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Research Article

A Focus on Types of Compatible Mappings in S-metric Space

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Abstract. In this article, we study unique common fixed point theorems for four self mappings in S-metric space using generalized concepts of compatible mappings such as compatible mappings of type (A), type (B) and type (C). Our results are generalizing the theorems proved by Sedghi *et al*. (Common fixed point of four maps in S-metric spaces, *Mathematical Sciences* **12** (2018), 137 – 143) in metric space.

Keywords. S-metric space, Compatible mappings of type (A), type (B) and type (C)

Mathematics Subject Classification (2020). 54H25, 47H10

1. Introduction

In the history of fixed point theory, contraction mapping principle established by Banach is most famous tool and it has been used to prove existence and uniqueness of solutions in analysis. Since then, one can solve the problems of various kinds that arises in dynamic programming, control theory, computer science, etc.

In complete metric space, the notion of compatible mappings of type (A) initiated by Jungck *et al*. [1], later Pathak and Khan [2] presented the notation of compatible mappings of type (B) and compared with compatible mappings of type (B). Further, Pathak *et al*. [3] developed the notion of compatible mappings of type (C) and established a common fixed point theorem for four mappings in metric space.

Recently, Sedghi *et al.* [6] proposed the notion of S-metric space, it is generalization of the concept of a G-metric space introduced by Mustafa and Sims [4] and also the notion of a D^*-metric spaces presented by Sedghi *et al.* [5]. Many researchers studied the common fixed point results in S-metric space (see [6–10]).

In this paper, we present unique common fixed point theorems for four self mappings defined on complete S-metric space using various types of compatible mappings. Our results are generalizing the theorem proved by of the Sedghi *et al.* [9].

Now, we present some needful definitions with examples and lemma's for S-metric space.

Definition 1.1 ([6])**.** Let X be a nonempty set, a mapping $S : X \times X \times X \to [0,\infty)$ is said to be S-metric space on X, if $S(\alpha,\beta,\gamma) = 0 \Leftrightarrow \alpha = \beta = \gamma$ and $S(\alpha,\beta,\gamma) \le S(\alpha,\alpha,a) + S(\beta,\beta,a) + S(\gamma,\gamma,a)$ for each $\alpha,\beta,\gamma,a \in X$. The pair (X,S) is called S-metric space.

Example 1.2 ([6])**.** Let (X,d) be usual metric then $S(\alpha,\beta,\gamma) = d_1(\alpha,\gamma) + d_2(\beta,\gamma)$, is satisfying the properties of Definition 1.1.

Lemma 1.3 ([7])**.** *Let (X,S) be an S-metric space, then $S(\alpha,\alpha,\beta) = S(\beta,\beta,\alpha)$ for $\alpha,\beta \in X$.*

Definition 1.4 ([10])**.** Let (X,S) be S-metric space and $\{\alpha_\eta\}$ be a sequence in X and αX. Then

(a) a sequence $\alpha_\eta \in X$ is converges to α if $S(\alpha_\eta,\alpha_\eta,\alpha) \to 0$ as $\eta \to \infty$.

(b) A sequence α_η in X is said to be Cauchy sequence if for each $\epsilon > 0$ there exists $\eta_0 \in \mathbb{N}$ such that $S(\alpha_\eta,\alpha_\eta,\alpha_\mu) < \varepsilon$ for each $\eta,\mu \ge \eta_0$.

(c) If every Cauchy sequence in S-metric space is convergent then (X,S) is complete.

Definition 1.5. Let two self mappings f,g be defined on (X,S). If $\{\alpha_\eta\} \in X$ exists such that $\lim_{\eta\to\infty} f\alpha_\eta = \lim_{\eta\to\infty} g\alpha_\eta = t$ for some $t \in X$ then a pair (f,g) is said to be

(i) ([9]) compatible if and only if $\lim_{\eta\to\infty} S(fg\alpha_\eta, fg\alpha_\eta, gf\alpha_\eta) = 0$,

(ii) compatible of type (A) if and only if $\lim_{\eta\to\infty} S(fg\alpha_\eta, fg\alpha_\eta, gg\alpha_\eta) = 0$, and

$$\lim_{\eta\to\infty} S(gf\alpha_\eta, gf\alpha_\eta, ff\alpha_\eta) = 0,$$

(iii) compatible of type (B) if and only if

$$\lim_{\eta\to\infty} S(fg\alpha_\eta, fg\alpha_\eta, gg\alpha_\eta) \le \frac{1}{2}\left\{\lim_{\eta\to\infty} S(fg\alpha_\eta, fg\alpha_\eta, ft) + \lim_{\eta\to\infty} S(ft, ft, ff\alpha_\eta)\right\}$$

and

$$\lim_{\eta\to\infty} S(gf\alpha_\eta, gf\alpha_\eta, ff\alpha_\eta) \le \frac{1}{2}\left\{\lim_{\eta\to\infty} S(gf\alpha_\eta, gf\alpha_\eta, gt) + \lim_{\eta\to\infty} S(gt, gt, gg\alpha_\eta)\right\}$$

