**Communications in Mathematics and Applications**
Vol. 13, No. 3, pp. 877–892, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i3.1818

http://www.rgnpublications.com

Research Article

# Maximum Total Irregularity of Totally Segregated Extended Bicyclic Graphs

T.F. Jorry

*Department of Mathematics, Mercy College, Palakkad 678006, Kerala, India*
srjorrytf@gmail.com

**Received:** February 16, 2022 **Accepted:** August 8, 2022

**Abstract.** In this paper, we determine maximum total irregularity of two types of connected totally segregated bicyclic graphs on $n$ vertices: extended $\infty$ bicyclic graph and $\Theta$-bicyclic graph and also characterize those extremal graphs.

**Keywords.** Total irregularity, Totally segregated graph, Extended bicyclic graph

**Mathematics Subject Classification (2020).** 05C30



## 1. Introduction

In this paper, we consider only simple undirected connected graphs. As well known, a graph whose vertices have equal degrees is said to be regular. Then, a graph in which all the vertices do not have equal degrees can be viewed as somehow deviating from regularity. In mathematical literature, several measures of such 'irregularity' were proposed [3] [6] [5] [4]. A measure of 'irregularity' was put forward by Albertson [2]. Albertson defines irregularity of $G$ as

$$irr(G) = \sum_{uv \in E(G)} |\deg_G(u) - \deg_G(v)|. \tag{1.1}$$

The most investigated irregularity measure is the *Total Irregularity* of a graph. It is found by Abdo *et al.* in [1], as:

$$irr_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)|. \tag{1.2}$$

In this paper, we focus on two types of totally segregated bicyclic graphs on n vertices with maximum total irregularity. The notion of totally segregated graph is defined in [7]. A connected graph $G$ is said to be totally segregated, if $uv \in E(G)$, $\deg_G(u) \neq \deg_G(v)$. Three types of bicyclic graphs are introduced in [9].

Minimum total irregularity of totally segregated $\infty$ bicyclic graph is found in [8]. In this paper we find maximum total irregularity of two types of totally segregated bicyclic graphs on $n$ vertices.

## 2. Totally Segregated Extended Bicyclic Graphs

A bicyclic graph is a simple connected graph in which the number of edges is exactly one more than the number of vertices. Here our focus is on extended bicyclic graph. Extended $\infty$ bicyclic graph is a bicyclic graph constructed by attaching trees to the basic bicycle denoted by $\infty(p,q,l)$ (see Figure 1), is obtained from two vertex-disjoint cycles $C_p$ and $C_q$ by connecting one vertex of $C_p$ and one vertex of $C_q$ with a path $P_l$ of length $l-1$ ($l \geq 2$), where $p,q \geq 3$; and $\Theta$-bicyclic graph, is a bicyclic graph constructed by attaching trees to the basic bicycle denoted by $\theta(p,q,l)$ (see Figure 2), is a graph on $p+q-l$ vertices with the two cycles $C_p$ and $C_q$ having $l$ common vertices, where $p,q \geq 3$ and $l \geq 2$.

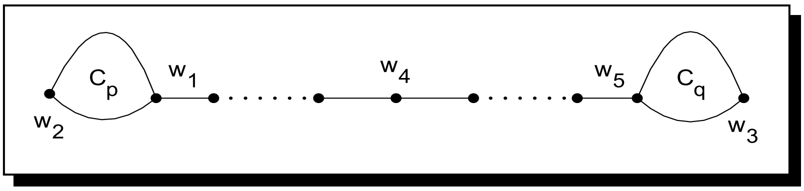


**Figure 1.** The graph $\infty(p,q,l)$ with $p \geq 3$, $q \geq 3$ and $l \geq 2$

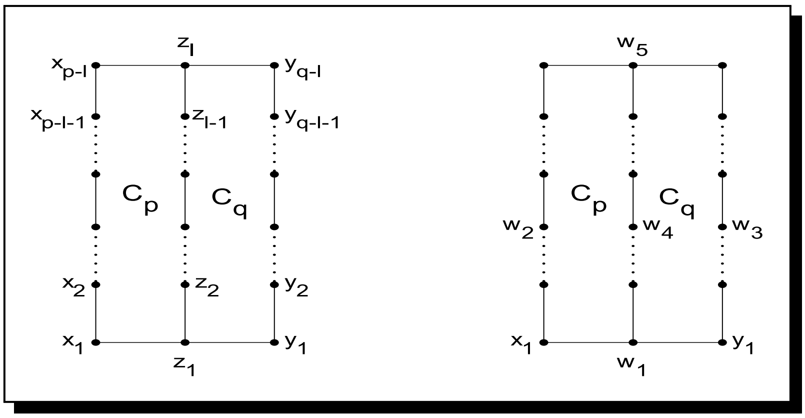


**Figure 2.** The graph $\theta(p,q,l)$ with $p \geq 3$, $q \geq 3$ and $l \geq 2$

In Figure 1, let $w_1$ be the common vertex of $P_l$ and $C_p$ and let $w_5$ be the common vertex of $P_l$ and $C_q$. Let $w_2 \in V(C_p) \setminus \{w_1\}$, $w_3 \in V(C_q) \setminus \{w_5\}$ and $w_4 \in V(P_l) \setminus \{w_1, w_5\}$ if $l \geq 3$.

In Figure 2, let $w_1 = z_1$, $w_2 \in \{x_1, x_2, \cdots, x_{p-l}\}$, $w_4 \in \{z_2, \cdots, z_{l-1}\}$ if $l \geq 3$, $w_3 \in \{y_1, y_2, \cdots, y_{q-l}\}$, and $w_5 = z_l$. Let $P_n$, $C_n$ and $S_n$ be the path, cycle, and star on $n$ vertices, respectively. A rooted graph has one of its vertices, called the root, distinguished from the others. Root of the star $S_n$ is its central vertex. Let $G_1$ and $G_2$ be two graphs: $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. The graph $G = (G_1, v_1) \circ (G_2, v_2)$ denote the resultant graph by identifying $v_1$ with $v_2$. Let $x \in V(\infty(p,q,l))$ and $v$ be the root of the rooted tree $T$. Take $\infty(p,q,l,x \circ T) = (\infty(p,q,l,x)) \circ (T, v)$. In this case we say that tree $T$ is attached to the graph $\infty(p,q,l)$ at $x$ (for example, see Figure 3).

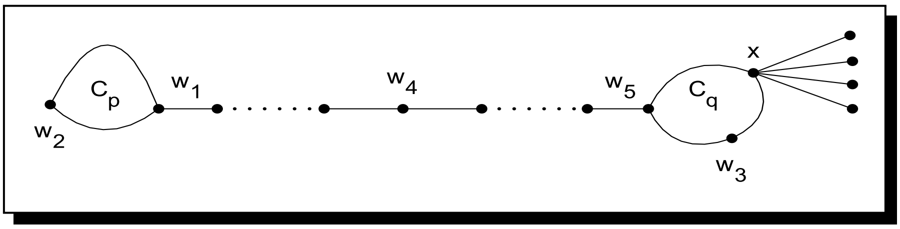


**Figure 3.** The graph $\infty(p,q,l,x \circ S_5)$

**Remark 2.1.** Let $S_n$ be a star on $n$ vertices. If a star $S_2$ is attached to $\infty(p,q,l)$ ($p \geq 3$, $q \geq 3$) at $w_2$, the resultant graph $G$ is denoted by $\infty(p,q,l,w_2 \circ S_2)$.
Note that $\Theta(p,q,l,w_1 \circ T) \cong \Theta(p,q,l,w_5 \circ T)$, ($p \geq 3$, $q \geq 3$, $l \geq 2$).

