(x) + \frac{3}{x}C'_{n+1}(x) + \left(1 - \frac{(n+1)^2-1}{x^2}\right)C_{n+1}(x) = 0$$

This is the new special equation (5) of order $n+1$. Similarly, equation (9) can be expressed as

$$C_n(x) = \frac{2n(n-2)}{x}C_{n-1}(x) - 2nC'_{n-1}(x) \tag{13}$$

and by substituting this equation into equation (8), we get the new special equation (5) of order $n-1$.

Recurrences (8) and (9) are the first-order linear non homogeneous differential equations which can be solved by using their integrating factors, respectively as below:

$$(x^{n+1}C_n)' = 2nx^{n+1}C_{n-1}, \tag{14}$$

$$(x^{-(n-1)}C_n)' = -\frac{1}{2(n+1)}x^{-(n-1)}C_{n+1}. \tag{15}$$

Now, multiply both sides of equations (8) and (9) by x^n and x^{-n}, respectively and differentiate both sides of two equations, respectively we get

$$\frac{1}{2n}[x^nC'_n + (n+1)x^{n-1}C_n]' = (x^nC_{n-1})', \tag{16}$$

$$-2(n+1)[x^{-n}C'_n - (n-1)x^{-(n+1)}C_n]' = (x^{-n}C_{n+1})'. \tag{17}$$

Substituting the values of $(x^nC_{n-1})'$ and $(x^{-n}C_{n+1})'$ in the equations (16) and (17) from the equations (14) and (15) respectively, and simplifying them, we obtain

$$C''_n + \frac{2n+1}{x}C'_n + \frac{(n+1)(n-1)}{x^2}C_n = 4n(n-1)C_{n-2}, \tag{18}$$

$$C''_n - \frac{2n-1}{x}C'_n + \frac{(n-1)(n+1)}{x^2}C_n = \frac{1}{4(n+1)(n+2)}C_{n+2}. \tag{19}$$

Subtracting the equation (5) from equations (18) and (19) to eliminate C''_n from them, we get the new recurrences as

$$C'_n + \left(\frac{n+1}{x} - \frac{x}{2(n-1)}\right)C_n = 2nxC_{n-2}, \tag{20}$$

$$C'_n + \left(\frac{x}{2(n+1)} - \frac{n-1}{x}\right)C_n = -\frac{x}{8(n+1)^2(n+2)}C_{n+2}. \tag{21}$$

Multiply both sides of equations (20) and (21) by x^{n-1} and $x^{-(n-1)}$, respectively. Then differentiate them and subtract the equation (5) from them to eliminate C''_n, we get the new recurrences respectively as,

$$C_n' + \left[\frac{n+1}{x} - \frac{2(n-2)x}{4(n-1)(n-2)-x^2}\right]C_n = \frac{8n(n-1)(n-2)x^2}{4(n-1)(n-2)-x^2}C_{n-3},$$

$$C_n' + \left[\frac{-2(n+2)x}{x^2-4(n+1)(n+2)} - \frac{n-1}{x}\right]C_n = \frac{x^2}{8\left[x^2-4(n+1)(n+2)\right](n+1)(n+2)(n+3)}C_{n+3}.$$

Theorem 3.3. *For a real number $n \notin \{-1,-2,-3,\dots\}$, if $J_n(x)$ is the Bessel function* (*Bell* [5]) *of first kind, then*

$$C_n(x) = 2^n\Gamma(n+1)\frac{J_n(x)}{x}. \tag{22}$$

Proof. For a real number $n \notin \{-1,-2,-3,\dots\}$, the Bessel function of the first kind of order n is defined by