(iv) compatible of type (C), if and only if

$$\lim_{\eta\to\infty} S(fg\alpha_\eta, fg\alpha_\eta, gg\alpha_\eta) \le \frac{1}{3}\left\{\lim_{\eta\to\infty} S(fg\alpha_\eta, fg\alpha_\eta, ft) + \lim_{\eta\to\infty} S(ft, ft, ff\alpha_\eta) + \lim_{\eta\to\infty} S(ft, ft, gg\alpha_\eta)\right\}$$

and

$$\lim_{\eta\to\infty} S(gfa_\eta, gfa_\eta, ffa_\eta)$$
$$\leq \frac{1}{3}\left\{\lim_{\eta\to\infty} S(gfa_\eta, gfa_\eta, gt) + \lim_{\eta\to\infty} S(gt, gt, gga_\eta) + \lim_{\eta\to\infty} S(gt, gt, ffa_\eta)\right\}$$

Proposition 1.6. *Let f, g be two self mappings of (X,S). If f and g are compatible of type (A) and $fa_\eta, ga_\eta \to t$ for some $t \in X$. Then we have the following.*

(i) $\lim_{\eta\to\infty} gfa_\eta = ft$ *if f is continuous at t.*

(ii) $fg(t) = gf(t)$ *and* $f(t) = g(t)$ *if f and g are continuous at t.*

Lemma 1.7. *Let f and g be two self mappings of (X,S), the pair of mappings are either compatible of type (A) or type (B) or type (C) and also satisfying $f(t) = g(t)$ for some $t \in X$ then $fg(t) = gg(t) = gf(t) = gg(t)$.*

Lemma 1.8. *Let f and g be two self maps defined on (X,S) and the pair (f,g) is compatible mappings of type (B). Suppose that $\lim_{\eta\to\infty} fa_\eta = \lim_{\eta\to\infty} ga_\eta = t$ for some $t \in X$, then*

(i) $\lim_{\eta\to\infty} gga_\eta = ft$ *if f is continuous at t.*

(ii) $\lim_{\eta\to\infty} ffa_\eta = gt$ *if g is continuous at t.*

Now we present examples for various types of compatible mappings.

Example 1.9. Consider (X,S) be a S-metric space where $X = [1,10]$ and $S(\alpha,\beta,\gamma) = |\alpha-\gamma| + |\beta-\gamma|$ for $\alpha,\beta,\gamma \in X$. The self maps f and g are defined as follows.

$$f(\alpha) = \begin{cases} 2 & \text{for } \alpha = 2 \text{ or } \alpha > 5, \\ 12 & \text{for } \alpha \in (2,5], \end{cases} \qquad g(\alpha) = \begin{cases} 2 & \text{for } \alpha = 2, \\ 12 & \text{for } \alpha \in (2,5], \\ \frac{\alpha+1}{3} & \text{for } \alpha > 5. \end{cases}$$

Choosing the sequence $\{\alpha_\eta\} = \left\{5 + \frac{1}{\eta}\right\}$,

$$fa_\eta = f\left(5 + \frac{1}{\eta}\right) \to 2, \quad ga_\eta = g\left(5 + \frac{1}{\eta}\right) \to 2.$$

Also, $ff(\alpha_\eta) \to 2, gf(\alpha_\eta) \to 2, fg(\alpha_\eta) \to 12, gg(\alpha_\eta) \to 12$ as $\eta \to \infty$. The pair is compatible of type (A), type (B) and type (C) but not compatible.

Example 1.10. Let (X,S) be S-metricspace where $X = \mathbb{R}$ and $S(\alpha,\beta,\gamma) = |\alpha-\gamma| + |\beta-\gamma|$ for $\alpha,\beta,\gamma \in X$.

The self maps f and g are defined as follows.

$$f(\alpha) = \begin{cases} \frac{1}{\alpha^4} & \text{for } \alpha \neq 0, \\ 1 & \text{for } \alpha = 0, \end{cases} \qquad g(\alpha) = \begin{cases} \frac{1}{\alpha^2} & \text{for } \alpha \neq 0, \\ 3 & \text{for } \alpha = 0. \end{cases}$$

Choosing the sequence $\{\alpha_\eta\} = \{\eta\}$, $f(\alpha_\eta) = f(\eta) \to 0$, $g(\alpha_\eta) = g(\eta) \to 0$ as $\eta \to \infty$.

$$\lim_{\eta\to\infty} S(fga_\eta, fga_\eta, gfa_\eta) = \lim_{\eta\to\infty} S(\eta^8, \eta^8, \eta^8) = 0.$$

For compatible mappings of type (A),

$$\lim_{\eta\to\infty} S(fga_\eta, fga_\eta, gga_\eta) = \lim_{\eta\to\infty} S(\eta^8,\eta^8,\eta^4) = \infty$$

and

$$\lim_{\eta\to\infty} S(gfa_\eta, gfa_\eta, ffa_\eta) = \lim_{\eta\to\infty} S(\eta^8,\eta^8,\eta^{12}) = \infty.$$

Now for compatible mappings of type (B),

$$\frac{1}{2}\left\{\lim_{\eta\to\infty} S(fga_\eta, fga_\eta, ft) + \lim_{\eta\to\infty} S(ft, ft, ffa_\eta)\right\}$$
$$= \frac{1}{2}\left\{\lim_{\eta\to\infty} S(\eta^8,\eta^8,1) + \lim_{\eta\to\infty} S(1,1,\eta^{16})\right\} = \infty$$

and

$$\frac{1}{2}\left\{\lim_{\eta\to\infty} S(gfa_\eta, gfa_\eta, gt) + \lim_{\eta\to\infty} S(gt, gt, gga_\eta)\right\}$$
$$= \frac{1}{2}\left\{\lim_{\eta\to\infty} S(\eta^8,\eta^8,2) + \lim_{\eta\to\infty} S(2,2,\eta^{4})\right\} = \infty.$$