The set denoted by $B_n^+(C_p \circ T_1, C_q \circ T_2)$ is the set of those graphs each of which is an $\infty^+$-bicyclic graph such that a tree is attached to at least one vertex (say $w_2$) in $V(C_p) \setminus \{w_1\}$ and a tree is attached to at least one vertex (say $w_3$) in $V(C_q) \setminus \{w_5\}$, where $w_1, w_2, w_3, w_5$ are as defined in Figure 1.

A totally segregated extended $\infty$ bicyclic graph is a extended $\infty$ bicyclic graph which is totally segregated (see Figure 4).

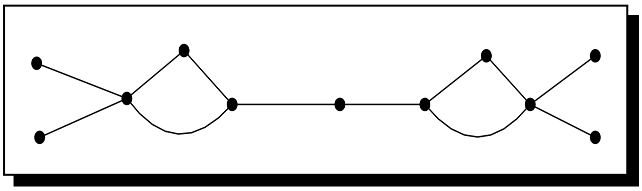

**Figure 4.** Totally segregated $\infty^+$ bicyclic graph with, basic bicycle $\infty(3,3,3)$

The extended bicyclic graph with basic bicycle $\infty(p,q,l)$, ($p \geq 3$, $q \geq 3$, $l \geq 2$) is called $\infty^+$-bicyclic graph in short and the bicyclic graph with basic bicycle $\Theta(p,q,l)$, ($p \geq 3$, $q \geq 3$, $l \geq 2$) is called $\Theta$-bicyclic graph in short.

**Remark 2.2.** For $n \leq 9$, a totally segregated $\infty^+$-bicyclic graph of order $n$ does not exist.
For $n \leq 4$, a totally segregated $\Theta$-bicyclic graph of order $n$ does not exist.

**Proposition 2.1.** *If $G$ is totally segregated $\infty^+$-bicyclic graph of order $n$, ($n \geq 10$), then* $\Delta(G) \leq n-6$.

*Proof.* Let $G$ be a totally segregated $\infty^+$-bicyclic graph on $n$ vertices. Let $C_p$ and $C_q$ ($p \geq 3$, $q \geq 3$) be two cycles in $G$ and let $P_l$, $l \geq 2$ be the path connecting one vertex of $C_p$ and one vertex of $C_q$ of length $l-1$ where $l \geq 2$.

Delete one edge $e$ of $P_l$ and get two components $C_1$ and $C_2$. Then each component has at least 5 vertices and $V(G) = V(C_1) \cup V(C_2)$, $E(G) = E(C_1) \cup E(C_2) \cup \{e\}$. Let $u$ be a vertex of maximum degree in $G$. If $u \in V(C_1)$, $u$ is not adjacent to at least 4 vertices of $V(C_2)$ and one vertex of $V(C_1)$ in $G$. In similar manner, if $u \in V(C_2)$, $u$ is not adjacent to at least 5 vertices of $G$. Hence $\Delta(G) \leq n-6$. □

Totally segregated $\infty^+$ bicyclic graph $G$ of order $n$ with basic bicycle $\infty(3,3,2)$ and $\Delta(G) = n-6$ for $n = 11$ is presented in Figure 5.

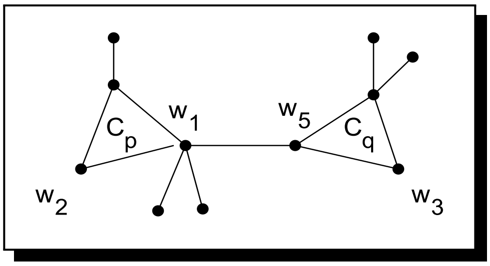


**Figure 5.** Totally segregated $\infty^+$ graph $G$ with, basic bicycle $\infty(3,3,2)$ and $\Delta(G) = n-6$ for $n = 11$

**Proposition 2.2.** *If $G$ is a totally segregated $\Theta$-bicyclic graph of order $n$, ($n \geq 5$), then $\Delta(G) \leq n-1$.*

*Proof.* For any graph $G$, $\Delta(G) \leq n-1$. There exists a totally segregated bicyclic graph $G$ with basic bicycle $\theta(p,q,l)$, ($p \geq 3$, $q \geq 3$, $l \geq 2$) and $\Delta(G) = n-1$ (see Figure 6).

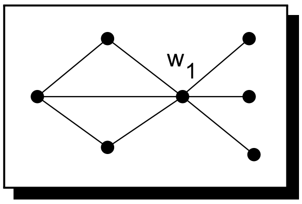


**Figure 6.** TSB graph $G$ of order $n$ with, basic bicycle $\theta(3,3,2)$ and $\Delta(G) = n-1$ for $n = 7$

□

## 3. Maximum Total Irregularity of Totally Segregated Extended Bicyclic Graphs

**Definition 3.1** ($\alpha$-Transformation [9]). Let $G = (V,E)$ be a bicyclic graph with basic bicycle $\infty(p,q,l)$ ($l \geq 1$), or $\Theta(p,q,l)$ ($l \geq 2$) with rooted trees $T_1,\cdots,T_k$ ($k \geq 1$) attached and let $u \in V$ be one of the maximal degree vertices of $G$ and let $w$ be any pendant vertex of $G$ which is adjacent to vertex $y (y \neq u)$. Let $G'$ be the graph obtained from $G$ by deleting the pendant edge $yw$ and adding a pendant edge $uw$. The transformation from $G$ to $G'$ is a $\alpha$-*transformation* on $G$ (for example, see Figure 7).

Note that in Figure 7, the edge $yw \in E(T_i)$ and $u \in V(T_i)$. In fact, $yw \in E(T_j)$ for any $j \in \{1,2,\cdots,k\}$.