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\cdot\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n} \tag{23}$$

$$\Rightarrow \qquad 2^n\Gamma(n+1)\frac{J_n(x)}{x} = 2^n\Gamma(n+1)\frac{1}{x}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\cdot\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$$

$$\Rightarrow \qquad 2^n\Gamma(n+1)\frac{J_n(x)}{x} = \frac{1}{x}\sum_{m=0}^{\infty}\frac{(-1)^m 2^n\Gamma(n+1)}{m!\cdot\Gamma(m+n+1)}\frac{x^{2m+n}}{2^{2m+n}}$$

$$\Rightarrow \qquad 2^n\Gamma(n+1)\frac{J_n(x)}{x} = \sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(n+1)}{4^m\cdot m!\cdot\Gamma(m+n+1)}x^{2m+n-1}$$

This proves that

$$C_n(x) = 2^n\Gamma(n+1)\frac{J_n(x)}{x}, \quad n \notin \{-1,-2,-3,\dots\}. \qquad \square$$

Theorem 3.4. *The general solution of a differential equation* (4) *defined by*

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \left(1 - \frac{n^2-1}{x^2}\right)y = 0, \quad x > 0$$

is

$$y(x) = C_n(x) = \frac{c_1 J_n(x)}{x} + \frac{c_2 Y_n(x)}{x}, \tag{24}$$

where n is not an integer.

Proof. The general solution of equation

$$xu'' + u' + \left(\mu^2 x - \frac{n^2}{x}\right)u = 0$$

is given by (Bell [5])

$$u(x) = c_n J_n(\mu x) + d_n Y_n(\mu x).$$

Hence, $u(x) = c_1 J_n(x) + c_2 Y_n(x)$ is the general solution of an equation

$$xu'' + u' + \left(x - \frac{n^2}{x}\right)u = 0.$$

Let

$$y(x) = \frac{u(x)}{x},$$

then

$$\begin{aligned}
\frac{d^2y}{dx^2}+\frac{3}{x}\frac{dy}{dx}+\left(1-\frac{n^2-1}{x^2}\right)y &= \frac{d^2}{dx^2}\left[\frac{u(x)}{x}\right]+\frac{3}{x}\frac{d}{dx}\left[\frac{u(x)}{x}\right]+\left(1-\frac{n^2-1}{x^2}\right)\frac{u(x)}{x}\\
&= \frac{x^3u''(x)-2x^2u'(x)+2xu(x)}{x^4}+\frac{3}{x}\frac{xu'(x)-u(x)}{x^2}+\left(1-\frac{n^2-1}{x^2}\right)\frac{u(x)}{x}\\
&= \frac{u''(x)}{x}-\frac{2u'(x)}{x^2}+\frac{2u(x)}{x^3}+\frac{3u'(x)}{x^2}-\frac{3u(x)}{x^3}+\frac{u(x)}{x}-\frac{n^2u(x)}{x^3}+\frac{u(x)}{x^3}\\
&= \frac{u''(x)}{x}+\frac{u'(x)}{x^2}+\frac{u(x)}{x}-\frac{n^2u(x)}{x^3}\\
&= \left[xu''+u'+\left(x-\frac{n^2}{x}\right)u\right]\frac{1}{x^2}=0\left(\frac{1}{x^2}\right)=0
\end{aligned}$$

$$\Rightarrow\quad \frac{d^2y}{dx^2}+\frac{3}{x}\frac{dy}{dx}+\left(1-\frac{n^2-1}{x^2}\right)y=0.$$

Hence, it is proved that

$$y(x)=C_n(x)=\frac{c_1J_n(x)}{x}+\frac{c_2Y_n(x)}{x},$$

is a general solution of differential equation (4). □

Theorem 3.5. *The new special function have an integral representation*

$$C_n(x)=\frac{2^n\Gamma(n+1)}{\pi x}\int_0^\pi \cos(nt-x\sin t)\,dt. \tag{25}$$

Proof. The integral representation of the Bessel function [5] is given by

$$J_n(x)=\frac{1}{\pi}\int_0^\pi \cos(nt-x\sin t)\,dt.$$

Hence by relation (22), we get the integral representation (25).

