**Communications in Mathematics and Applications**
Vol. 14, No. 1, pp. 105–115, 2023
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v14i1.1812

http://www.rgnpublications.com

Research Article

# Symmetric Division Degree Invariants of Join Total and Mid Graphs

P. Murugarajan

*Department of Mathematics, Velammal Institute of Technology, Chennai 601204, Tamil Nadu, India*
mathrajan.mp@gmail.com

**Received:** January 30, 2022 **Accepted:** May 5, 2022

**Abstract.** The symmetric division degree (*SDD*) invariant is one of the 200 discrete Adriatic indices introduced several years ago. This *SDD* invariant has already been proven a valuable invariant in the *QSAR* (*Quantitative Structure Activity Relationship*) and *QSPR* (*Quantitative Structure Property Relationship*) studies. In this article, we present the bounds for *SDD* invariant of join total graph and *SDD* invariant of mid graphs.

**Keywords.** Degree, Join total graph, Mid graph, Symmetric division deg invariant, Samundi invariant

**Mathematics Subject Classification (2020).** 05C11, 05C50



## 1. Introduction

Molecular descriptors have found applications in modelling several physicochemical properties in *QSAR* and *QSPR* studies ([3, 8]). Many molecular descriptors are defined as functions of the structure of the underlying molecular graph, such as the Wiener invariant [21], the Zagreb invariant [7], and Balaban invariants [2]. Vukicević *et al*. [20] proved that many of these descriptors are defined some of individual bond contributions. Among the 148 discrete Adriatic invariants studied in [20], whose predictive properties were evaluated against the benchmark datasets of the International Academy of Mathematical Chemistry[1], 20 invariants were selected as significant predictors of physicochemical properties. One of these useful discrete Adriatic indices is symmetric division degree (*SDD*) invariant which is defined as $SDD(\Gamma) = \sum_{xy\in E(\Gamma)} \left(\frac{\lambda_\Gamma(x)}{\lambda_\Gamma(y)} + \frac{\lambda_\Gamma(y)}{\lambda_\Gamma(x)}\right)$, where $\lambda_\Gamma(x)$ and $\lambda_\Gamma(y)$ are the degrees of vertices $x$ and $y$, respectively. Among all the existing molecular descriptors, *SDD* invariant has the best correlating ability for predicting the total surface area of polychlorobiphenys [20]. Vasilev [19] provided the different types of lower and upper bounds of symmetric division deg invariant in some classes of graphs and determined the corresponding extremal graphs. Palacios [13] found a new upper bound for the symmetric division deg invariant of a graph $\Gamma$ with $n$ vertices, in terms of the inverse degree invariant,that is attained by all regular,all complete multipartite graphs, $K_{b_1,b_2,\dots,b_l}$, and all $(s-1,t)$-regular graphs of order $s$, where $1 = t < s-1$. Several papers have been appeared in literature addressing the mathematical aspects of this descriptor (e.g., see [1, 5, 6, 10, 11]). In this article, we present on bounds of *SDD* invariant of join total graph and *SDD* invariant of mid graphs.

## 2. Preliminaries

Let $\Gamma$ be a finite simple connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We denote by $\delta$ and $\Delta$ the minimum and maximum vertex degrees of $\Gamma$, respectively.

Line graph is defined as the line graph $L(\Gamma)$ of $G$ is the graph in which the vertex set is the edge set of $\Gamma$, and there is an edge between two vertices of $L(\Gamma)$ if and only if their corresponding edges are incident in $\Gamma$.

The Zagreb invariants are among the oldest topological invariants introduced by Gutman and Trinajstic in 1972 [7]. These indices have since been used to study molecular complexity, chirality, ZE-isomerism and hetero-systems. They are defined as $M_1(\Gamma) = \sum_{xy\in E(\Gamma)} (\lambda_\Gamma(x) + \lambda_\Gamma(y))$ and $M_2(\Gamma) = \sum_{xy\in E(\Gamma)} (\lambda_\Gamma(x)\lambda_\Gamma(y))$. Randić [16] proposed a structure descriptor, based on the end -vertex degrees of edges in a graph, called branching invariant that later became the well-known Randić connectivity invariant. The Randić invariant of $\Gamma$ is defined as $R(\Gamma) = \sum_{xy\in E(\Gamma)} \frac{1}{\sqrt{\lambda_\Gamma(x)\lambda_\Gamma(y)}}$. It gave rise to a number of generalizations. The most common one arises by varying the exponent $\alpha$ in the edge contribution $(\lambda_\Gamma(x)\lambda_\Gamma(y))^\alpha$. The $\alpha$-Randić invariant is then defined as $R_\alpha(\Gamma) = \sum_{xy\in E(\Gamma)} (\lambda_\Gamma(x)\lambda_\Gamma(y))^\alpha$.

The $F$-invariant and multiplicative $F$-invariant of a connected graph $\Gamma$ are respectively, defined as $F(\Gamma) = \sum_{xy\in E(\Gamma)} (\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2)$ and $F^*(\Gamma) = \prod_{xy\in E(\Gamma)} (\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2)$. The $\alpha$-$F$-invariant of $\Gamma$ is defined as $F_\alpha(\Gamma) = \sum_{xy\in E(\Gamma)} (\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2)^\alpha$. *Inverse degree index* defined by $ID(\Gamma) =$

[1] Milano Chemometrics and QSAR Research Group, *Molecular descriptors dataset*, Department of Earth and Environmental Sciences, University of Milano-Bicocca, Italy, accessed: 18.04.14, URL: https://michem.unimib.it/.

$\sum_{v\in V(\Gamma)} \frac{1}{\lambda_\Gamma(v)}$, Harmonic index $=H(\Gamma) = \sum_{xy\in E(\Gamma)} \frac{2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}$, Hyper Zagreb index $= HM(\Gamma) = \sum_{xy\in E(\Gamma)} (\lambda_\Gamma(x)+\lambda_\Gamma(y))^2$, General sum connectivity index $= \chi_\alpha(\Gamma) = \sum_{xy\in E(\Gamma)} (\lambda_\Gamma(x)+\lambda_\Gamma(y))^\alpha$, put $\alpha=-2$ we get $\chi_{-2}(\Gamma) = \sum_{xy\in E(\Gamma)} (\lambda_\Gamma(x)+\lambda_\Gamma(y))^{-2}$. We introduce the new invariant called as Samundi invariant and it is denoted by $D_1(\Gamma) = \sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}$.