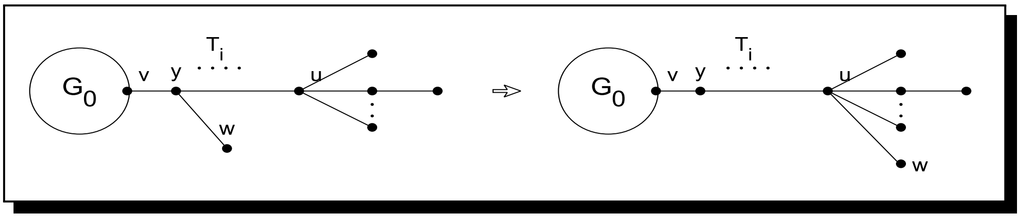


**Figure 7.** $\alpha$-transformation

**Lemma 3.1** ([9])**.** *Let $G=(V,E)$ be a $\infty^+$ bicyclic graph with basic bicycle $\infty(p,q,l)$ $(l \geq 2)$ or $\Theta(p,q,l)$ $(l \geq 2)$ with $k(\geq 1)$ rooted trees $T_1,T_2,\cdots,T_k$ attached and let $G'$ be the graph obtained from $G$ by $\alpha$-transformation. Then $irr_t(G) < irr_t(G')$.*

**Definition 3.2** ($\beta_1$ and $\beta_2$-transformation)**.** Let $G=(V,E)$ be a bicyclic graph, with basic bicycle $\infty(p,q,l)(l \geq 1)$, such that all trees attached to the basic bicycle are $S_2$ except $T$ where $T \in S^* \cup PS^*$ and $S_2$-s are attached to vertex $x$, $x \in V(G)\setminus\{w_1,w_5\}$.
Let $u \in V$ be a vertex of maximal degree and let $u_1,u_2,\cdots,u_t$ $(t \geq 1)$ be the pendant vertices adjacent to $u$. Let $G'$ be the graph obtained from $G$ by deleting the pendant edges $uu_1,uu_2,\cdots,uu_t$ and adding the pendant edges $w_1u_1,w_1u_2,\cdots,w_1u_t$. We call the transformation from $G$ to $G'$ a $\beta_1$-*transformation* on $G$ (for example, see Figure 8).
Let $u \in V$ be the vertex of maximal degree which is the root of the rooted star $S^*$ $(=S_{t+1})$ and $u_1,u_2,\cdots,u_t$ $(t \geq 2)$ be the pendant vertices adjacent to $u$. Let $G''$ be the graph obtained from $G$ by deleting the pendant edges $uu_2,\cdots,uu_t$ (except one edge) and adding the pendant edges $w_1u_2,\cdots,w_1u_t$. We call the transformation from $G$ to $G''$ a $\beta_2$-*transformation* on $G$ (for example, see Figure 9).

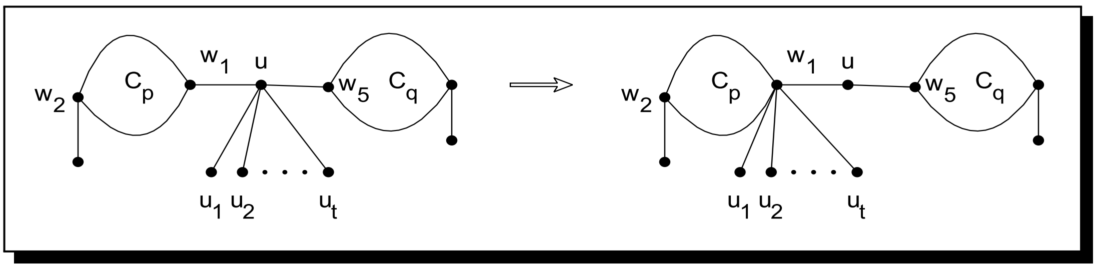


**Figure 8.** $\beta_1$-transformation on $\infty^+$-bicyclic graph with two $S_2$-s and $T \in S^*$ are attached

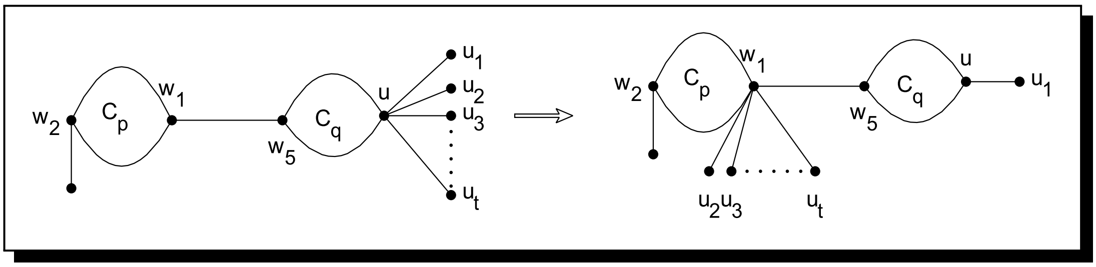


**Figure 9.** $\beta_2$-Transformation on $\infty^+$-bicyclic graph with one $S_2$ and $T \in S^*$ are attached

By Lemma 3.1 and by Definition 3.1, we have the following result.

**Lemma 3.2.** *Let $G = (V,E)$ be a totally segregated $\infty^+$-bicyclic graph on $n$ vertices*

(a) *Let $G_1 = (V,E') \in B_n^+(C_p \circ T_1, C_q \circ T_2)$ be the graph obtained from $G$ by repeating $\alpha$-transformation until one cannot get a new graph belongs to $B_n^+(C_p \circ T_1, C_q \circ T_2)$ from $G_1$. Then there exists some rooted tree $T$ such that*

$$G_1 \cong \infty(p,q,l,w\circ T, w_2\circ S_2, w_3\circ S_2), \text{ or } G_1 \cong \infty(p,q,l,w_2\circ T, w_3\circ S_2), \text{ or}$$

$$G_1 \cong \infty(p,q,l,w_2\circ S_2, w_3\circ T),$$

*where $T \in S^* \cup PS^*$, $w \in V(P_l)$, $w_2$ and $w_3$ are as defined in Figure 1 and also $irr_t(G) < irr_t(G_1)$.*

(b) *In case (a), let $u$ be a vertex of $T$ and let $u_1, u_2, \cdots, u_t$ be the pendant vertices adjacent to $u$. Then*

$$\deg_{G_1} u \geq \deg_{G_1} x, \quad \text{for all } x \in V.$$

**Lemma 3.3.** *Let $G$ be a $\infty^+$-bicyclic graph on $n$ vertices obtained as in Lemma 3.2. That is $G$ is a $\infty^+$-bicyclic graph and $G \cong \infty(p,q,l,w\circ T, w_2\circ S_2, w_3\circ S_2)$ or $G \cong \infty(p,q,l,w_2\circ T, w_3\circ S_2)$ or $G \cong \infty(p,q,l,w_2\circ S_2, w_3\circ T)$.*
*Let $T \in PS^*$ and let $v$ be the root of the rooted tree $T$. Let $u$ be a vertex of maximal degree and let $u_1, u_2, \cdots, u_t (t \geq 1)$ be the pendant vertices adjacent to $u$. If $G'$ is the graph obtained from $G$ by $\beta_1$ transformation (Figure 8) then, $irr_t(G) < irr_t(G')$.*

*Proof.* Let $G = (V,E)$. Note that the root $v$ of the rooted tree is not necessarily different from $w_1$. Clearly, we know that only the degrees of vertices $u$ and $w_1$ have been changed after the $\beta_1$-transformation; namely $\deg_{G'} u = 1$, $\deg_{G'} w_1 = \deg_G w_1 + \deg_G u - 1$ and $\deg_{G'} x = \deg_G x$ for any vertex $x \in V \setminus \{u, w_1\}$. Let $U = V \setminus \{u, w_1\}$.
It is given that the vertex $u$ is one of the maximal degree vertices of $G$; namely, $\deg_G u \geq \deg_G x$ for any vertex $x \in V$. Then

$$|\deg_{G'} u - \deg_{G'} w_1| - |\deg_G u - \deg_G w_1| = 2\deg_G w_1 - 2, \tag{3.1}$$

$$\sum_{x\in U} |\deg_{G'} u - \deg_{G'} x| - \sum_{x\in U} |\deg_G u - \deg_G x| = 2\sum_{x\in U} \deg_G x - (n-2)(\deg_G u + 1). \tag{3.2}$$

Now, we discuss $\sum_{x\in U} |\deg_{G'} w_1 - \deg_{G'} x| - \sum_{x\in U} |\deg_G w_1 - \deg_G x|$ as follows:

*Case* 1: $l \geq 2$.