Now we check compatible mappings of type (C),

$$\frac{1}{3}\left\{\lim_{\eta\to\infty} S(fga_\eta, fga_\eta, ft) + \lim_{\eta\to\infty} S(ft, ft, ffa_\eta) + \lim_{\eta\to\infty} S(ft, ft, gga_\eta)\right\}$$
$$= \frac{1}{3}\left\{\lim_{\eta\to\infty} S(\eta^8,\eta^8,1) + \lim_{\eta\to\infty} S(1,1,\eta^{16}) + \lim_{\eta\to\infty} S(\eta^8,\eta^8,\eta^4)\right\} = \infty$$

and

$$\frac{1}{3}\left\{\lim_{\eta\to\infty} S(gfa_\eta, gfa_\eta, gt) + \lim_{\eta\to\infty} S(gt, gt, gga_\eta) + \lim_{\eta\to\infty} S(gt, gt, ffa_\eta)\right\}$$
$$= \frac{1}{3}\left\{\lim_{\eta\to\infty} S(\eta^8,\eta^8,2) + \lim_{\eta\to\infty} S(2,2,\eta^4) + \lim_{\eta\to\infty} S(2,2,\eta^{16})\right\} = \infty.$$

In this example the pair (f,g) does not satisfies various types of compatible mappings mentioned above.

Example 1.11. Consider the S-metric space (X,S) where $X=[0,2]$ and $S(\alpha,\beta,\gamma)=|\alpha-\gamma|+|\beta-\gamma|$ for $\alpha,\beta,\gamma\in X$. The self maps f and g are defined as follows.

$$f(\alpha)=\begin{cases}\frac{1}{2}+\alpha & \text{for } \alpha\in[0,\frac{1}{2}),\\ 2 & \text{for } \alpha=\frac{1}{2},\\ 1 & \text{for } \alpha\in]\frac{1}{2},2],\end{cases} \qquad g(\alpha)=\begin{cases}\frac{1}{2}-\alpha & \text{for } \alpha\in[0,\frac{1}{2}),\\ 1 & \text{for } \alpha=\frac{1}{2},\\ 0 & \text{for } \alpha\in]\frac{1}{2},2].\end{cases}$$

Clearly, the defined functions are not continuous at $\alpha=1/2$.

By choosing, $\{\alpha_\eta\} = \frac{1}{\eta}$, $\eta\geq 1$, $fa_\eta\to\frac{1}{2}$, $ga_\eta\to\frac{1}{2}$.

Also, $ff(\alpha_\eta)\to 1$, $gf(\alpha_\eta)\to 0$, $fg(\alpha_\eta)\to 1$, $gg(\alpha_\eta)\to 0$ as $\eta\to\infty$.

The pair (f,g) is neither compatible nor compatible of type (A) but it satisfies type (B).

Example 1.12. Consider the S-metric space (X,S), where $X=[1,10]$ and $S(\alpha,\beta,\gamma)=|\alpha-\gamma|+|\beta-\gamma|$ for $\alpha,\beta\gamma\in X$. The self maps f and g are defined as follows.

$$f(\alpha)=\begin{cases}2 & \text{for } \alpha\in[1,2],\\ 4 & \text{for } \alpha\in(2,8],\\ \alpha-6 & \text{for } \alpha\in(8,10],\end{cases}\qquad g(\alpha)=\begin{cases}2 & \text{for } \alpha\in[1,2]\cup(8,10],\\ 3 & \text{for } \alpha\in(2,8].\end{cases}$$

Choosing the sequence $\{\alpha_\eta\}=8+\frac{1}{\eta}$, $\eta\geq 1$.
Also $ff(\alpha_\eta)\to 4$, $gf(\alpha_\eta)\to 3$, $fg(\alpha_\eta)\to 2$, $gg(\alpha_\eta)\to 2$ as $\eta\to\infty$.
Here the pair (f,g) is only compatible of type (C).

2. Main Result

Theorem 2.1. *Suppose that $A,B,P,$ and Q are self maps of a complete S-metric space (X,S) with satisfying the following conditions:*

(ζ-1) $A(X)\subseteq Q(X)$ *and* $B(X)\subseteq P(X)$

(ζ-2) $S(A\alpha,A\beta,B\gamma)\leq q\cdot\max\{S(P\alpha,P\beta,Q\gamma),S(A\alpha,A\alpha,P\alpha),S(B\gamma,B\gamma,Q\gamma),S(A\beta,A\beta,B\gamma)\}$,

for each $\alpha,\beta,\gamma\in X$ *with* $q\in(0,1)$, *and*

(ζ-3) *the pairs (A,P) and (B,Q) are compatible of type (A).*

Then a unique common fixed point exists in X for A,B,P and Q provided any one of these mappings is continuous.

