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Research Article

A New Integral Transform With Applications to Fractional Calculus

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Abstract. In this paper, an integral transform with the kernel being the Mittag-Leffler function in two parameters is introduced. Some properties of this integral transform are discussed. Also, its formulae for derivatives of the function are derived. The new integral transform is applied to derive the exact formula for the Laplace transform of fractional derivatives.

Keywords. Mittag-Leffler function, Integral transform, Laplace transform, Fractional derivative

Mathematics Subject Classification (2020). 26A33, 33E12, 44A10, 44A05

1. Introduction

For almost two centuries, integral transformations have been effectively used in solving many problems in applied mathematics, mathematical physics, and engineering science. The concept of integral transform originates from the famous Fourier integral formula. The significance of integral transforms is that they accommodate important operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations. The study of differential and integral equations occurring in applied mathematics, mathematical physics, and engineering was the main incentive for the development of the operational calculus of integral transforms. The operational calculus of integral transforms is also applied to difference equations, integral equations, fractional derivatives, and fractional integrals, [4] evaluation of definite integrals, summation of infinite series, and problems of probability and statistics.

The Laplace transform is a special case of the Fourier transform for a class of functions defined on the positive real axis, further it is more simple than the Fourier transform. The Fourier or Laplace transform methods based on the correct mathematical support are essentially equivalent to the modern operational calculus. The Laplace transform is a broadly used integral transform with many applications in physics and engineering. Appropriate choice of integral transforms helps to convert differential equations as well as integral equations in terms of an algebraic equation that can be solved easily. In this article, we present a new integral transform with the kernel being the Mittag-Leffler function in two parameters. It can be considered as a generalization of the Laplace transform because we can obtain the Laplace transform from it. The new integral transform is used to find the Laplace transform of fractional derivatives.

2. Preliminaries

Definition 2.1. The Mittag-Leffler function of two parameters [2, 3] is defined by

$$t^{\beta-1}E_{\alpha,\beta}(-t^{\alpha}) = t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{\Gamma(\alpha k+\beta)}. \tag{1}$$

Definition 2.2. If f is a real- or complex-valued function of the (time) variable $t > 0$ and s is a real or complex parameter, then the Laplace transform [5] of f is defined by

$$F(s) = \mathscr{L}\{f(t); s\} = \int_0^{\infty} e^{-st} f(t)dt. \tag{2}$$

Theorem 2.1. *The Laplace transform of Riemann-Liouville fractional derivative* [4] *of order* $p > 0$ *is given by*

$$\mathscr{L}\{{}_0D_t^p f(t); s\} = s^p \mathscr{L}\{f(t); s\} - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{p-k-1} f(t)\right]_{t=0}; \quad (n-1 \le p < n). \tag{3}$$

3. A New Integral Transform

Definition 3.1. Let $f(t)$ be a piecewise continuous function defined for $t > 0$ such that $|f(t)| \le Me^{-kt}$ for some $k > 0$. Then for $s > 0$, we define the new integral transform $\mathscr{C}_{\alpha,\beta}$ of $f(t)$ by the formula

$$\mathscr{C}_{\alpha,\beta}\{f(t); s\} = \bar{\mathscr{F}}(s) = \int_0^{\infty} t^{\beta-1}E_{\alpha,\beta}(-st^{\alpha})f(t)dt. \tag{4}$$

Theorem 3.1. *The new integral transform* $\mathscr{C}_{\alpha,\beta}\{f(t); s\} = \bar{\mathscr{F}}(s)$ *is bounded and continuous for all* $s > 0$.