Here $t \geq 2$, since $u$ is one of the maximal degree vertices of $G$. As,

$$\deg_G w_1 - \deg_G w_5 - |\deg_G w_1 - \deg_G w_5| = \begin{cases} -2, & \text{if } v = w_5; \\ 0, & \text{if } v \neq w_5, \end{cases}$$

and $\deg_G w_1 \geq \deg_G x$ for any $x \in U \setminus \{w_5\}$,

$$\begin{aligned}
&\sum_{x\in U} |\deg_{G'} w_1 - \deg_{G'} x| - \sum_{x\in U} |\deg_G w_1 - \deg_G x| \\
&\quad = \sum_{x\in U\setminus\{w_5\}} (\deg_G w_1 + \deg_G u - 1 - \deg_G x) - \sum_{x\in U\setminus\{w_5\}} (\deg_G w_1 - \deg_G x) \\
&\qquad + (\deg_G w_1 + \deg_G u - 1 - \deg_G w_5) - |\deg_G w_1 - \deg_G w_5| \\
&\quad = (n-2)(\deg_G u - 1) + (\deg_G w_1 - \deg_G w_5) - |\deg_G w_1 - \deg_G w_5| \\
&\quad \geq (n-2)(\deg_G u - 1) - 2.
\end{aligned}$$

Then, we have

$$\sum_{x\in U} |\deg_{G'} w_1 - \deg_{G'} x| - \sum_{x\in U} |\deg_G w_1 - \deg_G x| \geq (n-2)(\deg_G u - 1) - 2. \tag{3.3}$$

By equations (3.1), (3.2), (3.3) and since $\deg_G w_1 \geq 3$ and $\deg_G x \geq 1$ for any $x \in U$, we have

$$\begin{aligned}
irr_t(G') - irr_t(G) &= |\deg_{G'} u - \deg_{G'} w_1| + \sum_{x\in U} |\deg_{G'} u - \deg_{G'} x| + \sum_{x\in U} |\deg_{G'} w_1 - \deg_{G'} x| \\
&\quad - \left[ |\deg_G u - \deg_G w_1| + \sum_{x\in U} |\deg_G u - \deg_G x| + \sum_{x\in U} |\deg_G w_1 - \deg_G x| \right] \\
&\geq 2\deg_G w_1 - 2 + 2\sum_{x\in U} \deg_G x - (n-2)(\deg_G u + 1) + (n-2)(\deg_G u - 1) - 2 \\
&\geq 2\sum_{x\in U} \deg_G x \\
&> 0.
\end{aligned}$$

It follows the result. □

**Lemma 3.4.** *Let $G$ be the $\infty^+$-bicyclic graph obtained as in Lemma 3.2 and $G \cong \infty(p,q,l,u \circ T, w_2 \circ S_2, w_3 \circ S_2)$ where $u \in V(P_l), l \geq 2$ and $w_1, w_2, w_3, w_5$ are as defined in Figure 1. Let $T \in S^*$ and $u$ be the root of the rooted tree $T$ and $u_1, u_2, \cdots, u_t$ be the pendant vertices adjacent to $u$, $u \neq w_1$ and $\deg_G(u) \geq \deg_G(x)$, for all $x \in V(G)$. If $G'$ is the graph obtained from $G$ by $\beta_1$ transformation (Figure 8) then,*

(i) $G' \cong \infty(p,q,l,w_1 \circ T, w_2 \circ S_2, w_3 \circ S_2)$,

(ii) $irr_t(G) \leq irr_t(G')$ *and equality holds if and only if* $u = w_5$.

*Proof.* It is given that $u \neq w_1$. By the definition of $\beta_1$-transformation, result (i) is obvious. Now we show that result (ii) holds.

If $u = w_5$, then $G$ and $G'$ have the same degree sequence. Thus, they have the same total irregularity, i.e.,

$$irr_t(G) = irr_t(G').$$

Hence, we assume $u \neq w_5$. Note that the degrees of vertices $u$ and $w_1$ have been changed after $\beta_1$-transformation as follows:

$\deg_{G'} u = 2$, $\deg_{G'} w_1 = \deg_G w_1 + \deg_G u - 2$ and $\deg_{G'} x = \deg_G x$, for any vertex $x \in V \setminus \{u, w_1\}$.

Hence degree sequence of $G$ is $((t+2)^1, 3^4, 2^{n-t-7}, 1^{t+2})$ and degree sequence of $G'$ is

$$((t+3)^1, 3^3, 2^{n-t-6}, 1^{t+2}).$$

Then by equation (1.2) we have

$$irr_t(G') = irr_t(G) + 8.$$

It follows that

$$irr_t(G) \leq irr_t(G'). \qquad \square$$

**Lemma 3.5.** *Let $G$ be a $\infty^+$-bicyclic graph and $G \cong \infty(p,q,l,w_2 \circ T, w_3 \circ S_2)$ or $G \cong \infty(p,q,l,w_2 \circ S_2, w_3 \circ T)$, $l \geq 2$, where $T \in S^*$. Let $T = S_{t+1}$ and $u \in \{w_2, w_3\}$ be a vertex of maximal degree which is the root of the rooted tree $T$ and $u_1, u_2, \cdots, u_t$ $(t \geq 2)$ be the pendant vertices adjacent to $u$. If $G'$ is the graph obtained from $G$ by $\beta_2$ transformation (Figure 9) then*

(i) $G' \cong \infty^+(p,q,l,w_1 \circ S_t, w_2 \circ S_2, w_3 \circ S_2)$,

(ii) $irr_t(G) = irr_t(G')$.

*Proof.* By the definition of $\beta_2$-transformation (Definition 3.2) result (i) is obvious. Now, we show that result (ii) holds. Note that only the degrees of vertices $u$ and $w_1$ have been changed after the $\beta_2$-transformation; namely, $\deg_{G'} u = 3$, $\deg_{G'} w_1 = \deg_G w_1 + \deg_G u - 3$, and $\deg_{G'} x = \deg_G x$ for any vertex $x \in V \setminus \{u, w_1\}$. Let $U = V \setminus \{u, w_1\}$. Note that $t \geq 2$.

