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Research Article

Proximal Point Algorithm Based on AP Iterative Technique for Nonexpansive Mappings in CAT(0) Spaces

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Abstract. In this paper, we introduce the modified proximal point algorithm for solving minimization problems in CAT(0) spaces. We then show that the sequence converges to a common fixed point of nonexpansive mapping and a minimizer of a convex function. Finally, we present a numerical illustration for supporting our main result. The findings in this paper are a generalization of certain corresponding results given by some authors.

Keywords. Proximal point algorithm, Nonexpansive mapping, CAT(0) space, Convex minimization problem

Mathematics Subject Classification (2020). 47H10, 54H25, 54E50

1. Introduction

Kirk [21] was the first to investigate fixed point theory in a CAT(0) space. Fixed point theory for various forms of mappings in CAT(0) spaces has gotten a lot of attention since then. Dhompongsa and Panyanak [10] investigated the convergence of nonexpansive mappings in CAT(0) spaces in 2008. Following that, numerous authors investigated the convergence of nonexpansive mappings using various iteration procedures. Lamba and Panwar [23] recently developed new fixed point results in the context of CAT(0) spaces using AP iteration process, and they also interpreted the efficiency of new three step iteration process using a numerical example. AP iteration process is described as follows: $x_1 \in C$ and

$$\begin{cases} z_n = T((1-b_n)x_n \oplus b_n Tx_n), \\ y_n = T((1-a_n)Tx_n \oplus a_n Tz_n), \\ x_{n+1} = Ty_n, \end{cases}$$

for each $n \in \mathbb{N}$, where $\{a_n\}$, $\{b_n\}$ are sequences in $(0,1)$ and T is a self map defined on a nonempty subset C of a CAT(0) space.

For example, [14, 24, 25] has some interesting results for solving a fixed point problem of nonlinear mappings in the context of CAT(0) spaces.

Let (X,d) be a metric space and $f : X \to (-\infty,\infty]$ be a proper and convex function. One of the major problems in optimization is to find $x \in X$ such that

$$f(x) = \min_{y\in X} f(y).$$

The set of minimizers of f is denoted by $\operatorname{argmin}_{y\in X} f(y)$. The well-known proximal point algorithm (also known as the PPA) was developed by Martinet [28] in 1970 and has proven to be a successful and strong technique for tackling this problem. The convergence to a solution of the convex minimization problem in the framework of Hilbert spaces using PPA was studied by Rockafellar [31] in 1976.

Indeed, let f be a proper, convex, and lower semi-continuous(lsc) function on a Hilbert space H that reaches its minimum. The PPA is defined by $x_1 \in H$ and

$$x_{n+1} = \operatorname*{argmin}_{y\in H} \left(f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right),$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$. It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of f provided $\sum_{n=1}^{\infty} \lambda_n = \infty$. However, as Güler [16] has demonstrated, the PPA does not always converges strongly in general. In the year 2000, Kamimura and Takahashi [19] combined the PPA and Halpern's algorithm [17] to ensure strong convergence.

Bačák [5] introduced the PPA in a CAT(0) space (X,d) in 2013, as follows: $x_1 \in X$ and

$$x_{n+1} = \operatorname*{argmin}_{y\in X} \left(f(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right),$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$. It was proven that if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer based on the Fejer monotonicity idea (see also [3]). Bačák [4] used a split version of the PPA in complete CAT(0) spaces to minimize a sum of convex functions in 2014.

Many PPA convergence techniques for solving optimization problems have recently been extended to the setting of manifolds from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces [13, 26, 29, 30, 34]. In the field of analysis and geometry, minimizers of the objective convex functionals in nonlinear spaces play a vital role. A wide range of applications in computer vision, machine learning, electronic structure computation, system balance, and robot manipulation can be considered as addressing optimization problems on manifolds [1, 32, 33].