In the similar way, we get the following other integral representations of new special functions:

$$C_n(x)=\frac{2^n\Gamma(n+1)}{\pi x i^n}\int_0^\pi e^{ix\cos t}\cos(nt)\,dt=\frac{2^n\Gamma(n+1)}{2\pi x}\int_{-\pi}^{\pi} e^{i(n(-\frac{\pi}{2}+t)+x\cos t)}\,dt.$$ □

4. Behaviour of New Special Function

From equation (6), we note that $C_n(x)$ is an even function of x when n is odd, and odd when n is even. It has been verified by the graphs of $C_n(x)$ indicated in Figure 1 to Figure 7, respectively. The graphs have been obtained using equation (22) with the help of MATLAB software.

5. Applications of New Special Function

The separation of variables method is very important method to solve the problem of definite solution of partial differential equations, which is generally used in various definite solution problems. New special functions make it easier to solve the Laplace equation in 5-dimensional hyperspherical coordinate system (Andreev and Tsipenyuk [2], and Avery and Avery [4]). We also find the solution of 4-dimensional radial Schrödinger equation for two particles (Abu-Shady *et al*. [1]) in terms of new special function with the *New Cornell Potential*.

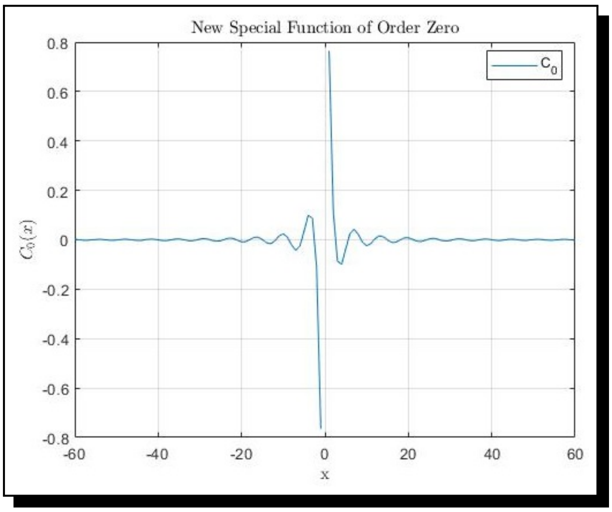

Figure 1. New special function of order zero