The vertex $u$ is one of the maximal degree vertices of $G$; namely, $\deg_G u \geq \deg_G x$ for any vertex $x \in V$ and $\deg_G w_1 \geq \deg_G x$ for any $x \in U$, $\deg_{G'} u \geq \deg_{G'} x$ for any vertex $x \in U$ and $\deg_{G'} w_1 \geq \deg_{G'} x$ for any $x \in U$. Then

$$|\deg_{G'} u - \deg_{G'} w_1| - |\deg_G u - \deg_G w_1| = 2\deg_G w_1 - 6, \tag{3.4}$$

$$\sum_{x \in U} |\deg_{G'} w_1 - \deg_{G'} x| - \sum_{x \in U} |\deg_G w_1 - \deg_G x| = (n-2)(t-1), \tag{3.5}$$

$$\sum_{x \in U} |\deg_{G'} u - \deg_{G'} x| - \sum_{x \in U} |\deg_G u - \deg_G x| = (n-2)(1-t). \tag{3.6}$$

By equations (3.4), (3.5), (3.6) and since $deg_G w_1 \geq 3$, $t \geq 2$ and $\deg_G u - 3 \geq 1$, we have

$$\begin{aligned} irr_t(G') - irr_t(G) &= |\deg_{G'} u - \deg_{G'} w_1| + \sum_{x \in U} |\deg_{G'} u - \deg_{G'} x| + \sum_{x \in U} |\deg_{G'} w_1 - \deg_{G'} x| \\ &\quad - [|\deg_G u - \deg_G w_1| + \sum_{x \in U} |\deg_G u - \deg_G x| + \sum_{x \in U} |\deg_G w_1 - \deg_G x|] \\ &= 2\deg_G w_1 - 6 + (n-2)(t-1) + (n-2)(1-t) = 0. \end{aligned} \tag{3.7}$$

since $\deg_G w_1 = 3$. It follows the result. $\square$

By Lemmas 3.1, 3.2, 3.3 and 3.4 we obtain:

If $p, q (\geq 3)$ are given, then

$$\max\{irr_t(G) : G \in B_n(C_p \circ T_1, C_q \circ T_2)\} = irr_t(\infty(p,q,1,w_1 \circ S_r, w_2 \circ S_2, w_3 \circ S_2)),$$

where $r = n-(p+q)$, and
if $p$, $q(\geq 3)$, $l(\geq 2)$ are given, then

$$\max\{irr_t(G) : G \in B_n^+(C_p \circ T_1, C_q \circ T_1)\} = irr_t(\infty(p,q,l,w_1 \circ S_r, w_2 \circ S_2, w_3 \circ S_2)),$$

where $r = n-(p+q+l)+1$.

**Remark 3.1.** The $\infty^+$-bicyclic graphs $\infty(p,q,l,w_1\circ S_r,w_2\circ S_2,w_3\circ S_2)$ and $\infty(p,q,l,w_5\circ S_r,w_2\circ S_2,w_3\circ S_2)$ have same degree sequence and hence same total irregularity ($w_1,w_2,w_3,w_5$ are as defined in Figure 1).

In the following theorem, the totally segregated $\infty^+$-bicyclic graph with the maximum total irregularity is determined.

Let $n \geq 10$ be a positive integer and $p,q,l$ be positive integers with $p \geq 3, q \geq 3, l \geq 2$ and $p+q+l+r-1 = n$ and $G = \infty_n(p,q,l,w_2\circ S_2,w_3\circ S_2,w_1\circ S_r)$ or $G = \infty_n(p,q,l,w_2\circ S_2,w_3\circ S_2,w_5\circ S_r)$. Clearly, the degree sequence of $G$ is $((r+2)^1,3^3,2^{p+q+l-6},1^{r+1})$. By simple calculation, by eq. (1.2) we have

$$irr_t(G) = (p+q+l-6)(2r+4)+(r+1)(r+1)+9r+3. \tag{3.8}$$

**Lemma 3.6.** *Let $n,p,q,l,r$ be positive integers with $p\geq 3$, $q\geq 3$, $l\geq 2$, $r\geq 2$ and $n=p+q+l+r-1$.*

(a) *If $p \geq 4$, then*

$$\begin{aligned} irr_t(\infty_n(p,q,l,w_2\circ S_2,w_3\circ S_2,w_1\circ S_r)) &< irr_t(\infty_n(p-1,q,l,w_2\circ S_2,w_3\circ S_3,w_1\circ S_r)) \\ &< irr_t(\infty_n(p-1,q,l,w_2\circ S_2,w_3\circ S_2,w_1\circ S_{r+1})). \end{aligned}$$

(b) *If $q \geq 4$, then*

$$\begin{aligned} irr_t(\infty_n(p,q,l,w_2\circ S_2,w_3\circ S_2,w_1\circ S_r)) &< irr_t(\infty_n(p,q-1,l,w_2\circ S_2,w_3\circ S_3,w_1\circ S_r)) \\ &< irr_t(\infty_n(p,q-1,l,w_2\circ S_2,w_3\circ S_2,w_1\circ S_{r+1})). \end{aligned}$$

(c) *If $l \geq 3$, then*

$$\begin{aligned} irr_t(\infty_n(p,q,l,w_2\circ S_2,w_3\circ S_2,w_1\circ S_r)) &< irr_t(\infty_n(p,q,l-1,w_2\circ S_2,w_3\circ S_3,w_1\circ S_r)) \\ &< irr_t(\infty_n(p,q,l-1,w_2\circ S_2,w_3\circ S_2,w_1\circ S_{r+1})). \end{aligned}$$

*Proof.* Clearly, proofs of the results (a), (b) and (c) are similar and hence we prove only the result (a).

Given the positive integers $n,p,q,r,l$ with $p \geq 3$, $q \geq 3$, $l \geq 2$, $r \geq 2$ and $p+q+l+r-1 = n$.
Let

$$G \cong \infty_n(p,q,l,w_2\circ S_2,w_3\circ S_2,w_1\circ S_r),$$

$$G_1 \cong \infty_n(p-1,q,l,w_2\circ S_2,w_3\circ S_3,w_1\circ S_r), \text{ and}$$

$$G_2 \cong \infty_n(p-1,q,l,w_2\circ S_2,w_3\circ S_2,w_1\circ S_{r+1}).$$

$G_1$ is obtained from $G$ by contracting any edge different from $w_1w_2$ of $C_p$ and adding one pendant edge at $w_3$. $G_2$ is obtained from $G$ by contracting any edge different from

$w_1w_2$ of $C_p$ and adding one pendant edge at $w_1$. Clearly, the degree sequence of $G$ is $((r+2)^1, 3^3, 2^{p+q+l-6}, 1^{r+1})$.
Degree sequence of $G_1$ is $((r+2)^1, 4^1, 3^2, 2^{p+q+l-7}, 1^{r+2})$.
Degree sequence of $G_2$ is $((r+3)^1, 3^3, 2^{p+q+l-7}, 1^{r+2})$.
By simple calculation, by equation (1.2) we have

$$irr_t(G) = (p+q+l-6)(2r+4)+(r+1)(r+1)+9r+3,$$

$$irr_t(G_1) = irr_t(G)+2(p+q+l-6)+4,$$

$$irr_t(G_2) = irr_t(G)+2(p+q+l-6)+6.$$

Hence the result. □

**Theorem 3.7.** *If $n \geq 10$ is a positive integer and $G$ is a totally segregated $\infty^+$-bicyclic graph on $n$ vertices with basic bicycle $\infty(p,q,l)$ ($p \geq 3$, $q \geq 3$, $l \geq 2$ and $p = 3$, $q = 3$, $l = 2$ does not hold simultaneously), then $irr_t(G) \leq n^2 + n - 46$ and the equality holds if and only if $G \cong \infty_n(3,3,2,w_1 \circ S_{n-8}, w_2 \circ S_2, w_3 \circ S_3)$.*