## 3. Join Total Graph

The join of $\Gamma_1$ and $\Gamma_2$ denoted by $\Gamma_1+\Gamma_2$, is the union $\Gamma_1\cup\Gamma_2$ together with all the edges joining $V(\Gamma_1)$ and $V(\Gamma_2)$. Total graph $T(\Gamma)$ of $\Gamma$ is obtained by inserting a new vertex corresponding to each edge of $G$, then join it to the end vertices of the corresponding edge and join those pairs of new vertices such that their respective edges share a common vertex in $\Gamma$. Let $I(\Gamma_1)$ be the collection of all new vertices those are inserted to $\Gamma_1$. The join total graph of $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ is the graph derived from $T(\Gamma_1)$, $\Gamma_2$ and $\Gamma_3$ by connecting every vertex of $\Gamma_1$ to every vertex of $\Gamma_2$ and every vertex of $I(\Gamma_1)$ to every vertex of $\Gamma_2$.

**Lemma 3.1** (Jensen's Inequality)**.** *Let $T$ be a convex function on an interval $J$ and $x_1,x_2,\dots,x_n\in J$. Then $T\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\le\frac{T(x_1)+T(x_2)+\ldots T(x_n)}{n}$, with equality if and only if $x_1=x_2=\ldots=x_n$.*

**Theorem 3.2.** *Let $\Gamma_i$ be a graph with $s_i$ vertices and $m_i$ edges, $i\in\{1,2,3\}$. Then $SDD(\Gamma_1+_T(\Gamma_2,\Gamma_3))\le\sum_{i=1}^{9}\frac{\alpha_i}{16}$, where*

$$\alpha_1 = SDD(\Gamma_1)+SDD(\Gamma_2)+SDD(\Gamma_3)+SDD(L(\Gamma_1)),$$

$$\alpha_2 = \frac{4F(L(\Gamma_1))}{(s_3+2)^2}+\frac{16F(\Gamma_1)}{s_2^2}+\frac{4F(\Gamma_2)}{s_1^1}+\frac{4F(\Gamma_3)}{m_1^2},$$

$$\alpha_3 = \frac{D_1(L(\Gamma_1))}{(s_3+2)}+\frac{2D_1(\Gamma_1)}{s_2}+\frac{D_1(\Gamma_2)}{s_1}+\frac{D_1(\Gamma_3)}{m_1},$$

$$\alpha_4 = \frac{8M_1(L(\Gamma_1))}{(s_3+2)}+M_1(\Gamma_1)\left(\frac{4ID(\Gamma_2)}{s_2}+\frac{4m_1+2s_1s_3}{s_1m_1}+3ID(\Gamma_3)+5\right)$$
$$+\left(\frac{ID(\Gamma_1)}{2s_1}+\frac{8s_2+s_1}{s_1s_2}\right)M_1(\Gamma_2)+\frac{m_1+8s_3}{s_3m_1}M_1(\Gamma_3),$$

$$\alpha_5 = \left(3m_2+s_1s_2+\frac{s_2(s_1^2+s_2^2)}{2s_1}+\frac{4s_3m_3+m_1+s_3^2+m_1^2}{s_3}\right)ID(\Gamma_1)$$
$$+\left(12m_1+2s_1s_2+\frac{(s_1^2+s_2^2)ID(\Gamma_1)}{2}+\frac{s_1(s_1^2+s_2^2)}{s_2}\right)ID(\Gamma_2)+2s_3m_1ID(\Gamma_3),$$

$$\alpha_6 = (s_3+2)H(L(\Gamma_1))+\left(\frac{8s_2+2s_3}{4}+\frac{4s_2^2+s_3^2}{4(s_3+s_2)}+\frac{s_2}{2}+\frac{M_1(\Gamma_3)}{2m_1}+\frac{2m_1+s_3^2+m_1^2}{2}\right.$$
$$\left.+\frac{2m_1m_3+s_3^2+m_1^2}{2m_1}\right)H(\Gamma_1)+s_1H(\Gamma_2)+m_1H(\Gamma_3),$$

$$\alpha_7 = 2(s_3+2)^2 R_{-1}(L(\Gamma_1)) + \frac{s_2^2}{2} R_{-1}(\Gamma_1) + 2s_1^2 R_{-1}(\Gamma_2) + 2m_1^2 R_{-1}(\Gamma_3),$$

$$\alpha_8 = \left(\frac{ID(\Gamma_3)}{s_3} + \frac{s_2 s_3 + 10m_1}{s_2 s_3 m_1}\right) HM(\Gamma_1) + \frac{(4s_2^2+s_3^2)}{2}\chi_{-2}(\Gamma_1),$$

$$\alpha_9 = 3(s_1^2+s_2^2+s_3^2) + 3s_1s_2 + \left(3s_3 + 3m_1 + \frac{12s_2}{s_1} + \frac{6m_3}{s_3} + \frac{13}{2}\right)m_1 + 10m_2 + 10m_3 + \frac{6m_2s_1}{s_2} + \frac{4m_1(s_2+2s_3)}{s_3+s_2} + \frac{2m_1(4s_2^2+s_3^2)}{2s_2s_3}.$$

*Proof.* Consider $\Gamma = \Gamma_1 +_T (\Gamma_2, \Gamma_3)$. By the definition of $SDD$ invariant of the graph $\Gamma$,

$$SDD(\Gamma) = \sum_{ab\in E(\Gamma)} \frac{\lambda_\Gamma(a)^2 + \lambda_\Gamma(b)^2}{\lambda_\Gamma(a)\lambda_\Gamma(b)}.$$

From the construction of join total graph, we obtain the following types of degrees:

- If $a \in V(\Gamma_1)$, then $\lambda_\Gamma(a) = 2\lambda_{\Gamma_1}(a) + s_2$.
- If $ab = c \in I(\Gamma_1)$, then $\lambda_\Gamma(a) = \lambda_{\Gamma_1}(a) + \lambda_{\Gamma_1}(b) + s_3$.
- If $a \in V(\Gamma_2)$, then $\lambda_\Gamma(a) = \lambda_{\Gamma_2}(a) + s_1$.
- If $a \in V(\Gamma_3)$, then $\lambda_\Gamma(a) = \lambda_{\Gamma_3}(a) + m_1$.