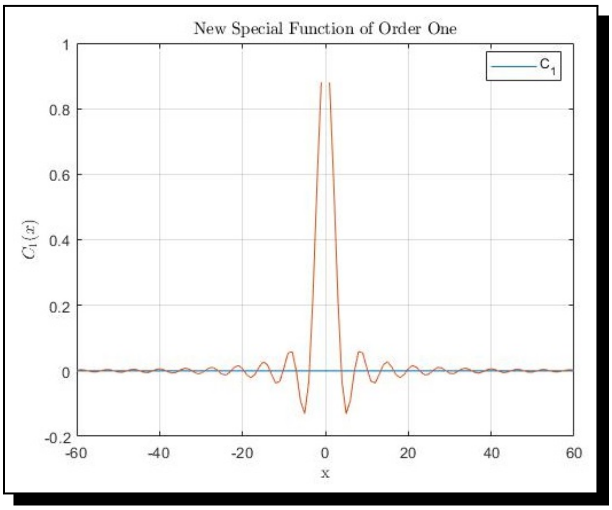

Figure 2. New special function of order one

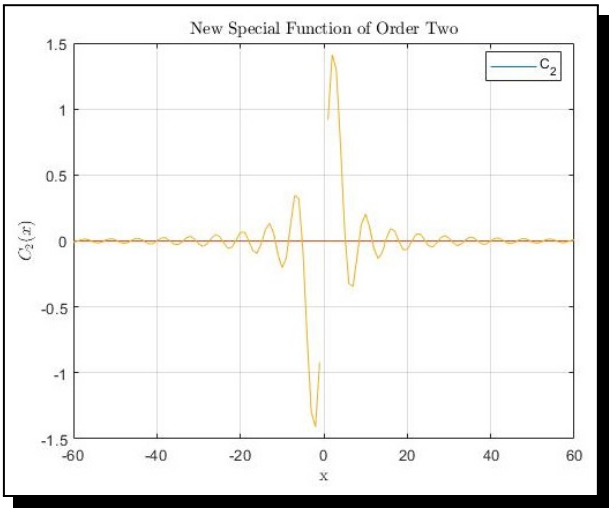

Figure 3. New special function of order two

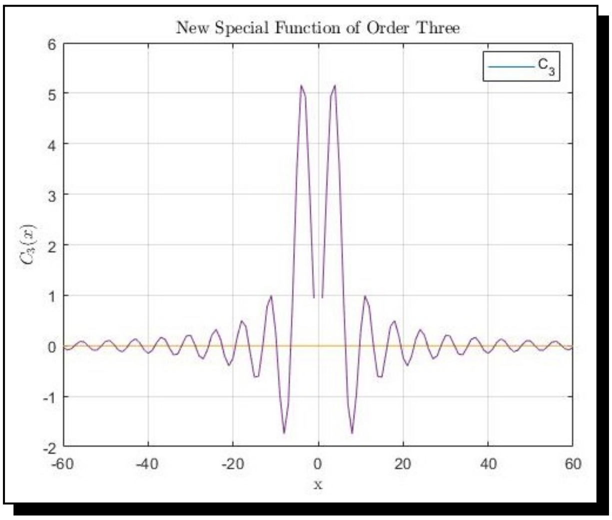

Figure 4. New special function of order three

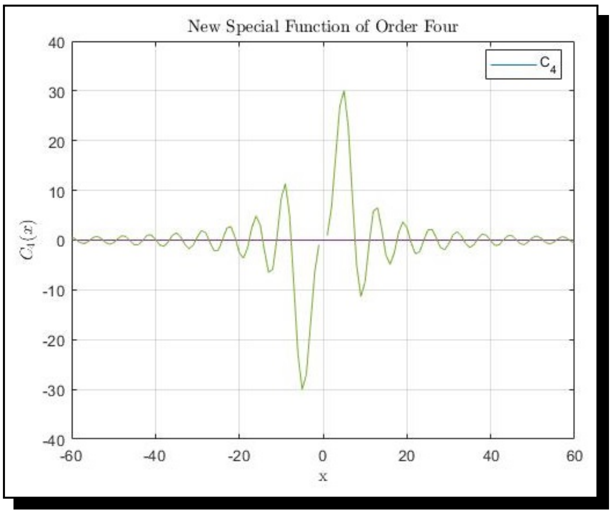

Figure 5. New special function of order four

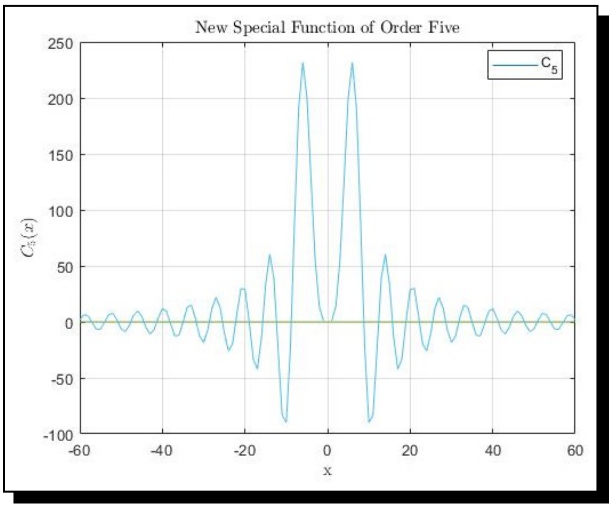

Figure 6. New special function of order five

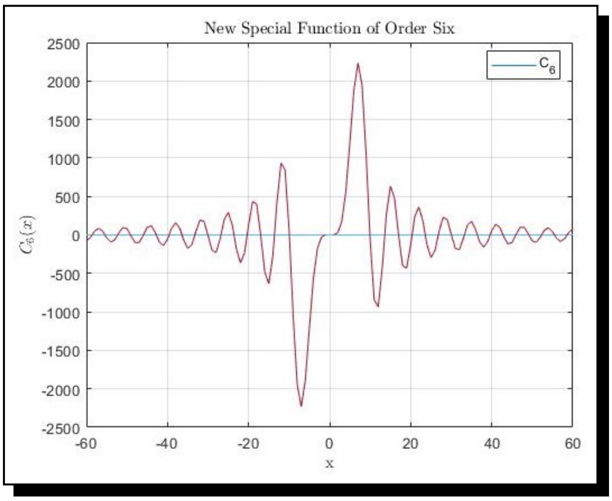

Figure 7. New special function of order six

5.1 Laplace's Equation

In $\mathbb{R}^5$, the Laplace's equation is the second order partial differential equation (Avery and Avery [4])

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial w^2}+\frac{\partial^2 u}{\partial t^2}=0. \tag{26}$$

The hyperspherical coordinates (r,θ,ϕ,ψ,t), defined by

$$x=r\sin\theta\sin\phi\cos\psi,\ y=r\sin\theta\sin\phi\sin\psi,\ z=r\sin\theta\cos\phi,\ w=r\cos\theta,\ t=t.$$

where $r\geq 0$, $-\pi<\theta\leq\pi$, $0\leq\phi<\pi$, $-\frac{\pi}{4}\leq\psi<\frac{\pi}{4}$ equation (26) is transformed into