*Proof.* Let $G$ be the given totally segregated $\infty^+$-bicyclic graph with basic bicycle $\infty(p,q,l)$, $p \geq 3$, $q \geq 3$, $l \geq 2$ on $n$ vertices where $p+q+l+r-1 = n$.
Since $G$ is totally segregated there exists a vertices

$$w_2 \in V(C_p) \setminus \{w_1\} \quad \text{with } \deg w_2 \geq 3$$

and

$$w_3 \in V(C_q) \setminus \{w_5\} \quad \text{with } \deg w_3 \geq 3.$$

We prove this theorem in two stages. In the first stage, we obtain bicyclic graph $G' \cong \infty_n^+(p,q,l,w_2 \circ S_2, w_3 \circ S_2, w_1 \circ S_r)$ from totally segregated bicyclic graph $G$, by repeating $\alpha$, $\beta_1$, $\beta_2$ transformations until a new graph which belongs to $B_n^+(C_p \circ T_1, C_q \circ T_2, l)$ cannot be obtained from $G'$ by these transformations. Then, by Lemmas 3.1, 3.3, 3.4, 3.5 we know that $irr_t(G) < irr_t(G')$.
In the second stage we obtain totally segregated bicyclic graph $G'' \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ from $G'$ by repeating replacement of edges until the lengths of the cycles $C_p$, $C_q$ and path $P_l$ cannot be reduced. By Lemma 3.6, $irr_t(G') < irr_t(G'')$, where $G''$ is totally segregated $\infty_n^+$ bicyclic graph.

**Stage 1:**
Let $G_1$ be the graph obtained from $G$ by repeating $\alpha$-transformation until we cannot get a new graph which belongs to $B_n^+(C_p \circ T_1, C_q \circ T_2, l)$ from $G_1$ by $\alpha$-transformation. Then $G_1 \cong \infty_n(p,q,l,w_2 \circ S_2, w_3 \circ T)$ or $G_1 \cong \infty_n(p,q,l,w_2 \circ T, w_3 \circ S_2)$ or $G_1 \cong \infty_n(p,q,l,w_2 \circ S_2, w_3 \circ S_2, u \circ T)$, $u \in V(P_l)$ where $T \in S^* \cup PS^*$ and $irr_t(G) < irr_t(G_1)$ by Lemma 3.1.

*Case* 1: $G_1 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ T$ where $T \in S^*$).
In this case, we can get a new graph $G_2 \cong \infty_n(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ S_r)$ by $\beta_2$-transformation on $G_1$ and $irr_t(G_1) < irr_t(G_2)$ by Lemma 3.5.

*Case* 2: $G_1 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ T)$ where $T \in PS^*$
Let $v$ be a vertex of $T$ such that the pendant vertices are adjacent to $v$ and $u$ be the root of rooted tree. If $d_{G_1}(u,v) = 1$, let $G_2$ be the graph obtained from $G_1$ by $\beta_1$-transformation. Then $G_2 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ S_r)$ and $irr_t(G_1) < irr_t(G_2)$ by Lemma 3.3.
If $d_{G_1}(u,v) > 1$, let $G_2$ be the graph obtained from $G_1$ by $\beta_1$-transformation and let $G_3$ be the graph obtained from $G_2$ by repeating $\alpha$-transformation until a new graph which belongs to $B_n^+(C_p \circ T_1,C_q \circ T_2,l)$ cannot be obtained from $G_3$ by $\alpha$-transformation. Then the resulting graph is $G_3 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ S_r)$. By Lemmas 3.1 and 3.3 $irr_t(G_1) < irr_t(G_2) < irr_t(G_3)$.

*Case* 3: $G_1 \cong \infty(p,q,l,w_2 \circ T,w_3 \circ S_2)$ where $T \in PS^* \cup S^*$
The proof is similar to the proof of *Cases* 1 and 2 and thus we omit it.

*Case* 4: $G_1 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,u \circ T)$ where $T \in PS^*$ and $u \in V(P_l)$
Let $v$ be a vertex of $T$ such that the pendant vertices are adjacent to $v$ and $u$ be the root of the rooted tree. If $d_{G_1}(u,v) = 1$, let $G_2$ be the graph obtained from $G_1$ by $\beta_1$-transformation. Then $G_2 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ S_r)$ and $irr_t(G_1) < irr_t(G_2)$ by Lemma 3.3.
If $d_{G_1}(u,v) > 1$, let $G_2$ be the graph obtained from $G_1$ by $\beta_1$-transformation and let $G_3$ be the graph obtained from $G_2$ by repeating $\alpha$-transformation until we cannot get a new graph which belongs to $B_n^+(C_p \circ T_1,C_q \circ T_2,l)$ from $G_3$ by $\alpha$-transformation. We know that $G_3 \cong \infty_n(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ S_r)$. Then $irr_t(G_1) < irr_t(G_2) < irr_t(G_3)$.

*Case* 5: $G_1 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,u \circ T)$, where $T \in S^*$ and $u \in V(P_l \setminus \{w_1\})$
In this case, if $u \neq w_5$ we can get a new graph $G_2 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ S_r)$ by $\beta_1$-transformation on $G_1$. Thus $irr_t(G_1) < irr_t(G_2)$ by Lemma 3.4. If $u = w_5$, $G_1$ and $G_2$ have same degree sequence. Hence $irr_t(G_1) = irr_t(G_2)$.

*Case* 6: $G_1 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ T)$ where $T \in S^*$
Then $G_1 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ S_r)$.
Combining the above arguments, we get a $\infty^+$ bicyclic graph $G' \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ S_r)$ and $irr_t(G) < irr_t(G') = (p+q+l-6)(2r+4)+(r+1)^2+9r+3$.

**Stage 2**
Given that $p \geq 4$ or $q \geq 4$ or $l \geq 3$.
Let $G_1 \cong \infty(p,q,l,w_2 \circ S_2,w_3 \circ S_2,w_1 \circ S_r)$ and $p \geq 4$ (or $q \geq 4$ or $l \geq 3$ )
Here we use two types of edge replacements.

*Type* A. Contract an edge of $C_p$ which is different from $w_1w_2$ (or contract an edge of $C_q$ which is different from $w_5w_3$ or contract an edge of path $P_l$ ) and add a pendant edge to $w_1$. In this case $p$ is reduced to $p-1$ and $r$ is increased to $r+1$.