Proof. Let $\alpha_0\in X$, since $A(X)$ is contained in $Q(X)$, there exist $\alpha_1\in X$ such that $A\alpha_0=Q\alpha_1$, and also as $B\alpha_1\in P(X)$, we choose $\alpha_2\in X$ such that $B\alpha_1=P\alpha_2$. In general, $\alpha_{2\eta+1}\in X$ is chosen such that $A\alpha_{2\eta}=Q\alpha_{2\eta+1}$ and $B\alpha_{2\eta+1}=P\alpha_{2\eta+2}$. We obtain a $\{\beta_\eta\}$ in X such that

$$\beta_{2\eta}=A\alpha_{2\eta}=Q\alpha_{2\eta+1},\quad \beta_{2\eta+1}=B\alpha_{2\eta+1}=P\alpha_{2\eta+2},\quad \eta\in 0.$$

Now, we show that $\{\beta_\eta\}$ is a Cauchy sequence. For this, we have

$$\begin{aligned}S(\beta_{2\eta},\beta_{2\eta}\beta_{2\eta+1})&=S(A\alpha_{2\eta},A\alpha_{2\eta},B\alpha_{2\eta+1})\\ &\leq q\cdot\max\{S(P\alpha_{2\eta},P\alpha_{2\eta},Q\alpha_{2\eta+1}),S(A\alpha_{2\eta},A\alpha_{2\eta},P\alpha_{2\eta}),\\ &\qquad S(B\alpha_{2\eta+1},B\alpha_{2\eta+1},Q\alpha_{2\eta+1}),S(A\alpha_{2\eta},A\alpha_{2\eta},B\alpha_{2\eta+1})\}\\ &=q\cdot\max\{S(\beta_{2\eta-1},\beta_{2\eta-1},\beta_{2n}),S(\beta_{2\eta},\beta_{2\eta},\beta_{2\eta+1})\}.\end{aligned}$$

If $S(\beta_{2\eta},\beta_{2\eta},\beta_{2\eta+1})>S(\beta_{2\eta-1},\beta_{2\eta-1},\beta_{2n})$ then from the above inequality we have

$$S(\beta_{2\eta},\beta_{2\eta},\beta_{2\eta+1})>q\cdot S(\beta_{2\eta},\beta_{2\eta}\beta_{2\eta+1})$$

which is a contradiction.
Therefore,

$$S(\beta_{2\eta},\beta_{2\eta}\beta_{2\eta+1})\leq S(\beta_{2\eta-1},\beta_{2\eta-1},\beta_{2n}).$$

Hence

$$S(\beta_{2\eta},\beta_{2\eta}\beta_{2\eta+1})\leq q\cdot S(\beta_{2\eta-1},\beta_{2\eta-1},\beta_{2n}). \tag{2.1}$$

By similar arguments,

$$\begin{aligned}S(\beta_{2\eta-1},\beta_{2\eta-1},\beta_{2n}) &\geq S(\beta_{2\eta},\beta_{2\eta},\beta_{2\eta-1})\\ &= S(A\alpha_{2\eta},A\alpha_{2\eta},B\alpha_{2\eta-1})\\ &\leq q\cdot\max\{S(P\alpha_{2\eta},P\alpha_{2\eta},Q\alpha_{2\eta-1}),S(A\alpha_{2\eta},A\alpha_{2\eta},P\alpha_{2\eta}),\\ &\qquad S(B\alpha_{2\eta-1},B\alpha_{2\eta-1},Q\alpha_{2\eta-1}),S(A\alpha_{2\eta},A\alpha_{2\eta},B\alpha_{2\eta-1})\}\\ &= q\cdot\max\{S(\beta_{2\eta-2},\beta_{2\eta-2},\beta_{2\eta-1}),S(\beta_{2\eta},\beta_{2\eta},\beta_{2\eta-1})\}.\end{aligned}$$

Now if, $S(\beta_{2\eta},\beta_{2\eta},\beta_{2\eta-1})>S(\beta_{2\eta-2},\beta_{2\eta-2},\beta_{2\eta-1})$ then from the above inequality, we have

$$S(\beta_{2\eta},\beta_{2\eta},\beta_{2\eta-1})<q\cdot S(\beta_{2\eta},\beta_{2\eta},\beta_{2\eta-1}),$$

which is a contradiction.

Hence

$$S(\beta_{2\eta-1},\beta_{2\eta-1},\beta_{2\eta})\leq S(\beta_{2\eta-2},\beta_{2\eta-2},\beta_{2\eta-1}).$$

Therefore, by the above inequality we have

$$S(\beta_{2\eta-1},\beta_{2\eta-1},\beta_{2\eta})\leq q\cdot S(\beta_{2\eta-2},\beta_{2\eta-2},\beta_{2\eta-1}). \tag{2.2}$$

From (2.1) and (2.2), we have $S(\beta_\eta,\beta_\eta,\beta_\eta-1)\leq q\cdot S(\beta_{\eta-1},\beta_{\eta-1},\beta_{\eta-2})$ for $\eta\geq 2$ where $q\in(0,1)$.
Hence for $\eta\geq 2$, it follows,

$$S(\beta_\eta,\beta_\eta,\beta_\eta-1)\leq q\cdot S(\beta_{\eta-1},\beta_{\eta-1},\beta_{\eta-2})\leq\leq q^{n-1}\cdot S(\beta_1,\beta_1,\beta_0). \tag{2.3}$$

By the triangular inequality of S-metric space, for $\eta>\mu$, we have

$$\begin{aligned}S(\beta_\eta,\beta_\eta,\beta_\mu) &\leq 2S(\beta_\mu,\beta_\mu,\beta_{\mu+1})+2\cdot S(\beta_{\mu+1},\beta_{\mu+1},\beta_{\mu+2})+\cdots+S(\beta y_{\eta-1},\beta_{\eta-1},\beta_\eta)\\ &< 2\cdot S(\beta_\mu,\beta_\mu,\beta_{\mu+1})+2\cdot S(\beta_{\mu+1},y\beta_{\mu+1},\beta y_{\mu+2})+\cdots+2S(\beta_{\eta-1},\beta_{\eta-1},\beta_\eta).\end{aligned}$$