$$\frac{\partial^2 u}{\partial r^2}+\frac{3}{r}\frac{\partial u}{\partial r}+\frac{2\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\cos\phi}{r^2\sin^2\theta\sin\phi}\frac{\partial u}{\partial\phi}$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}+\frac{1}{r^2\sin^2\theta\sin^2\phi}\frac{\partial^2 u}{\partial\psi^2}+\frac{\partial^2 u}{\partial t^2}=0. \tag{27}$$

We use separation of variables method to solve equation (27), so we assume that $u(r,\theta,\phi,\psi,t)$ is a product of a function of r, function of θ, function of ϕ and a function of ψ as

$$u(r,\theta,\phi,\psi,t)=R(r)\Theta(\theta)\Phi(\phi)\Psi(\psi)T(t).$$

After the appropriate differentiation, we obtained the following equations:

$$\frac{\partial u}{\partial r}=\Theta\Phi\Psi T\frac{dR}{dr},\quad \frac{\partial^2 u}{\partial r^2}=\Theta\Phi\Psi T\frac{d^2R}{dr^2},$$
$$\frac{\partial u}{\partial\theta}=R\Phi\Psi T\frac{d\Theta}{d\theta},\quad \frac{\partial^2 u}{\partial\theta^2}=R\Phi\Psi T\frac{d^2\Theta}{d\theta^2},$$
$$\frac{\partial u}{\partial\phi}=R\Theta\Psi T\frac{d\Phi}{d\phi},\quad \frac{\partial^2 u}{\partial\phi^2}=R\Theta\Psi T\frac{d^2\Phi}{d\phi^2},$$
$$\frac{\partial u}{\partial\psi}=R\Theta\Phi T\frac{d\Psi}{d\psi},\quad \frac{\partial^2 u}{\partial\psi^2}=R\Theta\Phi T\frac{d^2\Psi}{d\psi^2},$$
$$\frac{\partial u}{\partial t}=R\Theta\Phi\Psi\frac{dT}{dt},\quad \frac{\partial^2 u}{\partial t^2}=R\Theta\Phi\Psi\frac{d^2T}{dt^2}.$$