*Type* B. Contract an edge of the cycle $C_p$ which is different from $w_1w_2$ ( or contract an edge of $C_q$ which is different from $w_5w_3$ or contract an edge of path $P_l$ ) and add a pendant edge to $w_3$. Let $G_2'$ and $G_2''$ be the bicyclic graphs obtained from $G_1$ by applying edge replacements *type* A and *type* B, respectively. By Lemma 3.6, we have

$$irr_t(G_2') > irr_t(G_1), \quad irr_t(G_2'') > irr_t(G_1)$$

and

$$irr_t(G_2') - irr_t(G_1) > irr_t(G_2'') - irr_t(G_1).$$

Hence first we apply *type* A-edge replacement on $G_1$ maximum possible times and then *type* B. Let $p \geq 4$ and let $G_2$ be the bicyclic graph obtained from $G_1$ by repeating *type* A-edge replacement until length of the cycle $C_p$ is 4, length of the cycle $C_q$ is 3 and length of the path $P_l$ is 2, since further application of *type* A-edge replacement will lead to a non-totally segregated bicyclic graph. (If $p = 3$, $q \geq 4$, let $G_2$ be the bicyclic graph obtained from $G_1$ by repeating *type* A-edge replacement until length of the cycle $C_q$ is 4 and length of the path $P_l$ is 2; if $p = 3$, $q = 3$ $l \geq 3$ let $G_2$ be the bicyclic graph obtained from $G_1$ by repeating *type* A-edge replacement until length of the path $P_l$ is 3). Then

$$irr_t(G_1) < irr_t(G_2) = irr_t(\infty_n(4,3,2,w_2 \circ S_2, w_3 \circ S_2, w_1 \circ S_{n-8})), \quad \text{if } p \geq 4$$

or

$$irr_t(G_1) < irr_t(G_2) = irr_t(\infty(3,4,2,w_2 \circ S_2, w_3 \circ S_2, w_1 \circ S_{n-8})), \quad \text{if } p = 3,\ q \geq 4$$

or

$$irr_t(G_1) < irr_t(G_2) = irr_t(\infty(3,3,3,w_2 \circ S_2, w_3 \circ S_2, w_1 \circ S_{n-8})), \quad \text{if } p = 3,\ q = 3,\ l \geq 3.$$

Then

$$G_2 \cong \infty(4,3,2,w_2 \circ S_2, w_3 \circ S_2, w_1 \circ S_{n-8})$$

or

$$G_2 \cong \infty(3,4,2,w_2 \circ S_2, w_3 \circ S_2, w_1 \circ S_{n-8})$$

or

$$G_2 \cong \infty(3,3,3,w_2 \circ S_2, w_3 \circ S_2, w_1 \circ S_{n-8}).$$

Let $G_3$ be the totally segregated $\infty_n^+$ bicyclic graph obtained from $G_2$ by applying *type* B edge replacement till we cannot get a new totally segregated $\infty_n^+$ bicyclic graph from $G_3$.
$G_3 = \infty_n(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ and $irr_t(G_2) < irr_t(G_3)$.
Degree sequence of $G_3$ is $((n-6)^1, 4^1, 3^2, 2^2, 1^{n-6})$ and $irr_t(G_3) = n^2 + n - 46$. □

**Theorem 3.8.** *If $G$ is a totally segregated $\infty^+$-bicyclic graph with basic bicycle $\infty(3,3,2)$ on $n$ vertices, then $irr_t(G) \leq n^2 + n - 46$ and equality holds if and only if $G \cong \infty_n(3,3,2,w_1 \circ S_{n-8}, w_2 \circ S_2, w_3 \circ S_3)$.*

*Proof.* Let $G$ be a totally segregated $\infty^+$-bicyclic graph with basic bicycle $\infty(3,3,2)$ (Figure 10). Since $G$ is totally segregated, at least 3 trees are attached to $x$ where $x \in V(C_p) \cup V(C_q)$. Let $w_1w_2x$ and $w_3w_5y$ be two cycles and $w_1w_5$ be the edge joining these two cycles. Since $G$ is totally segregated, it satisfies the following conditions:

For the edge $xw_2$, $\deg x \geq 3$ or $\deg w_2 \geq 3$. Let $\deg w_2 \geq 3$. For the edge $yw_3$, $\deg y \geq 3$ or $\deg w_3 \geq 3$. Let $\deg w_3 \geq 3$. For the edge $w_1w_5$, $\deg w_1 \geq 4$ or $\deg w_5 \geq 4$.

*Case* 1. $T$ is not attached to $w_5$. Then,

$$\deg w_1 \geq 4,\ \deg w_3 \geq 4,\ \deg w_2 \geq 3,\ \deg x \geq 2,\ \deg y \geq 2. \tag{3.9}$$

Let $G_1$ be the graph obtained from $G$ by repeating $\alpha$-transformation till a new graph cannot be obtained from $G_1$ by $\alpha$-transformation so that it satisfies the required degree condition in (3.9). Then there exists a rooted tree $T$ attached to one of the five vertices $(w_1,w_2,w_3,x,y)$ where $T \in S^* \cup PS^*$ and $irr_t(G) < irr_t(G_1)$ by Lemma 3.1. Let $u$ be the root of the rooted tree.
Let $T \in S^*$.
If $u = w_1$, then $G_1 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ and $irr_t(G) < irr_t(G_1) = n^2+n-46$.
If $u = x$ or $y$, we can get a new graph $G_2 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ by $\beta_1$-transformation on $G_1$ and $irr_t(G_1) < irr_t(G_2) = n^2+n-46$.
If $u = w_2$, we can get a new graph $G_2 \cong \infty(3,3,2,w_2 S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ by $\beta_2$-transformation on $G_1$ and $irr_t(G_1) < irr_t(G_2) = n^2+n-46$.
If $u = w_3$, we can get a new graph $G_2 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ by deleting all pendant edges from $w_3$ except two and attaching to $w_1$ and $irr_t(G_1) = irr_t(G_2) = n^2+n-46$.
If $T \in PS^*$.
Let $v$ be vertex of $T$ such that pendant vertices are adjacent to $v$. If $d_{G_1}(u,v) = 1$, let $G_2$ be the graph obtained from $G_1$ by $\beta_1$-transformation. Then $G_2 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ and $irr_t(G_1) < irr_t(G_2) = n^2+n-46$ by Lemma 3.2. If $d_{G_1}(u,v) > 1$, let $G_2$ be the graph obtained from $G_1$ by $\beta_1$-transformation and let $G_3$ be the graph obtained from $G_2$ by repeating $\alpha$-transformation until we cannot get a new graph from $G_3$, which satisfies the degree condition (3.9). $G_3 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$. Then $irr_t(G_1) < irr_t(G_2) < irr_t(G_3) = n^2+n-46$.

*Case* 2: $T$ is not attached to $w_1$
The proof is similar to the proof of *Case* 1. In this case we get $G_3 \cong \infty(3,3,2,w_2 \circ S_3, w_3 \circ S_2, w_5 \circ S_{n-8})$.

*Case* 3: $T$ is attached to $w_1$ and $w_5$
Since $\deg w_1 \neq \deg w_5$, let $\deg w_1 > \deg w_5$. Then

$$\deg w_1 \geq 5,\ \deg w_5 \geq 4,\ \deg w_3 \geq 3,\ \deg w_2 \geq 3,\ \deg x \geq 2,\ \deg y \geq 2. \tag{3.10}$$

Let $G_1$ be the graph obtained from $G$ by repeating $\alpha$-transformation till a new graph from $G_1$ cannot be obtained by $\alpha$-transformation so that it satisfies the required degree condition (3.10). Then in $G_1$ there exists a rooted tree $T$ attached to one of the six vertices where $T \in S^* \cup PS^*$ and $irr_t(G) < irr_t(G_1)$ by Lemma 3.1. Let $u$ be the root of the rooted tree.