Proof. It follows from the definition of new integral transform that we have

$$|\mathscr{C}_{\alpha,\beta}\{f(t); s\}| = \left|\int_0^{\infty} t^{\beta-1}E_{\alpha,\beta}(-st^{\alpha})f(t)dt\right|$$

$$
\begin{aligned}
&\le \int_0^\infty |t^{\beta-1}E_{\alpha,\beta}(-st^\alpha)||f(t)|dt \\
&\le \int_0^\infty \left| \sum_{j=0}^{\infty} \frac{(-s)^j t^{\alpha j+\beta-1}}{\Gamma(\alpha j+\beta)} \right| Me^{-kt}dt \\
&\le M \sum_{j=0}^{\infty} \frac{s^j}{\Gamma(\alpha j+\beta)} \int_0^\infty e^{-kt} t^{\alpha j+\beta-1} dt \\
&= M \sum_{j=0}^{\infty} \frac{s^j}{\Gamma(\alpha j+\beta)} \frac{\Gamma(\alpha j+\beta)}{k^{\alpha j+\beta}} \\
&= \frac{M}{k^\beta} \sum_{j=0}^{\infty} \left(\frac{s}{k^\alpha}\right)^j \\
&= \frac{M}{k^\beta} \left[\frac{1}{1-\frac{s}{k^\alpha}}\right] \\
&= M \frac{k^{\alpha-\beta}}{k^\alpha - s}.
\end{aligned}
$$

This implies

$$
|\bar{\mathscr{F}}(s)| = |\mathscr{C}_{\alpha,\beta}\{f(t);s\}| \le M \frac{k^{\alpha-\beta}}{k^\alpha - s}. \tag{5}
$$

Hence

$$
\lim_{h\to 0} |\bar{\mathscr{F}}(s+h) - \bar{\mathscr{F}}(s)| = \lim_{h\to 0} \left| \int_0^\infty t^{\beta-1} E_{\alpha,\beta}(-(s+h-s)t^\alpha) f(t) dt \right| = 0.
$$

This shows that $\bar{\mathscr{F}}(s)$ is bounded and continuous for all $s > 0$. □

4. Properties of New Integral Transform

Theorem 4.1. *Let $f(t)$ is differentiable and s is positive, then*

$$
\mathscr{C}_{\alpha,\beta}\{f'(t);s\} = \begin{cases} s\mathscr{L}\{f(t);s\} - f(0), & \text{if } \alpha=\beta=1, \\ s\mathscr{C}_{\alpha,\alpha}\{f(t);s\} - f(0), & \text{if } \alpha>1,\ \beta=1, \\ -\mathscr{C}_{\alpha,\beta-1}\{f(t);s\}, & \text{if } \beta>1. \end{cases} \tag{6}
$$

Proof. Let $f(t)$ is differentiable and s is positive, then $f'(t)$ is differentiable and hence we have

$$
\mathscr{C}_{\alpha,\beta}\{f'(t);s\} = \int_0^\infty t^{\beta-1}E_{\alpha,\beta}(-st^\alpha)f'(t)dt.
$$

Using Integration by Parts rule, we get

$$
\begin{aligned}
\mathscr{C}_{\alpha,\beta}\{f'(t);s\} &= \left[t^{\beta-1}E_{\alpha,\beta}(-st^\alpha) \int f'(t)dt \right]_0^\infty - \int_0^\infty \left[\frac{d}{dt}(t^{\beta-1}E_{\alpha,\beta}(-st^\alpha)) \right] \left(\int f'(t)dt \right) dt \\
&= \left[t^{\beta-1} \sum_{k=0}^{\infty} \frac{(-st^\alpha)^k}{\Gamma(\alpha k+\beta)} f(t) \right]_0^\infty - \int_0^\infty f(t) \frac{d}{dt} \left[t^{\beta-1} \sum_{k=0}^{\infty} \frac{(-st^\alpha)^k}{\Gamma(\alpha k+\beta)} \right] dt.
\end{aligned} \tag{7}
$$

When $\alpha > 1$, $\beta = 1$ or $\beta > 1$, we have

$$
0 \le t^{\beta-1} \sum_{k=0}^{\infty} \frac{(-st^\alpha)^k}{\Gamma(\alpha k+\beta)} f(t) \le t^{\beta-1} f(t) \sum_{k=0}^{\infty} \frac{(-st^\alpha)^k}{\Gamma(k+1)} = t^{\beta-1} f(t) e^{-st^\alpha} = \frac{t^{\beta-1}f(t)}{e^{st^\alpha}}
$$

$$\Rightarrow \quad 0 \le \lim_{t\to\infty}\left[t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-st^{\alpha})^k}{\Gamma(\alpha k+\beta)}f(t)\right] \le \lim_{t\to\infty}\frac{t^{\beta-1}f(t)}{e^{st^{\alpha}}} = 0, \quad \forall\ s>0.$$