Substituting these derivatives into the equation (27), gives the result as

$$\Theta\Phi\Psi T\frac{d^2R}{dr^2}+\frac{3}{r}\Theta\Phi\Psi T\frac{dR}{dr}+\frac{2\cos\theta}{r^2\sin\theta}R\Phi\Psi T\frac{d\Theta}{d\theta}+\frac{1}{r^2}R\Phi\Psi T\frac{d^2\Theta}{d\theta^2}+\frac{\cos\phi}{r^2\sin^2\theta\sin\phi}R\Theta\Psi T\frac{d\Phi}{d\phi}$$
$$+\frac{1}{r^2\sin^2\theta}R\Theta\Psi T\frac{d^2\Phi}{d\phi^2}+\frac{1}{r^2\sin^2\theta\sin^2\phi}R\Theta\Phi T\frac{d^2\Psi}{d\psi^2}+R\Theta\Phi\Psi\frac{d^2T}{dt^2}=0.$$

Assuming $R(r)\Theta(\theta)\Phi(\phi)\Psi(\psi)T(t)\neq 0$ and dividing the above equation by $R\Theta\Phi\Psi T$, we get

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{3}{r}\frac{1}{R}\frac{dR}{dr}+\frac{2\cos\theta}{r^2\sin\theta}\frac{1}{\Theta}\frac{d\Theta}{d\theta}+\frac{1}{r^2}\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{\cos\phi}{r^2\sin^2\theta\sin\phi}\frac{1}{\Phi}\frac{d\Phi}{d\phi}+\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$
$$+\frac{1}{r^2\sin^2\theta\sin^2\phi}\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2}+\frac{1}{T}\frac{d^2T}{dt^2}=0.$$

This implies

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{3}{r}\frac{1}{R}\frac{dR}{dr}+\frac{2\cos\theta}{r^2\sin\theta}\frac{1}{\Theta}\frac{d\Theta}{d\theta}+\frac{1}{r^2}\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{\cos\phi}{r^2\sin^2\theta\sin\phi}\frac{1}{\Phi}\frac{d\Phi}{d\phi}$$
$$+\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+\frac{1}{r^2\sin^2\theta\sin^2\phi}\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2}=-\frac{1}{T}\frac{d^2T}{dt^2}. \tag{28}$$

The left hand side of this equation depends on r,θ,ϕ and ψ, while the right hand side depends on t. Assuming equation (28) holds over a domain Ω (open and connected set) of the $r\theta\phi\psi t$-space, each side must be a constant, which we denote $-a^2$. Thus, we obtain the pair of equations

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{3}{r}\frac{1}{R}\frac{dR}{dr}+\frac{2\cos\theta}{r^2\sin\theta}\frac{1}{\Theta}\frac{d\Theta}{d\theta}+\frac{1}{r^2}\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{\cos\phi}{r^2\sin^2\theta\sin\phi}\frac{1}{\Phi}\frac{d\Phi}{d\phi}$$
$$+\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+\frac{1}{r^2\sin^2\theta\sin^2\phi}\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2}=-a^2,$$
$$-\frac{1}{T}\frac{d^2T}{dt^2}=-a^2.$$

The second equation results into the differential equation

$$T''-a^2T=0.$$

While from the first equation, we have

$$\frac{r^2\sin^2\theta\sin^2\phi}{R}\frac{d^2R}{dr^2}+\frac{3r\sin^2\theta\sin^2\phi}{R}\frac{dR}{dr}+\frac{2\sin\theta\sin^2\phi\cos\theta}{\Theta}\frac{d\Theta}{d\theta}+\frac{\sin^2\theta\sin^2\phi}{\Theta}\frac{d^2\Theta}{d\theta^2}$$
$$+\frac{\sin\phi\cos\phi}{\Phi}\frac{d\Phi}{d\phi}+\frac{\sin^2\phi}{\Phi}\frac{d^2\Phi}{d\phi^2}+a^2r^2\sin^2\theta\sin^2\phi=-\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2}. \tag{29}$$