Hence

$$SDD(\Gamma) = \sum_{a,b\in I(\Gamma_1)} \frac{\lambda_\Gamma(a)^2 + \lambda_\Gamma(b)^2}{\lambda_\Gamma(a)\lambda_\Gamma(b)} + \sum_{a\in V(\Gamma_1), b\in I(\Gamma_1)} \frac{\lambda_\Gamma(a)^2 + \lambda_\Gamma(b)^2}{\lambda_\Gamma(a)\lambda_\Gamma(b)} + \sum_{ab\in E(\Gamma_1)} \frac{\lambda_\Gamma(a)^2 + \lambda_\Gamma(b)^2}{\lambda_\Gamma(a)\lambda_\Gamma(b)}$$
$$+ \sum_{ab\in I(\Gamma_2)} \frac{\lambda_\Gamma(a)^2 + \lambda_\Gamma(b)^2}{\lambda_\Gamma(a)\lambda_\Gamma(b)} + \sum_{a\in V(\Gamma_1)}\sum_{b\in V(\Gamma_2)} \frac{\lambda_\Gamma(a)^2 + \lambda_\Gamma(b)^2}{\lambda_\Gamma(a)\lambda_\Gamma(b)} + \sum_{ab\in E(\Gamma_3)} \frac{\lambda_\Gamma(a)^2 + \lambda_\Gamma(b)^2}{\lambda_\Gamma(a)\lambda_\Gamma(b)}$$
$$+ \sum_{a\in I(\Gamma_1)}\sum_{b\in V(\Gamma_3)} \frac{\lambda_\Gamma(a)^2 + \lambda_\Gamma(b)^2}{\lambda_\Gamma(a)\lambda_\Gamma(b)}.$$

Now substituting corresponding degrees to the vertices of the graph $\Gamma$, we get

$$SDD(\Gamma) = \sum_{ab,bc\in E(\Gamma_1)} \frac{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)^2 + (\lambda_{\Gamma_1}(b)+\lambda_{\Gamma_1}(c)+s_3)^2}{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)(\lambda_{\Gamma_1}(b)+\lambda_{\Gamma_1}(c)+s_3)}$$
$$+ \sum_{ab\in E(\Gamma_1)} \frac{(2\lambda_{\Gamma_1}(a)+s_2+2\lambda_{\Gamma_1}(b)+s_2)^2 + (\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)^2}{(2\lambda_{\Gamma_1}(a)+s_2+2\lambda_{\Gamma_1}(b)+s_2)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)}$$
$$+ \sum_{ab\in E(\Gamma_1)} \frac{(2\lambda_{\Gamma_1}(a)+s_2)^2 + (2\lambda_{\Gamma_1}(b)+s_2)^2}{(2\lambda_{\Gamma_1}(a)+s_2)(2\lambda_{\Gamma_1}(b)+s_2)} + \sum_{ab\in E(\Gamma_2)} \frac{(\lambda_{\Gamma_2}(a)+s_1)^2 + (\lambda_{\Gamma_2}(b)+s_1)^2}{(\lambda_{\Gamma_2}(a)+s_1)(\lambda_{\Gamma_2}(b)+s_1)}$$
$$+ \sum_{a\in V(\Gamma_1)}\sum_{b\in V(\Gamma_2)} \frac{(2\lambda_{\Gamma_1}(a)+s_2)^2 + (\lambda_{\Gamma_2}(b)+s_1)^2}{(2\lambda_{\Gamma_1}(a)+s_2)(\lambda_{\Gamma_2}(b)+s_1)} + \sum_{ab\in E(\Gamma_3)} \frac{(\lambda_{\Gamma_3}(a)+m_1)^2 + (\lambda_{\Gamma_3}(b)+m_1)^2}{(\lambda_{\Gamma_3}(a)+m_1)(\lambda_{\Gamma_3}(b)+m_1)}$$
$$+ \sum_{a\in I(\Gamma_1)}\sum_{b\in V(\Gamma_3)} \frac{(\lambda_{I(\Gamma_1)}(a)+s_3)^2 + (\lambda_{\Gamma_3}(b)+m_1)^2}{(\lambda_{I(\Gamma_1)}(a)+s_3)(\lambda_{\Gamma_3}(b)+m_1)}.$$

First, we find the sum $I_1$, where

$$I_1 = \sum_{ab,bc\in E(\Gamma_1)} \frac{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)^2+(\lambda_{\Gamma_1}(b)+\lambda_{\Gamma_1}(c)+s_3)^2}{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)(\lambda_{\Gamma_1}(b)+\lambda_{\Gamma_1}(c)+s_3)}$$
$$= \sum_{e,f\in L(\Gamma_1),e=ab,f=bc} \frac{(\lambda_{L(\Gamma_1)}(e)+s_3+2)^2+(\lambda_{L(\Gamma_1)}(f)+s_3+2)^2}{(\lambda_{L(\Gamma_1)}(e)+s_3+2)(\lambda_{L(\Gamma_1)}(f)+s_3+2)}$$
$$= \sum_{ef\in E(L(\Gamma_1))} \frac{(\lambda_{L(\Gamma_1)}(e)^2+\lambda_{L(\Gamma_1)}(f)^2)+2(s_3+2)^2+2(s_3+2)(\lambda_{L(\Gamma_1)}(e)+\lambda_{L(\Gamma_1)}(f))}{(\lambda_{L(\Gamma_1)}(e)\lambda_{L(\Gamma_1)}(f)+(s_3+2)^2+(s_3+2)(\lambda_{L(\Gamma_1)}(f)+\lambda_{L(\Gamma_1)}(f))}.$$