We present a modified proximal point algorithm for two nonexpansive mappings in CAT(0) spaces using the AP-type iteration process, and show various convergence results of the proposed process under some moderate conditions, based on earlier work. Our major findings extend the discoveries of Lamba and Panwar [23] from one nonexpansive mapping to two nonexpansive mappings in CAT(0) spaces involving the convex and lower semi-continuous functions.

2. Preliminaries

For the sake of simplicity, we recall a few definitions, exceptions and conclusions.

The researchers [7, 8, 15] provide a full overview of CAT(0) spaces and their importance in several disciplines of mathematics. We compose $(1-s)x \oplus sy$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = sd(x,y), \quad d(z,y) = (1-s)d(x,y).$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, i.e., $[x,y] = \{(1-s)x \oplus sy : s \in [0,1]\}$.

Example 2.1 ([6])**.** When endowed with the induced metric, a convex subset of Euclidean space $\mathbb{E}^n$ is CAT(0) and any real inner product space (not necessarily complete) is a CAT(0) space.

Example 2.2 ([8])**.** Attach together three copies of the ray $[0,\infty) \subset \mathbb{R}$ by gluing at the point 0. The resulting space has nonpositive curvature.

Lemma 2.3 ([6])**.** *Let X be a* CAT(0) *space. Then*

$$d((1-s)x \oplus sy, z) \le (1-s)d(x,z) + sd(y,z), \quad \text{for all } x,y,z \in X \text{ and } s \in [0,1].$$

Lemma 2.4 ([6])**.** *Let X be a* CAT(0) *space. Then*

$$d^2((1-s)x \oplus sy, z) \le (1-s)d^2(x,z) + sd^2(y,z) - s(1-s)d^2(x,y), \quad \text{for all } x,y,z \in X \text{ and } s \in [0,1].$$

Remember that a function $f : C \to (-\infty,\infty]$ defined on a convex subset C of a CAT(0) space is convex if the function $f \circ \gamma$ is convex for any geodesic $\gamma : [a,b] \to C$. We say that a function on C is lower semi-continuous at a point $x \in C$ if

$$f(x) \le \liminf_{n\to\infty} f(x_n),$$

for each sequence $x_n \to x$. A function f is said to be lower semi-continuous on C if it is lower semi-continuous at any point in C.

For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in CAT(0) spaces as

$$J_\lambda(x) = \operatorname*{argmin}_{y\in X}\left(f(y) + \frac{1}{2\lambda}d^2(y,x)\right)$$

for all $x \in X$. The mapping J_λ is well defined for all $\lambda > 0$ (see [18, 29]).

Lemma 2.5 ([3]). *Let $f : X \to (-\infty,\infty]$ be a proper, convex and lsc function, where (X,d) is a complete* CAT(0) *space. Then the set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $\operatorname{argmin}_{y\in X} f(y)$ of minimizers of f.*

A self map T defined on a nonempty subset C of a CAT(0) space is said to be nonexpansive if

$$d(Tx,Ty) \le d(x,y),$$

for all $x,y \in C$. Also, the fixed point set of T is denoted by $F(T)$ i.e., $F(T) = \{x \in C : x = Tx\}$.

Lemma 2.6 ([22]). *For any $\lambda > 0$, the resolvent J_λ of f is nonexpansive.*

Lemma 2.7 ([2]). *Let $f : X \to (-\infty,\infty]$ be a proper, convex and lsc function, where (X,d) is a complete* CAT(0) *space. Then, for all $x,y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}d^2(J_\lambda x,y) - \frac{1}{2\lambda}d^2(x,y) + \frac{1}{2\lambda}d^2(x,J_\lambda x) + f(J_\lambda x) \le f(y).$$

In 1976, Lim [27] introduced the concept of Δ-convergence in a general metric space. In 2008, Kirk and Panyanak [22] specialized Lim's concept to CAT(0) spaces and proved that it is similar to the weak convergence in Banach space setting. Since the notion of Δ-convergence has been widely studied. We now give the concept of Δ-convergence and collect some of its basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,\{x_n\}).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}.$$

The asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is given by

$$r_C(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in C\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\},$$

and the asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is the set

$$A_C(\{x_n\}) = \{x \in C : r(x,\{x_n\}) = r_C(\{x_n\})\}.$$

Proposition 2.8 ([12]). *Let X be a complete* CAT(0) *space, $\{x_n\}$ be a bounded sequence in X and C be a closed convex subset of X. Then*

(i) *there exists a unique point $u \in C$ such that $r(u,\{x_n\}) = \inf_{x\in C} r(x,\{x_n\})$;*

(ii) *$A(\{x_n\})$ and $A_C(\{x_n\})$ are both singleton.*

Definition 2.9 ([22]). Let X be a CAT(0) space. A sequence $\{x_n\}$ in E is said to Δ-converge to $p \in X$, if p is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_{n\to\infty} x_n = p$ and call p the Δ-limit of $\{x_n\}$.

Lemma 2.10 ([22]). *Every bounded sequence in a complete* CAT(0) *space admits a* Δ*-convergent subsequence.*

Lemma 2.11 ([11]). *Let X be a complete* CAT(0) *space, C be closed convex subset of X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.12 ([10]). *Let C be a closed and convex subset of a complete* CAT(0) *space X and T be a nonexpansive self mapping on C. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n\to\infty} x_n = x$. Then $x = Tx$.*

Lemma 2.13 ([10]). *If $\{x_n\}$ is a bounded sequence in a complete* CAT(0) *space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.14 (The resolvent identity, [18]). *Let (X, d) be a complete* CAT(0) *space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. Then, the following identity holds:*

$$J_\lambda x = J_\mu\left(\frac{\lambda-\mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x\right), \quad \text{for all } x \in X \text{ and } \lambda > \mu > 0.$$

3. Main Results

Theorem 3.1. *Consider $f : X \to (-\infty, \infty]$ is a proper, convex and lsc function, where (X, d) is a complete* CAT(0) *space. Let T_1, T_2 be nonexpansive self maps defined on X such that $\Omega = F(T_1) \cap F(T_2) \cap \operatorname{argmin}_{y\in X} f(y) \neq \phi$. Consider $\{a_n\}$ and $\{b_n\}$ are sequences with $0 < a \le a_n$, $b_n \le b < 1$ for all $n \in \mathbb{N}$ and for some $a, b \in (0,1)$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be generated in the following manner:*

$$\begin{cases} u_n = \operatorname*{argmin}_{y\in X}\left(f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\right), \\ z_n = T_1((1-b_n)x_n \oplus b_n T_1 u_n), \\ y_n = T_2((1-a_n)T_1 x_n \oplus a_n T_2 z_n), \\ x_{n+1} = T_2 y_n, \end{cases} \tag{3.1}$$

for each $n \in \mathbb{N}$. Then, we have the following:

(i) $\lim_{n\to\infty} d(x_n, q)$ *exists for all* $q \in \Omega$;

(ii) $\lim_{n\to\infty} d(x_n, u_n) = 0$;

(iii) $\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = 0$.

Proof. Suppose $q \in \Omega$. Then $q = T_1 q = T_2 q$ and since f is lsc function, therefore $f(q) \le f(y)$ for all $y \in X$. Also, $f(q) + \frac{1}{2\lambda_n} d^2(q,q) \le f(y) + \frac{1}{2\lambda_n} d^2(y,q)$ for all $y \in X$ and hence $q = J_{\lambda_n} q$ for all $n \in \mathbb{N}$.