Hence,

$$\left[t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-st^{\alpha})^k}{\Gamma(\alpha k+\beta)}\right]_{t=0} = \begin{cases} 1, & \text{if } \alpha>1,\ \beta=1,\\ 0, & \text{if } \beta>1 \end{cases}$$

$$\Rightarrow \quad \left[t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-st^{\alpha})^k}{\Gamma(\alpha k+\beta)}f(t)\right]_{t=0}^{\infty} = \begin{cases} -f(0), & \text{if } \alpha>1,\ \beta=1,\\ 0, & \text{if } \beta>1. \end{cases}$$

Also

$$\frac{d}{dt}\left[t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-st^{\alpha})^k}{\Gamma(\alpha k+\beta)}\right] = \begin{cases} \sum\limits_{k=1}^{\infty}\frac{(-s)^k t^{\alpha k-1}}{\Gamma(\alpha k)}, & \text{if } \alpha>1,\ \beta=1,\\ \sum\limits_{k=0}^{\infty}\frac{(-s)^k t^{\alpha k+\beta-2}}{\Gamma(\alpha k+\beta-1)}, & \text{if } \beta>1 \end{cases}$$

$$\Rightarrow \quad \frac{d}{dt}\left[t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-st^{\alpha})^k}{\Gamma(\alpha k+\beta)}\right] = \begin{cases} \sum\limits_{k=0}^{\infty}\frac{(-s)^{k+1} t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}, & \text{if } \alpha>1,\ \beta=1,\\ t^{\beta-2}\sum\limits_{k=0}^{\infty}\frac{(-st^{\alpha})^k}{\Gamma(\alpha k+\beta-1)}, & \text{if } \beta>1 \end{cases}$$

$$\Rightarrow \quad \frac{d}{dt}\left[t^{\beta-1}\sum_{k=0}^{\infty}\frac{(-st^{\alpha})^k}{\Gamma(\alpha k+\beta)}\right] = \begin{cases} (-s)t^{\alpha-1}E_{\alpha,\alpha}(-st^{\alpha}), & \text{if } \alpha>1,\ \beta=1,\\ t^{\beta-2}E_{\alpha,\beta-1}(-st^{\alpha}), & \text{if } \beta>1 \end{cases}$$

$$\Rightarrow \quad \mathscr{C}_{\alpha,\beta}\{f'(t);s\} = \begin{cases} -f(0)-\int_0^{\infty} f(t)[(-s)t^{\alpha-1}E_{\alpha,\alpha}(-st^{\alpha})]dt, & \text{if } \alpha>1,\ \beta=1,\\ 0-\int_0^{\infty} f(t)[t^{\beta-2}E_{\alpha,\beta-1}(-st^{\alpha})]dt, & \text{if } \beta>1 \end{cases}$$

$$\Rightarrow \quad \mathscr{C}_{\alpha,\beta}\{f'(t);s\} = \begin{cases} -f(0)+s\int_0^{\infty} f(t)[t^{\alpha-1}E_{\alpha,\alpha}(-st^{\alpha})]dt, & \text{if } \alpha>1,\ \beta=1,\\ -\int_0^{\infty} f(t)[t^{\beta-2}E_{\alpha,\beta-1}(-st^{\alpha})]dt, & \text{if } \beta>1 \end{cases}$$

$$= \begin{cases} s\mathscr{L}\{f(t);s\}-f(0), & \text{if } \alpha=\beta=1,\\ s\mathscr{C}_{\alpha,\alpha}\{f(t);s\}-f(0), & \text{if } \alpha>1,\ \beta=1,\\ -\mathscr{C}_{\alpha,\beta-1}\{f(t);s\}, & \text{if } \beta>1. \end{cases}$$