By Jensen's inequality, we have

$$I_1 \le \frac{1}{16}\Bigg[\sum_{ef\in E(L(\Gamma_1))} \frac{(\lambda_{L(\Gamma_1)}(e)^2+\lambda_{L(\Gamma_1)}(f)^2)}{\lambda_{L(\Gamma_1)}(e)\lambda_{L(\Gamma_1)}(f)} + \sum_{ef\in E(L(\Gamma_1))} \frac{4(\lambda_{L(\Gamma_1)}(e)^2+\lambda_{L(\Gamma_1)}(f)^2)}{(s_3+2)^2}$$
$$+ \sum_{ef\in E(L(\Gamma_1))} \frac{(\lambda_{L(\Gamma_1)}(e)^2+\lambda_{L(\Gamma_1)}(f)^2)}{(s_3+2)(\lambda_{L(\Gamma_1)}(f)+\lambda_{L(\Gamma_1)}(f))} + \sum_{ef\in E(L(\Gamma_1))} \frac{2(s_3+2)^2}{(\lambda_{L(\Gamma_1)}(e)\lambda_{L(\Gamma_1)}(f)}$$
$$+ \sum_{ef\in E(L(\Gamma_1))} \frac{8(s_3+2)^2}{(s_3+2)^2} + \sum_{ef\in E(L(\Gamma_1))} \frac{2(s_3+2)^2}{(s_3+2)(\lambda_{L(\Gamma_1)}(f)+\lambda_{L(\Gamma_1)}(f))}$$
$$+ \sum_{ef\in E(L(\Gamma_1))} \frac{2(s_3+2)(\lambda_{L(\Gamma_1)}(e)+\lambda_{L(\Gamma_1)}(f))}{(\lambda_{L(\Gamma_1)}(e)\lambda_{L(\Gamma_1)}(f)}$$
$$+ \sum_{ef\in E(L(\Gamma_1))} \frac{8(s_3+2)(\lambda_{L(\Gamma_1)}(e)+\lambda_{L(\Gamma_1)}(f))}{(s_3+2)^2} + \sum_{ef\in E(L(\Gamma_1))} \frac{2(s_3+2)(\lambda_{L(\Gamma_1)}(e)+\lambda_{L(\Gamma_1)}(f))}{(s_3+2)(\lambda_{L(\Gamma_1)}(f)+\lambda_{L(\Gamma_1)}(f))}\Bigg]$$
$$= \frac{1}{16}\Bigg[SDD(L(\Gamma_1))+\frac{4F(L(\Gamma_1))}{(s_3+2)^2}+\frac{D_1(L(\Gamma_1))}{(s_3+2)}+2(s_3+2)^2R_{-1}(L(\Gamma_1))$$
$$+10|E(L(\Gamma_1))|+(s_3+2)H(L(\Gamma_1))+2(s_3+2)|V(L(\Gamma_1))|+\frac{8M_1(L(\Gamma_1))}{(s_3+2)}\Bigg].$$

Since if $L(\Gamma_1)$ is the line graph of $\Gamma_1$, then $|E(L(\Gamma_1))| = \frac{M_1(\Gamma_1)}{2} - m_1$. Hence

$$I_1 \le \frac{1}{16}\Bigg[SDD(L(\Gamma_1))+\frac{4F(L(\Gamma_1))}{(s_3+2)^2}+\frac{K_1(L(\Gamma_1))}{(s_3+2)}+2(s_3+2)^2R_{-1}(L(\Gamma_1))$$
$$+(s_3+2)H(L(\Gamma_1))+\frac{8M_1(L(\Gamma_1))}{(s_3+2)}+10\left\{\frac{M_1(\Gamma_1)}{2}-m_1\right\}+2(s_3+2)m_1\Bigg]$$
$$= \frac{1}{16}\Bigg[SDD(L(\Gamma_1))+\frac{4F(L(\Gamma_1))}{(s_3+2)^2}+\frac{D_1(L(\Gamma_1))}{(s_3+2)}+2(s_3+2)^2R_{-1}(L(\Gamma_1))$$
$$+(s_3+2)H(L(\Gamma_1))+\frac{8M_1(L(\Gamma_1))}{(s_3+2)}+5M_1(\Gamma_1)+(s_3-6)m_1\Bigg].$$

Now, we shall obtain the sum $I_2$, where

$$I_2 = \sum_{ab\in E(\Gamma_1)} \frac{(2\lambda_{\Gamma_1}(a)+s_2+2\lambda_{\Gamma_1}(b)+s_2)^2+(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)^2}{(2\lambda_{\Gamma_1}(a)+s_2+2\lambda_{\Gamma_1}(b)+s_2)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)}$$
$$= \sum_{ab\in E(\Gamma_1)} \frac{5(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2+(8s_2+2s_3)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))+(4s_2^2+s_3^2)}{2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2+2(s_3+s_2)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))+2s_2s_3}.$$

By Jensen's inequality, we get

$$I_2 \le \frac{1}{16}\left[\sum_{ab\in E(\Gamma_1)} \frac{5(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2}{2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2} + \sum_{ab\in E(\Gamma_1)} \frac{5(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2}{2(s_3+s_2)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}\right.$$
$$+ \sum_{ab\in E(\Gamma_1)} \frac{20(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2}{2s_2s_3} + \sum_{ab\in E(\Gamma_1)} \frac{(8s_2+2s_3)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2}$$
$$+ \sum_{ab\in E(\Gamma_1)} \frac{(8s_2+2s_3)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{2(s_3+s_2)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))} + \sum_{ab\in E(\Gamma_1)} \frac{4(8s_2+2s_3)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{2s_2s_3}$$
$$+ \sum_{ab\in E(\Gamma_1)} \frac{(4s_2^2+s_3^2)}{2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2} + \sum_{ab\in E(\Gamma_1)} \frac{(4s_2^2+s_3^2)}{2(s_3+s_2)(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}$$
$$\left.+ \sum_{ab\in E(\Gamma_1)} \frac{4(4s_2^2+s_3^2)}{2s_2s_3}\right]$$
$$= \frac{1}{16}\left[\frac{5m_1}{2} + \frac{5M_1(\Gamma_1)}{2(s_3+s_2)} + \frac{20HM(\Gamma_1)}{2s_2s_3} + \frac{(8s_2+2s_3)H(\Gamma_1)}{4} + \frac{(8s_2+2s_3)m_1}{2(s_3+s_2)}\right.$$
$$\left.+ \frac{4(8s_2+2s_3)M_1(\Gamma_1)}{2s_2s_3} + \frac{(4s_2^2+s_3^2)\chi_{-2}(\Gamma_1)}{2} + \frac{(4s_2^2+s_3^2)H(\Gamma_1)}{4(s_3+s_2)} + \frac{4(4s_2^2+s_3^2)m_1}{2s_2s_3}\right].$$

Here we calculate the sum $I_3$, where

$$I_3 = \sum_{ab\in E(\Gamma_1)} \frac{(2\lambda_{\Gamma_1}(a)+s_2)^2+(2\lambda_{\Gamma_1}(b)+s_2)^2}{(2\lambda_{\Gamma_1}(a)+s_2)(2\lambda_{\Gamma_1}(b)+s_2)}$$
$$= \sum_{ab\in E(\Gamma_1)} \frac{4(\lambda_{\Gamma_1}(a)^2+\lambda_{\Gamma_1}(b)^2)+4s_2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))+2s_2^2}{4\lambda_{\Gamma_1}(a)\lambda_{\Gamma_1}(b)+2s_2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))+s_2^2}$$