Similarly as above, by assuming each side in the equation (29) equal to a constant b^2, it results into the following two equations:

$$\Psi''+b^2\Psi=0,$$
$$\frac{r^2\sin^2\theta}{R}\frac{d^2R}{dr^2}+\frac{3r\sin^2\theta}{R}\frac{dR}{dr}+\frac{2\sin\theta\cos\theta}{\Theta}\frac{d\Theta}{d\theta}+\frac{\sin^2\theta}{\Theta}\frac{d^2\Theta}{d\theta^2}+a^2r^2\sin^2\theta$$
$$=\frac{b^2}{\sin^2\phi}-\frac{\cos\phi}{\Phi\sin\phi}\frac{d\Phi}{d\phi}-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}. \tag{30}$$

Now by assuming each side in the equation (30) equal to a constant c^2, we get the following two equations:

$$\Phi'' + (\cot\phi)\Phi' + (c^2 - b^2\csc^2\phi)\Phi = 0,$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{3r}{R}\frac{dR}{dr} + a^2r^2 = c^2\csc^2\theta - \frac{2\cos\theta}{\Theta\sin\theta}\frac{d\Theta}{d\theta} - \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}. \tag{31}$$

Now by assuming each side in the equation (31) equal to a constant d^2, we get the following two equations:

$$\Theta'' + 2(\cot\theta)\Theta' + (d^2 - c^2\csc^2\theta)\Theta = 0,$$

$$R'' + \frac{3}{r}R' + \left(a^2 - \frac{d^2}{r^2}\right)R = 0.$$

Thus, the Laplace's equation (26) decomposes into

$$\begin{cases} T'' - a^2T = 0, \\ \Psi'' + b^2\Psi = 0, \\ \Phi'' + (\cot\phi)\Phi' + (c^2 - b^2\csc^2\phi)\Phi = 0, \\ \Theta'' + 2(\cot\theta)\Theta' + (d^2 - c^2\csc^2\theta)\Theta = 0, \\ R'' + \frac{3}{r}R' + \left(a^2 - \frac{d^2}{r^2}\right)R = 0. \end{cases} .$$

5.2 Solution of the 4-Dimensional Radial Schrödinger Equation in Terms of New Special Function With the New Cornell Potential

The 4-dimensional radial Schrödinger equation (Abu-Shady *et al.* [1]) for two particles interacting via symmetric potential has the form

$$\left[\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} - \frac{l(l+n-2)}{r^2} + 2\mu(E - V(r))\right]\Psi(r) = 0. \tag{32}$$

where $n = 4$ is the angular quantum number and

$$\mu = \frac{m_1m_2}{m_1 + m_2}$$

is the reduced mass of the two particles having masses m_1 and m_2. If we consider the Cornell potential

$$V(r) = E - \frac{1}{2\mu},$$

the equation (32) reduced to the equation

$$\frac{d^2\Psi}{dr^2} + \frac{3}{r}\frac{d\Psi}{dr} + \left(1 - \frac{(l+1)^2 - 1}{r^2}\right)\Psi(r) = 0.$$

This is in the form of equation (4). So, its solution is

$$\begin{aligned} \Psi(r) &= C_{l+1} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m\Gamma(l+2)}{4^m \cdot m! \cdot \Gamma(m+l+2)} r^{2m+l} \\ &= \frac{c_1J_{l+1}(r)}{r} + \frac{c_2Y_{l+1}(r)}{r}. \end{aligned}$$

Thus, the 4-dimensional radial Schrödinger equation for two particles is reduced to equation (4), when

$$V(r) = E - \frac{m_1 + m_2}{2m_1 m_2}$$

or

$$E = V(r) + \frac{m_1 + m_2}{2m_1 m_2}$$

is an energy eigenvalue (Abu-Shady *et al*. [1]).

6. Conclusions

The aim of this paper was to introduce a special functions that is a radial solution to the Laplace equation in the 5-dimensional hyperspherical coordinate system and a solution to 4-dimensional Radial Schrödinger equation. To achieve this goal, we make use of the Sturm-Liouville problem formed by assigning particular coefficient functions in the Sturm-Liouville differential equation. We derive the different properties of this function. The behaviour of this function for different integral values has been obtained using MATLABE software.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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