This is the desired proof. □

Corollary 4.1. *Let $f(t)$ is differentiable and s is positive, then*

$$\mathscr{C}_{\alpha,\beta}\{f''(t);s\} = \begin{cases} s^2\mathscr{L}\{f(t);s\}-sf(0)-f'(0), & \text{if } \alpha=\beta=1,\\ -s\mathscr{C}_{\alpha,\alpha-1}\{f(t);s\}-f'(0), & \text{if } \alpha>1,\ \beta=1,\\ \mathscr{C}_{\alpha,\beta-2}\{f(t);s\}, & \text{if } \beta>2. \end{cases} \tag{8}$$

Proof. As seen in Theorem 4.1, we have

$$\mathscr{C}_{\alpha,\beta}\{f''(t);s\} = \begin{cases} s\mathscr{C}_{\alpha,\alpha}\{f'(t);s\}-f'(0), & \text{if } \alpha>1,\ \beta=1,\\ -\mathscr{C}_{\alpha,\beta-1}\{f'(t);s\}, & \text{if } \beta>1 \end{cases}$$

$$\Rightarrow \quad \mathscr{C}_{\alpha,\beta}\{f''(t);s\} = \begin{cases} s^2\mathscr{C}_{1,1}\{f(t);s\}-sf(0)-f'(0), & \text{if } \alpha=\beta=1,\\ -s\mathscr{C}_{\alpha,\alpha-1}\{f(t);s\}-f'(0), & \text{if } \alpha>1,\ \beta=1,\\ \mathscr{C}_{\alpha,\beta-2}\{f(t);s\}, & \text{if } \beta>2 \end{cases}$$

$$\Rightarrow \quad \mathscr{C}_{\alpha,\beta}\{f''(t);s\} = \begin{cases} s^2\mathscr{L}\{f(t);s\} - sf(0) - f'(0), & \text{if } \alpha=\beta=1, \\ -s\mathscr{C}_{\alpha,\alpha-1}\{f(t);s\} - f'(0), & \text{if } \alpha>1,\ \beta=1, \\ \mathscr{C}_{\alpha,\beta-2}\{f(t);s\}, & \text{if } \beta>2. \end{cases}$$

This is the desired proof. □

Corollary 4.2. *Let $f(t)$ is differentiable and s is positive, then*

$$\mathscr{C}_{\alpha,\beta}\{f'''(t);s\} = \begin{cases} s^3\mathscr{L}\{f(t);s\} - s^2f(0) - sf'(0) - f''(0), & \text{if } \alpha=\beta=1, \\ -s^2\mathscr{C}_{2,2}\{f(t);s\} + sf(0) - f''(0), & \text{if } \alpha=2,\ \beta=1, \\ s\mathscr{C}_{\alpha,\alpha-2}\{f(t);s\} - f''(0), & \text{if } \alpha>2,\ \beta=1, \\ -\mathscr{C}_{\alpha,\beta-3}\{f(t);s\}, & \text{if } \beta>3. \end{cases} \tag{9}$$

Proof. As seen in Corollary 4.1, we have

$$\mathscr{C}_{\alpha,\beta}\{f'''(t);s\} = \begin{cases} s^2\mathscr{L}\{f'(t);s\} - sf'(0) - f''(0), & \text{if } \alpha=\beta=1, \\ -s\mathscr{C}_{\alpha,\alpha-1}\{f'(t);s\} - f''(0), & \text{if } \alpha>1,\ \beta=1, \\ \mathscr{C}_{\alpha,\beta-2}\{f'(t);s\}, & \text{if } \beta>2; \end{cases}$$

$$\mathscr{C}_{\alpha,\beta}\{f'''(t);s\} = \begin{cases} s^2[s\mathscr{L}\{f(t);s\} - f(0)] - sf'(0) - f''(0), & \text{if } \alpha=\beta=1, \\ -s[s\mathscr{C}_{2,2}\{f(t);s\} - f(0)] - f''(0), & \text{if } \alpha=2,\ \beta=1, \\ -s[-\mathscr{C}_{\alpha,\alpha-2}\{f(t);s\}] - f''(0), & \text{if } \alpha>2,\ \beta=1, \\ -\mathscr{C}_{\alpha,\beta-3}\{f(t);s\}, & \text{if } \beta>3; \end{cases}$$