By Jensen's inequality, we have

$$I_3 \le \frac{1}{16}\left[\sum_{ab\in E(\Gamma_1)} \frac{4(\lambda_{\Gamma_1}(a)^2+\lambda_{\Gamma_1}(b)^2)}{4\lambda_{\Gamma_1}(a)\lambda_{\Gamma_1}(b)} + \sum_{ab\in E(\Gamma_1)} \frac{4(\lambda_{\Gamma_1}(a)^2+\lambda_{\Gamma_1}(b)^2)}{2s_2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}\right.$$
$$+ \sum_{ab\in E(\Gamma_1)} \frac{16(\lambda_{\Gamma_1}(a)^2+\lambda_{\Gamma_1}(b)^2)}{s_2^2} + \sum_{ab\in E(\Gamma_1)} \frac{4s_2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{4\lambda_{\Gamma_1}(a)\lambda_{\Gamma_1}(b)}$$
$$+ \sum_{ab\in E(\Gamma_1)} \frac{4s_2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{2s_2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))} + \sum_{ab\in E(\Gamma_1)} \frac{16s_2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{s_2^2}$$
$$\left.+ \sum_{ab\in E(\Gamma_1)} \frac{2s_2^2}{4\lambda_{\Gamma_1}(a)\lambda_{\Gamma_1}(b)} + \sum_{ab\in E(\Gamma_1)} \frac{2s_2^2}{2s_2(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))} + \sum_{ab\in E(\Gamma_1)} \frac{8s_2^2}{s_2^2}\right]$$
$$= \frac{1}{16}\left[SDD(\Gamma_1) + \frac{4D_1(\Gamma_1)}{2s_2} + \frac{16F(\Gamma_1)}{s_2^2} + s_2|V(\Gamma_1)| + 10|E(\Gamma_1)| + \frac{16M_1(\Gamma_1)}{s_2}\right.$$
$$\left.+ \frac{2s_2^2R_{-1}(\Gamma_1)}{4} + \frac{s_2H(\Gamma_1)}{2}\right].$$

Here we find the sum $I_4$, where

$$I_4 = \sum_{ab\in E(\Gamma_2)} \frac{(\lambda_{\Gamma_2}(a)+s_1)^2+(\lambda_{\Gamma_2}(b)+s_1)^2}{(\lambda_{\Gamma_2}(a)+s_1)(\lambda_{\Gamma_2}(b)+s_1)}$$

$$= \sum_{ab\in E(\Gamma_2)} \frac{(\lambda_{\Gamma_2}(a)^2+\lambda_{\Gamma_2}(b)^2)+2s_1(\lambda_{\Gamma_2}(a)+\lambda_{\Gamma_2}(b))+2s_1^2}{\lambda_{\Gamma_2}(a)\lambda_{\Gamma_2}(b)+s_1(\lambda_{\Gamma_2}(a)+\lambda_{\Gamma_2}(b))+s_1^2}.$$

By Jensen's inequality, we have

$$\begin{aligned} I_4 \le \frac{1}{16}\Bigg[ &\sum_{ab\in E(\Gamma_2)} \frac{(\lambda_{\Gamma_2}(a)^2+\lambda_{\Gamma_2}(b)^2)}{\lambda_{\Gamma_2}(a)\lambda_{\Gamma_2}(b)} + \sum_{ab\in E(\Gamma_2)} \frac{(\lambda_{\Gamma_2}(a)^2+\lambda_{\Gamma_2}(b)^2)}{s_1(\lambda_{\Gamma_2}(a)+\lambda_{\Gamma_2}(b))} \\ &+ \sum_{ab\in E(\Gamma_2)} \frac{4(\lambda_{\Gamma_2}(a)^2+\lambda_{\Gamma_2}(b)^2)}{s_1^2} + \sum_{ab\in E(\Gamma_2)} \frac{2s_1(\lambda_{\Gamma_2}(a)+\lambda_{\Gamma_2}(b))}{\lambda_{\Gamma_2}(a)\lambda_{\Gamma_2}(b)} \\ &+ \sum_{ab\in E(\Gamma_2)} \frac{2s_1(\lambda_{\Gamma_2}(a)+\lambda_{\Gamma_2}(b))}{s_1(\lambda_{\Gamma_2}(a)+\lambda_{\Gamma_2}(b))} + \sum_{ab\in E(\Gamma_2)} \frac{8s_1(\lambda_{\Gamma_2}(a)+\lambda_{\Gamma_2}(b))}{s_1^2} \\ &+ \sum_{ab\in E(\Gamma_2)} \frac{2s_1^2}{\lambda_{\Gamma_2}(a)\lambda_{\Gamma_2}(b)} + \sum_{ab\in E(\Gamma_2)} \frac{2s_1^2}{s_1(\lambda_{\Gamma_2}(a)+\lambda_{\Gamma_2}(b))} + \sum_{ab\in E(\Gamma_2)} \frac{8s_1^2}{s_1^2}\Bigg] \\ = \frac{1}{16}\Bigg[&SDD(\Gamma_2)+\frac{D_1(\Gamma_2)}{s_1}+\frac{4F(\Gamma_2)}{s_1^2}+\frac{8M_1(\Gamma_2)}{s_1}+2s_1^2R_{-1}+s_1H(\Gamma_2)+2s_1s_2+10m_2\Bigg]. \end{aligned}$$

Now we obtain the sum $I_5$, where

$$\begin{aligned} I_5 &= \sum_{a\in V(\Gamma_1)}\sum_{b\in V(\Gamma_2)} \frac{(2\lambda_{\Gamma_1}(a)+s_2)^2+(\lambda_{\Gamma_2}(b)+s_1)^2}{(2\lambda_{\Gamma_1}(a)+s_2)(\lambda_{\Gamma_2}(b)+s_1)} \\ &= \sum_{a\in V(\Gamma_1)}\sum_{b\in V(\Gamma_2)} \frac{4\lambda_{\Gamma_1}(a)^2+4s_2\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_2}(b)^2+2s_1\lambda_{\Gamma_2}(b)+(s_2^2+s_1^2)}{2\lambda_{\Gamma_1}(a)\lambda_{\Gamma_2}(b)+2s_1d_{\Gamma_1}(a)+s_2\lambda_{\Gamma_2}(b)+s_1s_2}. \end{aligned}$$