If $T \in S^*$ and $u = w_1$, then

$$G_1 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_2, w_5 \circ S_2, w_1 \circ S^*),$$

$G_2$ be the TS graph obtained from $G_1$ by deleting the pendant edge from $w_5$ and attaching to $w_3$. Then $G_1$ and $G_2$ have same degree sequence. $G_2 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ and $irr_t(G_1) = irr_t(G_2) = n^2 + n - 46$.

If $T \in S^*$ and $u = x$ or $y$, we can get a new graph $G_2 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_2, w_5 \circ S_2, w_1 \circ S^*)$ by $\beta_1$-transformation on $G_1$ and $G_3$ be the TS graph obtained from $G_2$ by deleting the pendant edge from $w_5$ and attaching to $w_3$. Here $G_2$ and $G_3$ have same degree sequence. $G_3 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ and $irr_t(G_1) < irr_t(G_2) = irr_t(G_3) = n^2 + n - 46$.

If $T \in S^*$ and $u = w_2$ or $u = w_3$, we can get a new graph $G_2 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_2, w_5 \circ S_2, w_1 \circ S^*)$ by $\beta_2$-transformation on $G_1$ and $G_3$ be the totally segregated graph obtained from $G_2$ by deleting the pendant edge from $w_5$ and attaching to $w_3$. $G_3 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ and $irr_t(G_1) < irr_t(G_2) = irr_t(G_3) = n^2 + n - 46$.

Let $T \in PS^*$.

Let $v$ be a vertex of $T$ such that the pendant vertices are adjacent to $v$.

If $d_{G_1}(u,v) = 1$, let $G_2$ be the graph obtained from $G_1$ by $\beta_1$-transformation. Then $G_2 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_2, w_5 \circ S_2, w_1 \circ S^*)$ and $G_3$ be the TS graph obtained from $G_2$ by deleting the pendant edge from $w_5$ and attaching to $w_3$. $G_3 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$ and $irr_t(G_1) < irr_t(G_2) = irr_t(G_3) = n^2 + n - 46$.

If $d_{G_1}(u,v) > 1$, let $G_2$ be the graph obtained from $G_1$ by $\beta_1$-transformation and let $G_3$ be the graph obtained from $G_2$ by repeating $\alpha$-transformation until $G_3 \cong \infty(3,3,2,w_2 \circ S_2, w_3 \circ S_2, w_5 \circ S_2, w_1 \circ S^*)$. $G_4$ be the TS graph obtained from $G_2$ by deleting the pendant edge from $w_5$ and attaching to $w_3$. $G_3 \cong \infty(3,3,2,w_1 \circ S_{n-8}, w_2 \circ S_2, w_3 \circ S_3)$ and $irr_t(G) < irr_t(G_1) < irr_t(G_2) < irr_t(G_3) = irr_t(G_4) = n^2 + n - 46$.

Combining the above arguments, we complete the proof. □

**Theorem 3.9.** *If $n \geq 10$ is a positive integer and $G$ is a totally segregated $\infty^+$-bicyclic graph, with basic bicycle $\infty(p,q,l)$ ($p \geq 3$, $q \geq 3$, $l \geq 2$), on $n$ vertices, then $irr_t(G) \leq n^2 + n - 46$ and the equality holds if and only if $G \cong \infty_n(3,3,2,w_1 \circ S_{n-8}, w_2 \circ S_2, w_3 \circ S_3)$ (Figure 10).*

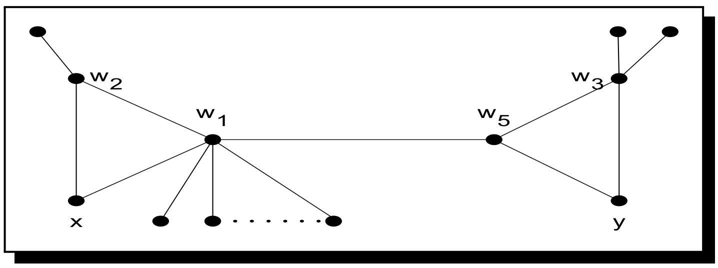


**Figure 10.** The graph $\infty(3,3,2,w_2 \circ S_2, w_3 \circ S_3, w_1 \circ S_{n-8})$

*Proof.* Combining Theorem 3.7 and Theorem 3.8, we complete the proof. □

**Theorem 3.10** ([9]). *Let $n \geq 4$ be a positive integer and let $G$ be a $\Theta$-bicyclic graph on $n$ vertices. Then $irr_t(G) \leq n^2+n-16$ and the equality holds if and only if $G \cong \theta(3,3,2,w_1 \circ S_{n-3})$ (Figure 11). Clearly, $\theta(3,3,2,w_1 \circ S_{n-3}) \cong \theta(3,3,2,w_5 \circ S_{n-3})$.*

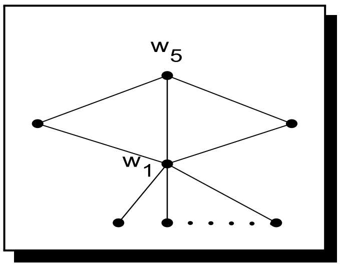


**Figure 11.** The graph $\theta(3,3,2,w_1 \circ S_{n-3}, w_3 \circ S_3, w_1 \circ S_{n-8})$

**Theorem 3.11.** *If $n \geq 4$ is a positive integer and $G$ is a totally segregated $\Theta$-bicyclic graph on $n$ vertices, then $irr_t(G) \leq n^2+n-16$ and the equality holds if and only if $G \cong \theta(3,3,2,w_1 \circ S_{n-3})$.*

*Proof.* By Theorem 3.10 we have if$G$ is a $\Theta$-bicyclic graph on $n$ vertices, $irr_t(G) \leq n^2+n-16$ and the equality holds if and only if $G \cong \theta(3,3,2,w_1 \circ S_{n-3})$.
Clearly, $G \cong \theta(3,3,2,w_1 \circ S_{n-3})$ is a totally segregated bicyclic graph. Hence the result. □

Denote by $\mathscr{B}_n$ the set of all totally segregated bicyclic graphs on $n$ vertices. Obviously, $\mathscr{B}_n$ consists of two types of graphs: first type denoted by $\mathscr{B}_n^+$ is the set of those graphs each of which is a totally segregated $\infty^+$-bicyclic graph and second type denoted by $\mathscr{B}_n^{++}$ is the set of those graphs each of which is a totally segregated $\Theta$-bicyclic graph. Then

$$\mathscr{B}_n = \mathscr{B}_n^+ \cup \mathscr{B}_n^{++}.$$

By Theorem 3.9 we have if $G \in \mathscr{B}_n^+$, $irr_t(G) \leq n^2+n-46$.
By Theorem 3.11 we have if $G \in \mathscr{B}_n^{++}$, $irr_t(G) \leq n^2+n-16$.

**Theorem 3.12.** *If $n \geq 4$ is a positive integer and $G \in \mathscr{B}_n$ is a totally segregated bicyclic graph on $n$ vertices, then, $irr_t(G) \leq n^2+n-16$ and the equality holds if and only if $G \cong \theta(3,3,2,w_1 \circ S_{n-3})$.*

## Competing Interests

The author declares that she has no competing interests.

## Authors' Contributions

The author wrote, read and approved the final manuscript.

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