$$\mathscr{C}_{\alpha,\beta}\{f'''(t);s\} = \begin{cases} s^3\mathscr{L}\{f(t);s\} - s^2f(0) - sf'(0) - f''(0), & \text{if } \alpha=\beta=1, \\ -s^2\mathscr{C}_{2,2}\{f(t);s\} + sf(0 - f''(0), & \text{if } \alpha=2,\ \beta=1, \\ s\mathscr{C}_{\alpha,\alpha-2}\{f(t);s\} - f''(0), & \text{if } \alpha>2,\ \beta=1, \\ -\mathscr{C}_{\alpha,\beta-3}\{f(t);s\}, & \text{if } \beta>3. \end{cases}$$

This is the desired proof. □

Corollary 4.3. *Let $f(t)$ is differentiable and s is positive, then*

$$\mathscr{L}\left\{{}_0D_t^{\frac{1}{2}}f(t);s\right\} = -is\mathscr{C}_{1,\frac{3}{2}}\{f(t);s\} - \left[{}_0D_t^{-\frac{1}{2}}f(t)\right]_{t=0}. \tag{10}$$

Proof. Form Theorem 4.1, Corollary 4.1 and Corollary 4.2, we can generalize the result and so, we get

$$\mathscr{C}_{\alpha,\beta}\left\{{}_0D_t^{\frac{1}{2}}f(t);s\right\} = (-1)^{\frac{3}{2}}s\mathscr{C}_{\alpha,\alpha+\frac{1}{2}}\{f(t);s\} - \left[{}_0D_t^{-\frac{1}{2}}f(t)\right]_{t=0}; \quad \alpha>\frac{1}{2},\ \beta=1$$

$$\Rightarrow \quad \mathscr{C}_{1,1}\left\{{}_0D_t^{\frac{1}{2}}f(t);s\right\} = -is\mathscr{C}_{1,1+\frac{1}{2}}\{f(t);s\} - \left[{}_0D_t^{-\frac{1}{2}}f(t)\right]_{t=0}.$$

This implies

$$\mathscr{L}\left\{{}_0D_t^{\frac{1}{2}}f(t);s\right\} = -is\mathscr{C}_{1,\frac{3}{2}}\{f(t);s\} - \left[{}_0D_t^{-\frac{1}{2}}f(t)\right]_{t=0}.$$ □

Corollary 4.4. *Let $f(t)$ is differentiable and s is positive, then*

$$\mathscr{C}_{1,\frac{3}{2}}\left\{{}_0D_t^{\frac{1}{2}}f(t);s\right\} = i\mathscr{L}\{f(t);s\}. \tag{11}$$

Proof. Form Theorem 4.1, Corollary 4.1 and Corollary 4.2, we can generalize the result and so, we get

$$\mathscr{C}_{\alpha,\beta}\{{}_0D_t^{\frac{1}{2}}f(t);s\} = (-1)^{\frac{1}{2}}\mathscr{C}_{\alpha,\beta-\frac{1}{2}}\{f(t);s\}; \quad \beta > \frac{1}{2}.$$

This implies

$$\mathscr{C}_{1,\frac{3}{2}}\{{}_0D_t^{\frac{1}{2}}f(t);s\} = (-1)^{\frac{1}{2}}\mathscr{C}_{1,\frac{3}{2}-\frac{1}{2}}\{f(t);s\}$$

$$\Rightarrow \quad \mathscr{C}_{1,\frac{3}{2}}\{{}_0D_t^{\frac{1}{2}}f(t);s\} = i\mathscr{C}_{1,1}\{f(t);s\}.$$

This proves the result

$$\Rightarrow \quad \mathscr{C}_{1,\frac{3}{2}}\{{}_0D_t^{\frac{1}{2}}f(t);s\} = i\mathscr{L}\{f(t);s\}.$$ □