(i) First, we prove that $\lim_{n\to\infty} d(x_n, q)$ exists. Writing $u_n = J_{\lambda_n} x_n$ $\forall$ $n \in \mathbb{N}$. Using Lemma 2.6, we have

$$d(u_n, q) = d(J_{\lambda_n} x_n, J_{\lambda_n} q) \le d(x_n, q)$$

Also, by Lemma 2.3 and (3.1)

$$\begin{aligned} d(z_n,q) &= d\left(T_1((1-b_n)x_n \oplus b_n T_1 u_n, q)\right) \\ &\le d\left((1-b_n)x_n \oplus b_n T_1 u_n, q\right) \\ &\le (1-b_n)d(x_n,q) + b_n d(T_1 u_n,q) \\ &\le (1-b_n)d(x_n,q) + b_n d(u_n,q) \\ &\le (1-b_n)d(x_n,q) + b_n d(x_n,q) \\ &= d(x_n,q). \end{aligned} \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned} d(y_n,q) &= d\left(T_2((1-a_n)T_1 x_n \oplus a_n T_2 z_n, q)\right) \\ &\le d((1-a_n)T_1 x_n \oplus a_n T_2 z_n, q) \\ &\le (1-a_n)d(T_1 x_n,q) + a_n d(T_2 z_n,q) \\ &\le (1-a_n)d(x_n,q) + a_n d(z_n,q) \\ &\le (1-a_n)d(x_n,q) + a_n d(x_n,q) \\ &= d(x_n,q). \end{aligned} \tag{3.3}$$

Using (3.3),

$$\begin{aligned} d(x_{n+1},q) &= d(T_2 y_n,q) \\ &\le d(y_n,q) \\ &\le d(x_n,q). \end{aligned} \tag{3.4}$$

Hence $\lim_{n\to\infty} d(x_n,q)$ exists and $\lim_{n\to\infty} d(x_n,q) = c$ for some c.

(ii) Now we prove $\lim_{n\to\infty} d(x_n,u_n) = 0$. Using Lemma 2.5, we see that

$$\begin{aligned} &\frac{1}{2\lambda_n} d^2(J_{\lambda_n}(x_n),q) - \frac{1}{2\lambda_n} d^2(x_n,q) + \frac{1}{2\lambda_n} d^2(x_n,J_{\lambda_n}(x_n)) + f(J_{\lambda_n}(x_n)) \le f(q), \\ &\frac{1}{2\lambda_n} d^2(u_n,q) - \frac{1}{2\lambda_n} d^2(x_n,q) + \frac{1}{2\lambda_n} d^2(x_n,u_n) + f(u_n) \le f(q), \\ &\frac{1}{2\lambda_n} d^2(u_n,q) - \frac{1}{2\lambda_n} d^2(x_n,q) + \frac{1}{2\lambda_n} d^2(x_n,u_n) \le f(q) - f(u_n). \end{aligned}$$

But $f(q) \le f(u_n)$ $\forall$ $n \in \mathbb{N}$, hence

$$\begin{aligned} &d^2(u_n,q) - d^2(x_n,q) + d^2(x_n,u_n) \le 0, \\ &d^2(x_n,u_n) \le d^2(x_n,q) - d^2(u_n,q). \end{aligned}$$

To prove $\lim_{n\to\infty} d(x_n,u_n) = 0$, suppose that $\lim_{n\to\infty} d(u_n,q) = c$ for $c > 0$.

Now,

$$d(x_{n+1},q) \le d(y_n,q).$$

So, we have

$$c = \liminf_{n\to\infty} d(x_n,q) = \liminf_{n\to\infty} d(x_{n+1},q) \le \liminf_{n\to\infty} d(y_n,q),$$