By Jensen's inequality, we obtain

$$\begin{aligned} I_5 \le \frac{1}{16}\sum_{a\in V(\Gamma_1)}\sum_{b\in V(\Gamma_2)}\Bigg[ &\frac{4\lambda_{\Gamma_1}(a)^2}{2\lambda_{\Gamma_1}(a)\lambda_{\Gamma_2}(b)} + \frac{4\lambda_{\Gamma_1}(a)^2}{2s_1\lambda_{\Gamma_1}(a)} + \frac{4\lambda_{\Gamma_1}(a)^2}{s_2\lambda_{\Gamma_2}(b)} + \frac{4\lambda_{\Gamma_1}(a)^2}{s_1s_2} + \frac{4s_2\lambda_{\Gamma_1}(a)}{2\lambda_{\Gamma_1}(a)\lambda_{\Gamma_2}(b)} \\ &+ \frac{4s_2\lambda_{\Gamma_1}(a)}{2s_1d_{\Gamma_1}(a)} + \frac{4s_2\lambda_{\Gamma_1}(a)}{s_2\lambda_{\Gamma_2}(b)} + \frac{4s_2\lambda_{\Gamma_1}(a)}{s_1s_2} + \frac{\lambda_{\Gamma_2}(b)^2}{2\lambda_{\Gamma_1}(a)\lambda_{\Gamma_2}(b)} + \frac{\lambda_{\Gamma_2}(b)^2}{2s_1\lambda_{\Gamma_1}(a)} + \frac{\lambda_{\Gamma_2}(b)^2}{s_2\lambda_{\Gamma_2}(b)} \\ &+ \frac{\lambda_{\Gamma_2}(b)^2}{s_1s_2} + \frac{2s_1\lambda_{\Gamma_2}(b)}{2\lambda_{\Gamma_1}(a)\lambda_{\Gamma_2}(b)} + \frac{2s_1\lambda_{\Gamma_2}(b)}{2s_1\lambda_{\Gamma_1}(a)} + \frac{2s_1\lambda_{\Gamma_2}(b)}{s_2\lambda_{\Gamma_2}(b)} + \frac{2s_1\lambda_{\Gamma_2}(b)}{s_1s_2} + \frac{(s_2^2+s_1^2)}{2\lambda_{\Gamma_1}(a)\lambda_{\Gamma_2}(b)} \\ &+ \frac{(s_2^2+s_1^2)}{2s_1\lambda_{\Gamma_1}(a)} + \frac{(s_2^2+s_1^2)}{s_2\lambda_{\Gamma_2}(b)} + \frac{(s_2^2+s_1^2)}{s_1s_2}\Bigg] \\ = \frac{1}{16}\Bigg[&4m_1ID(\Gamma_2)+\frac{4m_1s_2}{s_1}+\frac{4M_1(\Gamma_1)ID(\Gamma_2)}{s_2}+\frac{4M_1(\Gamma_1)}{s_1}+2s_1s_2ID(\Gamma_2) \\ &+2s_2^2+8m_1ID(\Gamma_2)+\frac{8m_1s_2}{s_1}+m_2ID(\Gamma_1)+\frac{ID(\Gamma_1)M_1(\Gamma_2)}{2s_1}+\frac{2m_2s_1}{s_2}+\frac{M_1(\Gamma_2)}{s_2} \\ &+s_1s_2ID(\Gamma_1)+2m_2ID(\Gamma_1)+2s_1^2+\frac{4m_2s_1}{s_2}+\frac{(s_2^2+s_1^2)ID(\Gamma_1)ID(\Gamma_2)}{2} \\ &+\frac{(s_2^2+s_1^2)s_2ID(\Gamma_1)}{2s_1}+\frac{(s_2^2+s_1^2)s_1ID(\Gamma_2)}{s_2}+(s_2^2+s_1^2)\Bigg]. \end{aligned}$$

Here we calculate the sum $I_6$, where

$$I_6 = \sum_{ab\in E(\Gamma_3)} \frac{(\lambda_{\Gamma_3}(a)+m_1)^2+(\lambda_{\Gamma_3}(b)+m_1)^2}{(\lambda_{\Gamma_3}(a)+m_1)(\lambda_{\Gamma_3}(b)+m_1)}$$

$$= \sum_{ab\in E(\Gamma_3)} \frac{(\lambda_{\Gamma_3}(a)^2+\lambda_{\Gamma_3}(b)^2)+2s_1(\lambda_{\Gamma_3}(a)+\lambda_{\Gamma_3}(b))+2m_1^2}{\lambda_{\Gamma_3}(a)\lambda_{\Gamma_3}(b)+m_1(\lambda_{\Gamma_3}(a)+\lambda_{\Gamma_3}(b))+m_1^2}.$$

By Jensen's inequality, we have

$$I_6 \le \frac{1}{16}\Bigg[\sum_{ab\in E(\Gamma_3)} \frac{(\lambda_{\Gamma_3}(a)^2+\lambda_{\Gamma_3}(b)^2)}{\lambda_{\Gamma_3}(a)\lambda_{\Gamma_3}(b)} + \sum_{ab\in E(\Gamma_3)} \frac{(\lambda_{\Gamma_3}(a)^2+\lambda_{\Gamma_3}(b)^2)}{m_1(\lambda_{\Gamma_3}(a)+\lambda_{\Gamma_3}(b))}$$
$$+ \sum_{ab\in E(\Gamma_3)} \frac{4(\lambda_{\Gamma_3}(a)^2+\lambda_{\Gamma_3}(b)^2)}{m_1^2} + \sum_{ab\in E(\Gamma_3)} \frac{2m_1(\lambda_{\Gamma_3}(a)+\lambda_{\Gamma_3}(b))}{\lambda_{\Gamma_3}(a)\lambda_{\Gamma_3}(b)}$$
$$+ \sum_{ab\in E(\Gamma_3)} \frac{2m_1(\lambda_{\Gamma_3}(a)+\lambda_{\Gamma_3}(b))}{m_1(\lambda_{\Gamma_3}(a)+\lambda_{\Gamma_3}(b))} + \sum_{ab\in E(\Gamma_3)} \frac{8m_1(\lambda_{\Gamma_3}(a)+\lambda_{\Gamma_3}(b))}{m_1^2}$$
$$+ \sum_{ab\in E(\Gamma_3)} \frac{2m_1^2}{\lambda_{\Gamma_3}(a)\lambda_{\Gamma_3}(b)} + \sum_{ab\in E(\Gamma_3)} \frac{2m_1^2}{m_1(\lambda_{\Gamma_3}(a)+\lambda_{\Gamma_3}(b))} + \sum_{ab\in E(\Gamma_3)} \frac{8m_1^2}{m_1^2}\Bigg]$$
$$= \frac{1}{16}\left[SDD(\Gamma_3)+\frac{D_1(\Gamma_3)}{m_1}+\frac{4F(\Gamma_3)}{m_1^2}+\frac{8M_1(\Gamma_3)}{m_1}+2m_1^2R_{-1}+m_1H(\Gamma_3)+2m_1s_3+10m_3\right].$$