Theorem 4.2 (Convolution)**.** *Let $f_1(t)$ and $f_2(t)$ have new integral transform $\bar{\mathscr{F}}_1(s)$ and $\bar{\mathscr{F}}_2(s)$. Then the new integral transform of the convolution of f_1 and f_2 is*

$$\mathscr{C}_{\alpha,\beta}\{f_1 \star f_2\} = \bar{\mathscr{F}}_1(s)\bar{\mathscr{F}}_2(s), \tag{12}$$

where

$$f_1 \star f_2 = \int_0^t f_1(\tau)f_2(t-\tau)d\tau. \tag{13}$$

Proof. Form Definition 3.1 of new integral transform $\mathscr{C}_{\alpha,\beta}$, we have

$$\mathscr{C}_{\alpha,\beta}\{f_1 \star f_2\} = \int_0^\infty t^{\beta-1}E_{\alpha,\beta}(-st^\alpha)\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right]dt$$

$$\Rightarrow \quad \mathscr{C}_{\alpha,\beta}\{f_1 \star f_2\} = \int_0^\infty \int_0^t f_1(\tau)f_2(t-\tau)t^{\beta-1}E_{\alpha,\beta}(-st^\alpha)d\tau\, dt.$$

By changing the order of integration, we have

$$\mathscr{C}_{\alpha,\beta}\{f_1 \star f_2\} = \int_0^\infty \int_\tau^\infty f_1(\tau)f_2(t-\tau)t^{\beta-1}E_{\alpha,\beta}(-st^\alpha)dt\, d\tau$$

$$\Rightarrow \quad \mathscr{C}_{\alpha,\beta}\{f_1 \star f_2\} = \int_0^\infty f_1(\tau)\int_\tau^\infty t^{\beta-1}E_{\alpha,\beta}(-st^\alpha)f_2(t-\tau)dt\, d\tau.$$

Put $t-\tau = v$. Then we have

$$\begin{aligned}
\mathscr{C}_{\alpha,\beta}\{f_1 \star f_2\} &= \int_0^\infty f_1(\tau)\int_0^\infty (v+\tau)^{\beta-1}E_{\alpha,\beta}(-s(v+\tau)^\alpha)f_2(v)dv\, d\tau \\
&= \int_0^\infty f_1(\tau)\int_0^\infty \sum_{j=0}^\infty \frac{(-s)^j(v+\tau)^{\alpha j+\beta-1}}{\Gamma(\alpha j+\beta)}f_2(v)dv\, d\tau \\
&= \int_0^\infty f_1(\tau)\int_0^\infty \left[\sum_{j=0}^\infty \frac{(-s)^j v^{\alpha j+\beta-1}}{\Gamma(\alpha j+\beta)}\sum_{j=0}^\infty \frac{(-s)^j \tau^{\alpha j+\beta-1}}{\Gamma(\alpha j+\beta)}\right]f_2(v)dv\, d\tau \\
&= \int_0^\infty \sum_{j=0}^\infty \frac{(-s)^j \tau^{\alpha j+\beta-1}}{\Gamma(\alpha j+\beta)}f_1(\tau)d\tau \int_0^\infty \sum_{j=0}^\infty \frac{(-s)^j v^{\alpha j+\beta-1}}{\Gamma(\alpha j+\beta)}f_2(v)dv \\
&= \int_0^\infty \tau^{\beta-1}E_{\alpha,\beta}(-s\tau^\alpha)f_1(\tau)d\tau \int_0^\infty v^{\beta-1}E_{\alpha,\beta}(-sv^\alpha)f_2(v)dv \\
&= \mathscr{C}_{\alpha,\beta}\{f_1(t);s\}\mathscr{C}_{\alpha,\beta}\{f_2(t);s\} \\
&= \bar{\mathscr{F}}_1(s)\bar{\mathscr{F}}_2(s).
\end{aligned}$$ □

5. Illustrations

Example 5.1. Form Definition 3.1 of new integral transform $\mathscr{C}_{\alpha,\beta}$, we have

$$\begin{aligned}\mathscr{C}_{1,2}\{e^{-at};s\} = \bar{\mathscr{F}}(s) &= \int_0^\infty tE_{1,2}(-st)e^{-at}dt \\ &= \int_0^\infty t\frac{e^{-st}-1}{-st}e^{-at}dt \\ &= \frac{1}{-s}\int_0^\infty (e^{-st}-1)e^{-at}dt \\ &= \frac{1}{-s}\int_0^\infty (e^{-(s+a)t}-e^{-at})dt \\ &= \frac{1}{-s}\left[\frac{1}{s+a}-\frac{1}{a}\right] \\ &= \frac{1}{a(s+a)}; \quad s>0,\ a>0. \end{aligned} \tag{14}$$