Hence from (2.3), and as $q\in(0,1)$, we have

$$\begin{aligned}S(\beta_\eta,\beta_\eta,\beta_\mu) &\leq 2\cdot(q^\mu+q^{\mu+1}+\ldots+q^{\eta-1})\,S(\beta_1,\beta_1,\beta_0)\\ &\leq 2q^\mu(1+q+q^2+\ldots)\,S(\beta_1,\beta_1,\beta_0)\\ &\leq 2\left(\frac{q^m}{1-q}\right)\,S(\beta_1,\beta_1,\beta_0)\to 0,\ \text{as } \mu\to\infty.\end{aligned}$$

This gives $\{\beta_\eta\}$ is a Cauchy sequence in (X,S).
Since X is a complete S-metric space, then there is some $t\in X$ such that

$$\lim_{\eta\to\infty}Aa_{2\eta}=\lim_{\eta\to\infty}Q\alpha_{2\eta+1}=\lim_{\eta\to\infty}B\alpha_{2\eta+1}=\lim_{\eta\to\infty}P\alpha_{2\eta+2}=t.$$

Now, we prove that t is a common fixed point for the mappings A,B,P and Q.

Case 1: Let Q be a continuous function it follows that $\lim_{\eta\to\infty}QQ\alpha_{2\eta+1}=Qt$. Since the pair (B,Q) is compatible of type (A), then

$$\lim_{\eta\to\infty}S(BQ\alpha_{2\eta+1},BQ\alpha_{2\eta+1},QQ\alpha_{2\eta+1})=0$$

and

$$\lim_{\eta\to\infty}S(QB\alpha_{2\eta+1},QB\alpha_{2\eta+1},BB\alpha_{2\eta+1})=0.$$

Using the continuity condition, we get

$$\lim_{\eta\to\infty} BQ\alpha_{2\eta+1} = Qt.$$

By substituting $\alpha = \beta = \alpha_{2\eta}$ and $\gamma = Q\alpha_{2\eta+1}$ in (ζ-2), we obtain

$$S(A\alpha_{2\eta},A\alpha_{2\eta},BQ\alpha_{2\eta+1}) \le q\cdot\max\{S(P\alpha_{2\eta},P\alpha_{2\eta},QQ\alpha_{2\eta+1}),S(A\alpha_{2\eta},A\alpha_{2\eta},P\alpha_{2\eta}), S(BQ\alpha_{2\eta+1},BQ\alpha_{2\eta+1},QQ\alpha_{2\eta+1}), S(A\alpha_{2\eta},A\alpha_{2\eta},BQ\alpha_{2\eta+1})\}.$$

On taking limit as η, we obtain $S(t,t,Qt) \le q\cdot\max\{S(t,t,Qt),0,0,S(t,t,Qt)\}$, this gives

$$Qt = t. \tag{2.4}$$

Now by substituting $\alpha = \beta = \alpha_{2\eta}$ and $\gamma = t$ in (ζ-2), then we have

$$S(A\alpha_{2\eta},A\alpha_{2\eta},Bt) \le q\cdot\max\{S(P\alpha_{2\eta},P\alpha_{2\eta},Qt),S(A\alpha_{2\eta},A\alpha_{2\eta},P\alpha_{2\eta}),S(Bt,Bt,Qt), S(A\alpha_{2\eta},A\alpha_{2\eta},Bt)\}$$

$$\Rightarrow\quad S(t,t,Bt) \le q\cdot\max\{S(t,t,Bt)\} \qquad (\text{since } S(\alpha,\alpha,\gamma) = S(\gamma,\gamma,\alpha)).$$

This proves

$$Bt = t. \tag{2.5}$$

Since from the second condition of (ζ-1) that is $B(X) \subseteq P(X)$, there exists a point $u \in X$ such that

$$Bt = Pu = t. \tag{2.6}$$

Putting $\alpha = \beta = u$ and $\gamma = t$ in (ζ-2), then we have

$$S(Au,Au,Bt) \le q\cdot\max\{S(Pu,Pu,Qt),S(Au,Au,Pu),S(Bt,Bt,Qt),S(Au,Au,Bt)\}$$

$$\Rightarrow\quad S(t,t,Au) \le q\cdot\max\{S(t,t,Au)\}.$$

This proves that $Au = t$. Therefore,

$$Au = Pu = t. \tag{2.7}$$

Since the pair (A,P) is compatible of type (A), we have

$$\lim_{\eta\to\infty} S(AP\alpha_{2\eta},AP\alpha_{2\eta},PP\alpha_{2\eta}) = 0$$

and

$$\lim_{\eta\to\infty} S(PA\alpha_{2\eta},PA\alpha_{2\eta},AA\alpha_{2\eta}) = 0.$$

Let $\{\alpha_{2\eta}\}$ be a sequence such that $\lim_{n\to\infty} x_{2\eta} = u$.

On putting $\alpha_{2\eta} = u$, we get

$$S(APu,APu,PPu) = 0 \text{ and } S(PAu,PAu,AAu) = 0$$

and since $Au = Pu = t$ which gives

$$At = Pt. \tag{2.8}$$

Now put $\alpha = \beta = \gamma = t$ in (ζ-2), we have

$$S(At,At,Bt) \le q\cdot\max\{S(Pt,Pt,Qt),S(At,At,Pt),S(Bt,Bt,Qt),S(At,At,Bt)\}.$$

This proves

$$At = tt. \tag{2.9}$$

Therefore, we have $At = Pt = Qt = Bt = t$, which proves 't' is a common fixed point of these four mappings of S-metric space.