Finally, we obtain the sum $I_7$, where

$$I_7 = \sum_{a\in I(\Gamma_1)}\sum_{b\in V(\Gamma_3)} \frac{(\lambda_{I(\Gamma_1)}(a)+s_3)^2+(\lambda_{\Gamma_3}(b)+m_1)^2}{(\lambda_{I(\Gamma_1)}(a)+s_3)(\lambda_{\Gamma_3}(b)+m_1)}$$
$$= \sum_{ab\in E(\Gamma_1)}\sum_{a\in V(\Gamma_3)} \frac{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)^2+(\lambda_{\Gamma_3}(a)+m_1)^2}{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b)+s_3)(\lambda_{\Gamma_3}(a)+m_1)}$$
$$= \sum_{ab\in E(\Gamma_1)}\sum_{a\in V(\Gamma_3)} \frac{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2+2s_3(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))+\lambda_{\Gamma_3}(a)^2+2m_1\lambda_{\Gamma_3}(a)+(s_3^2+m_1^2)}{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))\lambda_{\Gamma_3}(a)+m_1(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))+s_3\lambda_{\Gamma_3}(a)+s_3m_1}.$$

By Jensen's inequality, we get

$$I_7 \le \frac{1}{16}\sum_{ab\in E(\Gamma_1)}\sum_{a\in V(\Gamma_3)}\Bigg[\frac{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2}{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))\lambda_{\Gamma_3}(a)} + \frac{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2}{m_1(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}$$
$$+ \frac{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2}{s_3\lambda_{\Gamma_3}(a)} + \frac{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))^2}{s_3m_1} + \frac{2s_3(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))\lambda_{\Gamma_3}(a)}$$
$$+ \frac{2s_3(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{m_1(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))} + \frac{2s_3(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{s_3\lambda_{\Gamma_3}(a)} + \frac{2s_3(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))}{s_3m_1}$$
$$+ \frac{\lambda_{\Gamma_3}(a)^2}{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))\lambda_{\Gamma_3}(a)} + \frac{\lambda_{\Gamma_3}(a)^2}{m_1(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))} + \frac{\lambda_{\Gamma_3}(a)^2}{s_3\lambda_{\Gamma_3}(a)} + \frac{\lambda_{\Gamma_3}(a)^2}{s_3m_1}$$
$$+ \frac{2m_1\lambda_{\Gamma_3}(a)}{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))\lambda_{\Gamma_3}(a)} + \frac{2m_1d_{\Gamma_3}(a)}{m_1(\lambda_{\Gamma_1}(a)} + \frac{2m_1\lambda_{\Gamma_3}(a)}{s_3\lambda_{\Gamma_3}(a)} + \frac{2m_1\lambda_{\Gamma_3}(a)}{s_3m_1}$$
$$+ \frac{(s_3^2+m_1^2)}{(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))\lambda_{\Gamma_3}(a)} + \frac{(s_3^2+m_1^2)}{m_1(\lambda_{\Gamma_1}(a)+\lambda_{\Gamma_1}(b))} + \frac{(s_3^2+m_1^2)}{s_3\lambda_{\Gamma_3}(a)} + \frac{(s_3^2+m_1^2)}{s_3m_1}\Bigg]$$
$$= \frac{1}{16}\Bigg[\left(3ID(\Gamma_3)+\frac{2s_3}{m_1}\right)M_1(\Gamma_1)+\left(\frac{ID(\Gamma_3)}{s_3}+\frac{1}{m_1}\right)HM(\Gamma_1)$$

$$+\left(\frac{M_1(\Gamma_3)}{2m_1}+\frac{(2m_1+(s_3^2+m_1^2))}{2}+\frac{2m_1m_3+(s_3^2+m_1^2)}{2m_1}\right)H(\Gamma_1)$$
$$+\left(\frac{4s_3m_3+m_1+(s_3^2+m_1^2)}{s_3}\right)ID(\Gamma_1)$$
$$+2s_3m_1ID(\Gamma_3)+\frac{M_1(\Gamma_3)}{s_3}+\frac{6m_1m_3)}{s_3}+3(s_3^2+m_1^3)\Bigg].$$

Adding the sums $I_1$ to $I_7$, we get the desired upper bounds. □

## 4. The Mid Graph

The mid graph $Z(\Gamma)$ of a given graph $\Gamma$ is a graph which is obtained by subdividing each edge of $\Gamma$ exactly once and joining all the non-adjacent vertices $\Gamma$ in $Z(\Gamma)$. If $\Gamma$ is a graph with $s$ vertices and $m$ edges, then we observe that $|V(Z(\Gamma))| = |V(S(\Gamma))| = s+m$ and $|E(Z(\Gamma))| = |E(S(\Gamma))|\cup|\{x_ix_j \mid x_ix_j \notin E(\Gamma)\}| = \frac{s(s-1)}{2}+m$. Now, we find the exact value of *SDD* invariant of mid graphs.

**Theorem 4.1.** *For the* $(s,m)$ *graph* $\Gamma$, $SDD(Z(\Gamma)) = s(s-1)-m+\frac{m(s^2-2s+5)}{s-1}$.

*Proof.* Let $x_i$ and $x_j$ be adjacent in $\Gamma$. Then there is a pair of incident edges in $Z(\Gamma)$, that is, $x_ix_{ij}$ and $x_jx_{ij}$ are incident edges in $Z(\Gamma)$. Therefore every pair of incident edges in $\Gamma$, we have two incident edges in $Z(\Gamma)$ and total number of them is $2m$. Moreover, the total number of remaining incident edges of $Z(\Gamma)$ are the number of non-adjacent edges in $\Gamma$, that is, $\frac{1}{2}\sum_{x_i\in V(G)}(s-\lambda_\Gamma(x_i)-1)$ edges are non-adjacent in $\Gamma$. Hence by the definition of *SDD* invariant, we have