and also,

$$\limsup_{n\to\infty} d(y_n,q) \le \limsup_{n\to\infty} d(x_n,q) = c.$$

Thus,

$$\lim_{n\to\infty} d(y_n,q) = c$$

and

$$d(z_n,q) \le (1-b_n)d(x_n,q) + b_n d(x_n,q),$$

$$d(x_n,q) \le \frac{d(x_n,q)-d(z_n,q)}{b_n} + d(u_n,q).$$

It gives that

$$c = \liminf_{n\to\infty} d(x_n,q) \le \liminf_{n\to\infty} d(u_n,q).$$

Also,

$$\limsup_{n\to\infty} d(u_n,q) \le c.$$

It shows that

$$\lim_{n\to\infty} d(x_n,u_n) = 0.$$

(iii) To show

$$\lim_{n\to\infty} d(x_n,T_1x_n) = \lim_{n\to\infty} d(x_n,T_2x_n) = 0.$$

We observe that

$$\begin{aligned} d^2(z_n,q) &= d^2(T_1((1-b_n)x_n \oplus b_nT_1u_n,q)) \\ &= d^2((1-b_n)x_n \oplus b_nT_1u_n,q) \\ &= (1-b_n)d^2(x_n,q) + b_nd^2(T_1u_n,q) - b_n(1-b_n)d^2(x_n,T_1u_n) \\ &= d^2(x_n,q) - a(1-b)d^2(x_n,T_1u_n), \end{aligned}$$

$$\begin{aligned} d^2(x_n,T_1u_n) &\le \frac{1}{a(1-b)}(d^2(x_n,q)-d^2(z_n,q)) \\ &\to 0 \ \text{ as } \ n\to\infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} d(x_n,T_1u_n) = 0.$$

It follows that

$$\begin{aligned} d(x_n,T_1x_n) &\le d(x_n,T_1u_n) + d(T_1u_n + T_1x_n) \\ &\to 0 \ \text{ as } \ n\to\infty. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} d^2(y_n,q) &= d^2(T_2((1-a_n)T_1x_n \oplus a_nT_2z_n,q)) \\ &\le d^2((1-a_n)T_1x_n \oplus a_nT_2z_n,q) \\ &\le (1-a_n)d^2(T_1x_n,q) + a_nd^2(T_2z_n,q) - a_n(1-a_n)d^2(T_1x_n,T_2z_n) \end{aligned}$$

$$
\begin{aligned}
&\le (1-a_n)d^2(x_n,q)+a_nd^2(z_n,q)-a_n(1-a_n)d^2(T_1x_n,T_2z_n)\\
&\le d^2(x_n,q)-a_n(1-a_n)d^2(T_1x_n,T_2z_n)\\
d^2(T_1x_n,T_2z_n) &\le \frac{1}{a(1-b)}(d^2(x_n,q)-d^2(y_n,q))\\
\Rightarrow \lim_{n\to\infty} d(T_1x_n,T_2z_n) &= 0. \qquad (3.5)
\end{aligned}
$$

Also,

$$
\begin{aligned}
d(u_n,T_1u_n) &\le d(u_n,x_n)+d(x_n,T_1u_n)\\
&\to 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

Now,

$$
\begin{aligned}
d(z_n,u_n) &= d(T_1((1-b_n)x_n\oplus b_nT_1u_n),u_n)\\
&\le d((1-b_n)x_n\oplus b_nT_1u_n,u_n)\\
&\le (1-b_n)d(x_n,u_n)+b_nd(T_1u_n,u_n)\\
&\to 0 \ \text{ as } \ n\to\infty
\end{aligned}
$$

and

$$
\begin{aligned}
d(x_n,z_n) &\le d(x_n,u_n)+d(u_n,z_n)\\
&\ 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

So, it follows that

$$
\begin{aligned}
d(x_n,T_2x_n) &\le d(x_n,T_1x_n)+d(T_1x_n,T_2z_n)+d(z_n,x_n)\\
&\to 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

So, this completes the proof. □

Next, we prove the Δ-convergence of our iteration.

Theorem 3.2. *Consider $f : X \to (-\infty,\infty]$ is a proper, convex and lsc function, where (X,d) is a complete* CAT(0) *space. Let T_1,T_2 be nonexpansive self maps defined on X such that $\Omega = F(T_1)\cap F(T_2)\cap \operatorname{argmin}_{y\in X} f(y) \ne \phi$. Consider $\{a_n\}$ and $\{b_n\}$ are sequences with $0 < a \le a_n$, $b_n \le b < 1$ for all $n\in\mathbb{N}$ and for some $a,b\in(0,1)$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n\in\mathbb{N}$ and for some λ. If $\{x_n\}$ is the sequence formed by* (3.1)*, then $\{x_n\}$ Δ-converges to an element of Ω.*