Case 2: Let B is continuous it follows $\lim_{\eta\to\infty} BB\alpha_{2\eta+1} = \lim_{\eta\to\infty} BQ\alpha_{2\eta+1} = Bt$.

Since (B,Q) is compatible of type (A), we have

$$\lim_{\eta\to\infty} S(BQ\alpha_{2\eta+1}, BQ\alpha_{2\eta+1}, QQ\alpha_{2\eta+1}) = \lim_{\eta\to\infty} S(QB\alpha_{2\eta+1}, QB\alpha_{2\eta+1}, BB\alpha_{2\eta+1}) = 0.$$

Using the continuity condition which gives

$$\lim_{\eta\to\infty} QQ\alpha_{2\eta+1} = Bt.$$

Put $\alpha = \beta = \alpha_{2\eta}$ and $\gamma = Q\alpha_{2\eta+1}$ in condition (ζ-2), we obtain

$$\begin{aligned} S(A\alpha_{2\eta}, A\alpha_{2\eta}, BQ\alpha_{2\eta+1}) \le q \cdot \max\{ & S(P\alpha_{2\eta}, P\alpha_{2\eta}, QQ\alpha_{2\eta+1}), S(A\alpha_{2\eta}, A\alpha_{2\eta}, P\alpha_{2\eta}), \\ & S(BQ\alpha_{2\eta+1}, BQ\alpha_{2\eta+1}, QQ\alpha_{2\eta+1}), \\ & S(A\alpha_{2\eta}, A\alpha_{2\eta}, BQ\alpha_{2\eta+1})\}. \end{aligned}$$

Now on taking limit as $\eta \to \infty$,

$$\Rightarrow \quad S(t,t,Bt) \le q \cdot \max\{S(t,t,Bt)\}$$

it proves that

$$Bt = t. \tag{2.10}$$

Since from (ζ-1) we have, $B(X) \subseteq P(X)$ then there exists a point $u \in X$ such that

$$Bt = Pu = t.$$

Put $\alpha = \beta = u$ and $\gamma = Q\alpha_{2\eta+1}$ in (ζ-2)

$$\begin{aligned} S(Au, Au, BQ\alpha_{2\eta+1}) \le q \cdot \max\{ & S(Pu, Pu, QQ\alpha_{2\eta+1}), S(Au, Au, Pu), \\ & S(BQ\alpha_{2\eta+1}, BQ\alpha_{2\eta+1}, QQ\alpha_{2\eta+1}), S(Au, Au, BQ\alpha_{2\eta+1})\} \end{aligned}$$

$$\Rightarrow \quad S(t,t,Au) \le q \cdot \max\{S(t,t,Au)\}.$$

This gives that

$$Au = t. \tag{2.11}$$

So, we have $Au = Pu = t$. Since the pair (A,P) is compatible of type (A), then we obtain

$$At = Pt. \tag{2.12}$$

Put $\alpha = \beta = t$ and $\gamma = \alpha_{2\eta+1}$ in (ζ-2) then, we have

$$\begin{aligned} S(At, At, B\alpha_{2\eta+1}) \le q \cdot \max\{ & S(Pt, Pt, Q\alpha_{2\eta+1}), S(At, At, Pt), S(Bt, Bt, Q\alpha_{2n+1}), \\ & S(At, At, B\alpha_{2\eta+1})\} \end{aligned}$$

$$\Rightarrow \quad S(t,t,At) \le q \cdot \max\{S(t,t,At)\}$$

this gives

$$At = t. \tag{2.13}$$

Since $A(X)\subseteq Q(X)$, there exists a point $v\in X$ such that

$$At=Qv=t. \tag{2.14}$$

Put $\alpha=\beta=t$ and $\gamma=v$ in (ζ-2), then

$$S(At,At,Bv)\leq q\cdot\max\{S(Pt,Pt,Qv),S(At,At,Pt),S(Bv,Bv,Qv),S(At,At,Bv)\}$$
$$\Rightarrow\quad S(t,t,Bv)\leq q\cdot\max\{S(t,t,Bv)\}.$$

This gives that $Bv=t$. Since the pair (B,Q) is compatible of type (A) then

$$Bt=Qt. \tag{2.15}$$

From (2.10), (2.12), (2.13) and (2.15), we get $At=Pt=Bt=Qt=t$.

Similarly, we can complete the proof when either A or P is continuous mapping.

Assume that there exists another common fixed point γ^*X, such that $P\gamma^*=Q\gamma^*=A\gamma^*=B\gamma^*=\gamma^*$, then to prove the uniqueness of fixed point, $\alpha=\beta=\gamma^*$ and $\gamma=t$

$$\begin{aligned}S(\gamma^*,\gamma^*,t)&=S(A\gamma^*,A\gamma^*,Bt)\\&\leq q\cdot\max\{S(P\gamma^*,P\gamma^*,Qt),S(A\gamma^*,A\gamma^*,P\gamma^*),S(Bt,Bt,Qt),S(A\gamma^*,A\gamma^*,Bt)\}\\&=q\cdot S(\gamma^*,\gamma^*,t)\end{aligned}$$

which shows $\gamma^*=t$. Thus t is the unique common fixed point for mappings A,B,P, and Q. □

Theorem 2.2. *Suppose the self mappings A,B,P, and Q satisfying the conditions (ζ-1), (ζ-2) and also the pairs (A,P) and (B,Q) are compatible of type (B). Then a unique common fixed point exists in X provided the one these mappings is continuous.*