$$\begin{aligned}
SDD(Z(\Gamma)) &= \sum_{x_ix_j\in E(Z(\Gamma))}\frac{\lambda_{Z(\Gamma)}(x_i)^2+\lambda_{Z(\Gamma)}(x_j)^2}{\lambda_{Z(\Gamma)}(x_i)\lambda_{Z(\Gamma)}(x_j)}\\
&= \sum_{x_ix_j\in E(\Gamma)}\frac{\lambda_{Z(\Gamma)}(x_i)^2+\lambda_{Z(\Gamma)}(x_j)^2}{\lambda_{Z(\Gamma)}(x_i)\lambda_{Z(\Gamma)}(x_j)}+\sum_{x_ix_{ij}\in E(S(\Gamma))}\frac{\lambda_{Z(\Gamma)}(x_i)^2+\lambda_{Z(\Gamma)}(x_{ij})^2}{\lambda_{Z(\Gamma)}(x_i)\lambda_{Z(\Gamma)}(x_{ij})}\\
&= \sum_{x_i\in V(\Gamma)}\left(\frac{s-\lambda_\Gamma(x_i)-1}{2}\right)\left(\frac{2(s-1)^2}{(s-1)^2}\right)+\sum_{x_ix_{ij}\in E(S(\Gamma))}\frac{2^2+(s-1)^2}{2(s-1)}\\
&= \frac{2(s-1)^2}{(s-1)^2}\left(\frac{s(s-1)}{2}\right)-\sum_{x_i\in V(\Gamma)}\frac{\lambda_\Gamma(x_i)}{2}+2m\left(\frac{s^2-2s+5}{2(s-1)}\right)\\
&= s(s-1)-m+\frac{m(s^2-2s+5)}{s-1}.
\end{aligned}$$
□

## 5. Conclusion

We obtain the bounds for *SDD* invariant of join total graph using Jensen's inequality and *SDD* invariant of mid graphs.

### Competing Interests

The author declares that he has no competing interests.

### Authors' Contributions

The author wrote, read and approved the final manuscript.

## References


[1] R. Aruldoss and P. Murugarajan, Upper bounds for symmetric division deg index of graphs, *South East Asian Journal of Mathematics and Mathematical Sciences* **16** (2020), 157 – 162, URL: https://rsmams.org/journals/articleinfo.php?articleid=447&tag=seajmams.

[2] A. T. Balaban, Highly discriminating distance-based topological index, *Chemical Physics Letters* **89** (1982), 399 – 404, DOI: 10.1016/0009-2614(82)80009-2.

[3] J. Devillers and A. T. Balaban (editor), *Topological Indices and Related Descriptors in QSAR and QSPR*, Gordon and Breach Science Publishers, Amsterdam, The Netherlands, x + 811pp. (1999).

[4] B. Furtula, K. Ch. Das and I. Gutman, Comparative analysis of symmetric division deg index as potentially useful molecular descriptor, *International Journal of Quantum Chemistry* **118** (2018), e25659, DOI: 10.1002/qua.25659.

[5] C. K. Gupta, V. Lokesha, B. S. Shetty and P. S. Ranjini, Graph operations on the symmetric division deg invariant of graphs, *Palestine Journal of Mathematics* **6**(1) (2017), 280 – 286, URL: https://pjm.ppu.edu/sites/default/files/papers/PJM_MAR_2017_32_0.pdf.

[6] C. K. Gupta, V. Lokesha, B. S. Shetty and P. S. Ranjini, On the symmetric division deg index of graph, *Southeast Asian Bulletin of Mathematics* **40** (2016), 59 – 80, URL: http://www.seams-bull-math.ynu.edu.cn/downloadfile.jsp?filemenu=_201601&filename=06_40(1).pdf.

[7] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. Total $\varphi$-electron energy of alternant hydrocarbons, *Chemical Physics Letters* **17** (1972), 535 – 538, DOI: 10.1016/0009-2614(72)85099-1.

[8] M. Karelson, *Molecular Descriptors in QSAR/QSPR*, Wiley-Interscience, 448 pp., New York (2000), URL: https://www.wiley.com/en-us/Molecular+Descriptors+in+QSAR+QSPR+-p-9780471351689.

[9] H. Kober, On the arithmetic and geometric means and on Hölder's inequality, *Proceedings of the American Mathematical Society* **9** (1958), 452 – 459, DOI: 10.1090/S0002-9939-1958-0093564-7.

[10] G. Mohanappriya and D. Vijayalakshmi, Symmetric division degree index and inverse sum index of transformation graph, *Journal of Physics: Conference Series* **1139** (2018), 012048, DOI: 10.1088/1742-6596/1139/1/012048.

[11] P. Murugarajan and R. Aruldoss, Lower bounds for symmetric division degree invariant of graphs, *International Journal of Mathematics Trends and Technology* **68**(4) (2022), 38 – 42, DOI: 10.14445/22315373/IJMTT-V68I4P507.

[12] S. Nikolić, G. Kovacević, A. Milicević and N. Trinajstić, The Zagreb indices 30 years after, *Croatica Chemica Acta* **76** (2003), 113 – 124, URL: https://www.cmm.ki.si/~FAMNIT-knjiga/wwwANG/web-pages/CCA_76_2003_113-124_nikolic[1].pdf.

[13] J. L. Palacios, New upper bounds for the symmetric division deg index of graphs, *Discrete Mathematics Letters* **2** (2019), 52 – 56, URL: http://www.dmlett.com/archive/DML19_v2_p.52_56.pdf.

[14] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Springer-Verlag, Berlin (1972), DOI: 10.1007/978-1-4757-1640-5.

[15] J. Radon, *Theorie und anwendungen der absolut additiven mengenfunktionen*, Hölder, Wien, 144 pp. (1913) (in German).

[16] M. Randic, Characterization of molecular branching, *Journal of American Chemical Society* **97** (1975), 6609 – 6615, DOI: 10.1021/ja00856a001.

[17] B. C. Rennie, On a class of inequalities, *Journal of the Australian Mathematical Society* **3**(1963), 442 – 448, DOI: 10.1017/S1446788700039057.

[18] R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors*, WILEY-VCH Verlag GmbH, Weinheim (2000), DOI: 10.1002/9783527613106.

[19] A. Vasilyev, Upper and lower bounds of symmetric division deg index, *Iranian Journal of Mathematical Chemistry* **5**(2) (2014), 91 – 98, DOI: 10.22052/ijmc.2014.7357.

[20] D. Vukičević and M. Gašperov, Bond additive modeling 1. Adriatic indices, *Croatica Chemica Acta* **83** (2010), 243 – 260, URL: https://hrcak.srce.hr/file/93215.

[21] H. Wiener, Structural determination of paraffin boiling points, *Journal of the American Chemical Society* **69** (1947), 17 – 20, DOI: 10.1021/ja01193a005.