Example 5.2. Let us take $a = 1$ in (14), we have

$$\mathscr{C}_{1,2}\{e^{-t};s\} = \frac{1}{1+s}; \quad s>0.$$

Therefore, we have

$$\int_0^\infty tE_{1,2}(-st)e^{-t}dt = \frac{1}{1+s}; \quad s>0. \tag{15}$$

Example 5.3. Using the integral representation

$$\int_0^\infty e^{-s\zeta}\zeta^{m\alpha+\beta-1}E^{(m)}_{\alpha,\beta}(\pm a\zeta^\alpha)d\zeta = \frac{m!s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}; \quad \mathscr{R}(s)>0,\ \mathscr{R}(\alpha)>0,\ \mathscr{R}(\beta)>0. \tag{16}$$

As given in [3], we have

$$\mathscr{C}_{\alpha,\beta}\{e^{-at};s\} = \bar{\mathscr{F}}(s) = \int_0^\infty t^{\beta-1}E_{\alpha,\beta}(-st^\alpha)e^{-at}dt \tag{17}$$

$$= \frac{a^{\alpha-\beta}}{(a^\alpha+s)}; \quad a>0. \tag{18}$$

Therefore, we have

$$\mathscr{C}_{\alpha,\beta}\{e^{-t};s\} = \bar{\mathscr{F}}(s) = \int_0^\infty t^{\beta-1}E_{\alpha,\beta}(-st^\alpha)e^{-t}dt \tag{19}$$

$$= \frac{1}{1+s}. \tag{20}$$

Example 5.4. Using the results given in [3], we have

$$\frac{2}{\pi}\int_0^\infty \cos(t)\cos(kt)dt = \delta(k-1). \tag{21}$$

Therefore,

$$\begin{aligned}\mathscr{C}_{2,1}\{\cos(kt);s\}) &= \int_0^\infty E_{2,1}(-st^2)\cos(kt)dt \\ &= \int_0^\infty \cosh(i\sqrt{s}t)\cos(kt)dt\end{aligned}$$

$$= \int_0^\infty \cos(\sqrt{s}t)\cos(kt)dt$$
$$= \frac{1}{\sqrt{s}}\int_0^\infty \cos(t)\cos\left(\frac{k}{\sqrt{s}}t\right)dt$$
$$= \frac{\pi}{2\sqrt{s}}\delta\left(\frac{k}{\sqrt{s}}-1\right), \tag{22}$$

where $\delta(x)$ is the Dirac-delta function defined by

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}dk = \frac{1}{\pi}\int_0^\infty \cos(kx)dk. \tag{23}$$

Example 5.5. Form Definition 3.1 of new integral transform $\mathscr{C}_{\alpha,\beta}$, we have

$$\mathscr{C}_{1,3}\{e^{-at};s\} = \bar{\mathscr{F}}(s) = \int_0^\infty t^2 E_{1,3}(-st)e^{-at}dt$$
$$= \int_0^\infty t^2 \frac{e^{-st}-1+st}{(-st)^2}e^{-at}dt$$
$$= \frac{1}{s^2}\int_0^\infty (e^{-st}-1+st)e^{-at}dt$$
$$= \frac{1}{s^2}\int_0^\infty (e^{-(s+a)t}-e^{-at}+ste^{-at})dt$$
$$= \frac{1}{s^2}\left[\frac{1}{s+a}-\frac{1}{a}+s\frac{1}{a^2}\right]$$
$$= \frac{1}{s^2}\left[\frac{-s}{a(s+a)}+\frac{s}{a^2}\right]$$
$$= \frac{1}{s}\left[\frac{-a}{a^2(s+a)}+\frac{s+a}{(s+a)a^2}\right]$$
$$= \frac{1}{a^2(s+a)}; \quad s>0,\ a>0. \tag{24}$$