Proof. Choose a Cauchy sequence $\{\beta_\eta\}$ in X such that

$$\beta_{2\eta}=A\alpha_{2\eta}=Q\alpha_{2\eta+1},\ \beta_{2\eta+1}=B\alpha_{2\eta+1}=P\alpha_{2\eta+2},\quad \eta\geq 0. \tag{2.16}$$

Case 1: Let Q is continuous it follows that $\lim\limits_{\eta\to\infty}QQ\alpha_{2\eta+1}=\lim\limits_{\eta\to\infty}QB\alpha_{2\eta+1}=Qt$. Since the pair (B,Q) is compatible of type (B), then we have

$$\begin{aligned}\lim_{\eta\to\infty}S(BQ\alpha_{2\eta+1},BQ\alpha_{2\eta+1},QQ\alpha_{2\eta+1})\leq\frac{1}{2}\Big\{&\Big[\lim_{\eta\to\infty}S(QB\alpha_{2\eta+1},QB\alpha_{2\eta+1},Qt)\Big]\\&+\Big[\lim_{\eta\to\infty}S(Qt,Qt,QQ\alpha_{2\eta+1})\Big]\Big\}\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\lim_{\eta\to\infty}S(QB,QB\alpha_{2\eta+1},BB\alpha_{2\eta+1})\leq\frac{1}{2}\Big\{&\Big[\lim_{\eta\to\infty}S(QB\alpha_{2\eta+1},QB\alpha_{2\eta+1},Qt)\Big]\\&+\Big[\lim_{\eta\to\infty}S(Qt,Qt,QQ\alpha_{2\eta+1})\Big]\Big\},\end{aligned} \tag{2.18}$$

from the condition (2.18),

$$\lim_{\eta\to\infty}S(Qt,Qt,BB\alpha_{2\eta+1})\leq\frac{1}{2}\Big\{\Big[\lim_{\eta\to\infty}S(Qt,Qt,Qt)\Big]+\Big[\lim_{\eta\to\infty}S(Qt,Qt,Qt)\Big]\Big\}.$$

So

$$\lim_{\eta\to\infty}BB\alpha_{2\eta+1}=Qt. \tag{2.19}$$

Now putting $\alpha=\beta=\alpha_{2\eta}$ and $\gamma=B\alpha_{2\eta+1}$ in (ζ-2), we obtain

$$S(A\alpha_{2\eta},A\alpha_{2\eta},BB\alpha_{2\eta+1})\le q\cdot\max\{S(P\alpha_{2\eta},P\alpha_{2\eta},QB\alpha_{2\eta+1}),S(A\alpha_{2\eta},A\alpha_{2\eta},P\alpha_{2\eta}),\\ S(BB\alpha_{2\eta+1},BB\alpha_{2\eta+1},QB\alpha_{2\eta+1}),\\ S(A\alpha_{2\eta},A\alpha_{2\eta},BB\alpha_{2\eta+1})\}.$$

On taking limit as η, and using (2.19) it gives

$$Qt=t. \tag{2.20}$$

Put $\alpha=\beta=\alpha_{2\eta}$ and $\gamma=t$ in (ζ-2) and applying limit as η, we get

$$S(t,t,Bt)\le q\cdot\max\{S(t,t,Bt)\}.$$

This gives

$$Bt=t. \tag{2.21}$$

Since $B(X)\subseteq P(X)$, there exists a point $u\in X$ such that $Bt=Pu=t$.

Putting, $\alpha=\beta=u$ and $\gamma=t$ in (ζ-2), then we have $Au=t$. Since the pair (A,P) is compatible of type (B), we have

$$\lim_{\eta\to\infty}S(AP\alpha_{2\eta},AP\alpha_{2\eta},PP\alpha_{2\eta})\le\frac{1}{2}\left\{\left[\lim_{\eta\to\infty}S(AP\alpha_{2\eta},AP\alpha_{2\eta},At)\right]+\left[\lim_{\eta\to\infty}S(At,At,AA\alpha_{2\eta})\right]\right\}.$$

Let $\langle\alpha_{2\eta}\rangle$ be a sequence such that $\lim_{\eta\to\infty}\alpha_{2\eta}=u$. Putting $\alpha_{2\eta}=u$ in the above, then

$$S(APu,APu,PPu)\le\frac{1}{2}\{[S(APu,APu,At)]+[(At,At,AAu)]\}$$

$$\Rightarrow\quad S(At,At,Pt)\le\frac{1}{2}\{[S(At,At,At)]+[S(At,At,At)]\}$$

which proves

$$At=Pt. \tag{2.22}$$

Now put $\alpha=\beta=\gamma=t$ in (ζ-2), we have $S(t,t,At)\le q\cdot\max\{S(t,t,At)\}$ this gives

$$At=t. \tag{2.23}$$

From (2.20), (2.21), (2.22) and (2.23), we get $At=Pt=Bt=Qt=t$.

Correspondingly, we can prove when the mappings A or P or B is continuous. We can easily prove the uniqueness of the fixed point for these mappings.