Proof. In fact, it follows from Lemma 2.14 and Theorem 3.1(ii), that

$$
\begin{aligned}
d(x_n,J_\lambda x_n) &\le d(x_n,u_n)+d(u_n,J_\lambda x_n)\\
&= d(J_\lambda x_n,J_{\lambda_n}x_n)+d(x_n,u_n)\\
&= d\left(J_\lambda x_n, J_\lambda\left(\frac{\lambda_n-\lambda}{\lambda_n}J_{\lambda_n}x_n\oplus\frac{\lambda}{\lambda_n}x_n\right)\right)+d(x_n,u_n)\\
&\le d\left(x_n,\left(1-\frac{\lambda}{\lambda_n}\right)J_{\lambda_n}x_n\oplus\frac{\lambda}{\lambda_n}x_n\right)+d(x_n,u_n)
\end{aligned}
$$

$$\leq \left(1-\frac{\lambda}{\lambda_n}\right)d(x_n,J_{\lambda_n}x_n)+\frac{\lambda}{\lambda_n}d(x_n,x_n)+d(x_n,u_n)$$
$$=\left(1-\frac{\lambda}{\lambda_n}\right)d(x_n,u_n)+d(x_n,u_n)$$
$$\to 0$$

as $n\to\infty$. Now, this theorem can easily be proved in the similar fashion in [9, Theorem 3.2]. □

If $T_1=T_2=T$ in Theorem 3.2, then we obtain the following result.

Corollary 3.3. *Consider $f:X\to(-\infty,\infty]$ is a proper, convex and lsc function, where (X,d) is a complete* CAT(0) *space. Let T be nonexpansive self map defined on X such that $\Omega = F(T)\cap \operatorname{argmin}_{y\in X} f(y)\neq\phi$. Consider $\{a_n\}$ and $\{b_n\}$ are sequences with $0<a\leq a_n,\ b_n\leq b<1$ for all $n\in\mathbb{N}$ and for some $a,b\in(0,1)$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n\geq\lambda>0$ for all $n\in\mathbb{N}$ and for some λ. Let $\{x_n\}$ be generated in the following manner:*

$$\begin{cases} u_n=\operatorname{argmin}_{y\in X}\left(f(y)+\frac{1}{2\lambda_n}d^2(y,x_n)\right),\\ z_n=T((1-b_n)x_n\oplus b_nTu_n),\\ y_n=T((1-a_n)Tx_n\oplus a_nTz_n),\\ x_{n+1}=Ty_n,\end{cases}$$

for each $n\in\mathbb{N}$, then $\{x_n\}$ Δ-converges to an element of Ω.

Now, we prove strong convergence theorem.

Theorem 3.4. *Suppose all the assumptions are same as of Theorem 3.1, then $\{x_n\}$ strongly converges to a point of Ω if and only if*

$$\liminf_{n\to\infty} d(x_n,\Omega)=0,$$

where $d(x,\Omega)=\inf\{d(x,p^*):p^*\in\Omega\}$.

Proof. The necessity is obvious from Theorem 3.1. Conversely, let $\liminf_{n\to\infty} d(x_n,\Omega)=0$. Since

$$d(x_{n+1},p^*)\leq d(x_n,p^*)$$

for all $p^*\in\Omega$. Hence

$$d(x_{n+1},\Omega)\leq d(x_n,\Omega).$$

So, $\lim_{n\to\infty} d(x_n,\Omega)$ exists. Following [20, Proof of Theorem 2], we can easily show that $\{x_n\}$ is a Cauchy sequence in X. This implies that $\{x_n\}$ converges to a point p^* in X and hence $d(p^*,\Omega)=0$. Since Ω is closed, $p^*\in\Omega$. This completes the proof. □

A family $\{P,Q,R\}$ of mappings is said to satisfy the condition (Ω) if there exists a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0, f(r)>0$ for all $r\in(0,\infty)$ such that $d(x,Px)\geq f(d(x,F))$ or $d(x,Qx)\geq f(d(x,F))$ or $d(x,Rx)\geq f(d(x,F))$ for all $x\in X$. Here, $F=F(P)\cap F(Q)\cap F(R)$.