Example 5.6. Form Definition 3.1 of new integral transform $\mathscr{C}_{\alpha,\beta}$, we have

$$\mathscr{C}_{2,1}\{e^{-at};s\} = \bar{\mathscr{F}}(s) = \int_0^\infty E_{2,1}(-st^2)e^{-at}dt$$
$$= \int_0^\infty \cosh(i\sqrt{s}t)e^{-at}dt$$
$$= \int_0^\infty \cos(\sqrt{s}t)e^{-at}dt$$
$$= \frac{a}{a^2+(\sqrt{s})^2}. \tag{25}$$

Example 5.7. Form Definition 3.1 of new integral transform $\mathscr{C}_{\alpha,\beta}$, we have

$$\mathscr{C}_{2,1}\{e^{at};s\} = \bar{\mathscr{F}}(s) = \int_0^\infty E_{2,1}(-st^2)e^{at}dt$$
$$= \int_0^\infty \cosh(i\sqrt{s}t)e^{at}dt$$
$$= \int_0^\infty \cos(\sqrt{s}t)e^{at}dt$$

$$= \frac{-a}{(-a)^2 + (\sqrt{s})^2}. \tag{26}$$

Example 5.8. Form Corollary 4.4, we have

$$\begin{aligned}
\mathscr{C}_{1,\frac{3}{2}}\{t^n;s\} &= i\mathscr{L}\{{}_0D_t^{-\frac{1}{2}}\left(t^n\right);s\}\\
&= i\mathscr{L}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2}+1)}t^{n+\frac{1}{2}};s\right\}\\
&= i\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2}+1)}\mathscr{L}\{t^{n+\frac{1}{2}};s\}\\
&= i\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}+1\right)}\frac{\Gamma\left(n+\frac{1}{2}+1\right)}{s^{(n+\frac{1}{2}+1)}}\\
&= i\frac{\Gamma(n+1)}{s^{(n+\frac{3}{2})}}.
\end{aligned} \tag{27}$$

6. Applications

This paper deals with the application of the new integral transform to find the exact formula for the Laplace transform of fractional derivatives of a function. Form Corollary 4.3, we have

$$\mathscr{L}\{f'(t);s\} = -is\mathscr{C}_{1,\frac{3}{2}}\{{}_0D_t^{\frac{1}{2}}f(t);s\} - f(0).$$

Also, from Corollary 4.4, we have

$$\mathscr{C}_{1,\frac{3}{2}}\{{}_0D_t^{\frac{1}{2}}f(t);s\} = i\mathscr{L}\{f(t);s\}.$$

This implies

$$\mathscr{L}\{f'(t);s\} = -is[i\mathscr{L}\{f(t);s\}] - f(0).$$

Hence, we get

$$\mathscr{L}\{f'(t);s\} = s\mathscr{L}\{f(t);s\} - f(0).$$

This verifies the formula for the Laplace transform of the derivative of a function.

The Laplace transform of Riemann-Liouville fractional derivative [4] of order $p > 0$ is given by

$$\mathscr{L}\{{}_0D_t^p f(t);s\} = s^p\mathscr{L}\{f(t);s\} - \sum_{k=0}^{n-1} s^k\left[{}_0D_t^{p-k-1}f(t)\right]_{t=0}; \quad (n-1 \le p < n)$$

$$\Rightarrow \quad \mathscr{L}\{f'(t);s\} = \mathscr{L}\{{}_0D_t^p({}_0D_t^p f(t));s\} = s\mathscr{L}\{f(t);s\} - f(0) - s^{\frac{1}{2}}\left[{}_0D_t^{-\frac{1}{2}}f(t)\right]_{t=0}. \tag{28}$$

7. Conclusion

This paper aimed to introduce a new integral transformation like the Laplace transform. To achieve this goal, we made use of the Mittag-Leffler function in two parameters as its kernel. We derive the different properties of this integral transform. The new integral transform was used to find the Laplace transform of the fractional derivatives. It is found that the formula for the Laplace transform of half derivative of a function derived in Corollary 4.3 is more accurate than that of the formula (28) derived from the formula given in [4].

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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