Theorem 2.3. *Suppose the self maps $A,B,P,$ and Q of a complete S-metric space (X,S), are satisfying (ζ-1), (ζ-2) with pairs (A,P) and (B,Q) are compatible of type (C). Then a unique common fixed point exists in X for these mappings provided any one these mappings is continuous.*

Proof. Choose a cauchy sequence $\{\beta_\eta\}$ in X such that

$$\beta_{2\eta}=A\alpha_{2\eta}=Q\alpha_{2\eta+1},\ \beta_{2\eta+1}=B\alpha_{2\eta+1}=P\alpha_{2\eta+2},\quad \eta\ge 0.$$

Case 1: Let Q is continuous it follows that

$$\lim_{\eta\to\infty}QQ\alpha_{2\eta+1}=\lim_{\eta\to\infty}QB\alpha_{2\eta+1}=Qt. \tag{2.24}$$

Since the pair (B,Q) is compatible of type (C), we have

$$\lim_{\eta\to\infty} S(BQ\alpha_{2\eta+1},BQ\alpha_{2\eta+1},QQ\alpha_{2\eta+1}) \le \frac{1}{3}\left\{\left[\lim_{\eta\to\infty} S(BQ\alpha_{2\eta+1},BQ\alpha_{2\eta+1},Bt)\right] + \left[\lim_{\eta\to\infty} S(Bt,Bt,BB\alpha_{2\eta+1})\right] + \left[\lim_{\eta\to\infty} S(Bt,Bt,QQ\alpha_{2\eta+1})\right]\right\}. \tag{2.25}$$

and

$$\lim_{\eta\to\infty} S(QB\alpha_{2\eta+1},QB\alpha_{2\eta+1},BB\alpha_{2\eta+1}) \le \frac{1}{3}\left\{\left[\lim_{\eta\to\infty} S(QB\alpha_{2\eta+1},QB\alpha_{2\eta+1},Qt)\right] + \left[\lim_{\eta\to\infty} S(Qt,Qt,QQ\alpha_{2\eta+1})\right] + \left[\lim_{\eta\to\infty} S(Qt,Qt,BB\alpha_{2\eta+1})\right]\right\}. \tag{2.26}$$

From the condition (2.26) and (2.24) we get

$$\lim_{\eta\to\infty} S(Qt,Qt,BB\alpha_{2\eta+1}) \le \frac{1}{3}\left\{\left[\lim_{\eta\to\infty} S(Qt,Qt,Qt)\right] + \left[\lim_{\eta\to\infty} S(Qt,Qt,Qt)\right] + \left[\lim_{\eta\to\infty} S(Qt,Qt,BB\alpha_{2\eta+1})\right]\right\}.$$

So

$$\lim_{\eta\to\infty} BB\alpha_{2\eta+1} = Qt. \tag{2.27}$$

Now putting $\alpha=\beta=\alpha_{2\eta}$ and $\gamma=B\alpha_{2\eta+1}$ in (ζ-2), we obtain

$$S(A\alpha_{2\eta},A\alpha_{2\eta},BB\alpha_{2\eta+1}) \le q\cdot\max\{S(P\alpha_{2\eta},P\alpha_{2\eta},QB\alpha_{2\eta+1}),S(A\alpha_{2\eta},A\alpha_{2\eta},P\alpha_{2\eta}), S(BB\alpha_{2\eta+1},BB\alpha_{2\eta+1},QB\alpha_{2\eta+1}), S(A\alpha_{2\eta},A\alpha_{2\eta},BB\alpha_{2\eta+1})\}.$$

On taking limit as η, we obtain $S(t,t,Qt)\le q\cdot\max\{S(t,t,Qt),0,0,S(t,t,Qt)\}$, it gives

$$Qt = t. \tag{2.28}$$

Put $\alpha=\beta=\alpha_{2\eta}$ and $\gamma=t$ in (ζ-2), then we have $S(t,t,Bt)\le q\cdot\max\{S(t,t,Bt)\}$ this gives

$$Bt = t. \tag{2.29}$$

Since $B(X)\subseteq P(X)$, there exists a point $u\in X$ such that $Bt=Pu=t$.

In (ζ-2), now by putting, $\alpha=\beta=u$ and $\gamma=t$ in (ζ-2), then we have $Au=t$.

Since the pair (A,P) is compatible of type (C), we have

$$\lim_{\eta\to\infty} S(AP\alpha_{2\eta},AP\alpha_{2\eta},PP\alpha_{2\eta}) \le \frac{1}{3}\left\{\left[\lim_{\eta\to\infty} S(AP\alpha_{2\eta},AP\alpha_{2\eta},At)\right] + \left[\lim_{\eta\to\infty} S(At,At,AA\alpha_{2\eta})\right] + \left[\lim_{\eta\to\infty} S(At,At,PP\alpha_{2\eta})\right]\right\}.$$

Let $\langle \alpha_{2\eta} \rangle$ be a sequence such that $\lim_{\eta\to\infty} \alpha_{2\eta} = u$. On putting $\alpha_{2\eta} = u$ in the above, then

$$S(APu, APu, PPu) \leq \frac{1}{3}\{[S(APu, APu, At)] + [S(At, At, AAu)] + [S(Pt, Pt, PPu)]\} \tag{2.30}$$

which gives

$$At = Pt. \tag{2.31}$$

Now put $\alpha = \beta = \gamma = t$ in (ζ-2), then we have

$$At = t. \tag{2.32}$$

Therefore, from (2.28), (2.29), (2.31) and (2.32) we have $At = Pt = Qt = Bt = t$.

Similarly, we can prove when other three functions are continuous. □

3. Conclusion

In this paper, we proved three unique common fixed point theorems for four self mappings in the complete S-metric space. Two pair of mappings are satisfying compatible mappings of type (A) in Theorem 2.1, type (B) in Theorem 2.2 and type (C) in Theorem 2.3 with one of the mappings is considered as continuous.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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