Theorem 3.5. *Suppose all the assumptions are same as in Theorem* 3.1. *If* $\{T_1, T_2, J_\lambda\}$ *satisfies the condition* (Ω)*, then* $\{x_n\}$ *converges strongly to a point of* Ω*.*

Proof. From Theorem 3.1, we know that $\lim_{n\to\infty} d(x_n, p^*)$ exists for all $p^* \in \Omega$. This implies that $\lim_{n\to\infty} d(x_n, \Omega)$ exists.

Also, by the condition (Ω), we have

$$\lim_{n\to\infty} f(d(x_n,\Omega)) \le \lim_{n\to\infty} d(x_n, T_1 x_n) = 0,$$

or

$$\lim_{n\to\infty} f(d(x_n,\Omega)) \le \lim_{n\to\infty} d(x_n, T_2 x_n) = 0,$$

or

$$\lim_{n\to\infty} f(d(x_n,\Omega)) \le \lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0.$$

Thus, we have

$$\lim_{n\to\infty} f(d(x_n,\Omega)) = 0.$$

By using the property of f, we obtain $\lim_{n\to\infty} d(x_n,\Omega) = 0$. Thus, the proof follows from Theorem 3.4. □

A mapping $T : C \to C$ is said to be semi-compact if any sequence $\{x_n\}$ in C satisfying $d(x_n, Tx_n) \to 0$ has a convergent subsequence.

Theorem 3.6. *Under the hypothesis of Theorem* 3.1*, suppose that* T_1 *or* T_2 *or* J_λ *is semi-compact, then the sequence* $\{x_n\}$ *generated by* (3.1) *strongly converges to a common element of* Ω*.*

Proof. Suppose that T_1 is semi-compact. By Theorem 3.1, we have $d(x_n, T_1x_n) \to 0$ as $n \to \infty$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p^* \in X$. Since $d(x_n, J_\lambda x_n) \to 0$ and $d(x_n, T_i x_n) \to 0$ for all $i \in \{1,2\}$, we have $d(p^*, J_\lambda p^*) = 0$, and $d(p^*, T_1 p^*) = d(p^*, T_2 p^*) = 0$, which shows that $p^* \in \Omega$. In other cases, we can prove the strong convergence of $\{x_n\}$ to an element of Ω. This completes the proof. □

Now, we give the numerical example to show the convergence of our iteration scheme and support our main theorem in a space of real numbers.

Example 3.7. Let $X = \mathbb{R}$ with the Euclidean norm and $C = \{x : -4 \le x \le 4\}$. For each $x \in C$, we define mappings T_1 and T_2 on C as follows:

$$T_1 x = x,$$

and

$$T_2 x = \frac{x}{5}.$$

Clearly, T_1 and T_2 are nonexpansive mappings.

Also, for each $x \in C$, we define $f : C \to (-\infty, \infty]$ by

$$f(x) = x^2.$$

We can easily check that f is a proper, convex and lower semi-continuous function.

We choose $a_n = \frac{n+3}{n+4}$ and $b_n = \frac{n}{n+3}$. Also, we set $\lambda = \frac{1}{2}$ $\forall$ n. It can be observed that all the assumptions of Theorem 3.4 are satisfied. Hence, the sequence $\{x_n\}$ generated by (3.1) converges to 0 which is the fixed point of T_1, T_2 and minimizer of $f(x)$.

4. Conclusion

Our primary findings build on the results of Khan and Abbas [20], Cholamjiak *et al*. [9] and Lamba and Panwar [23]. Indeed, for two nonexpansive mappings in CAT(0) spaces, we provide a new modified proximal point algorithm for addressing convex minimization problems as well as common fixed point problems. Finally, we presented a numerical example to back up our main result.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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