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Research Article

Fixed Point Theorems by Using Altering Distance Function in S-Metric Spaces

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Abstract. In the present paper, we prove some fixed point theorems by using the non-decreasing mapping $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ known as altering distance function or control function, in the context of S-metric space. Further, we explore the property P for these contractive mappings.

Keywords. Fixed point, S-metric space, Contractive mappings, Altering distance function

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

In 1922, the notion of fixed point theorem was introduced by Banach [3] which plays a significant role to figure out many complications in mathematical study such as variational inequalities, hypothesis of presence of solutions of nonlinear differential, functional and integral equations etc. This principle was amplified and progressed in numerous ways and different fixed point results were obtained.

The concept of altering distance function was established by Khan *et al*. [12] in 1984, for self mapping on a metric space. In research of fixed point theory Guttia and Kumssa [9] generalized the notion of altering distance function where they called them control function. Pupa and Mocanu [16] introduced altering distance and common fixed points under implicit relations. Using control function many authors extended the Banach Contraction Principle.

Gähler [8] in 1960, presented the idea of 2-metric, simply taking as motivation the area determined by three points. After that more attention were devoted in the generalization of 2-metric spaces, D^*-metric spaces, D-metric spaces etc. Mustafa and Sims [14] in 2003, declared that some results in D-metric spaces were not correct. In 2006 Mustafa and Sims also established the concept of a G-metric space by altering axioms of D-metric. Sedghi *et al.* [19] introduced the concept of S-metric and emphasize that S-metric space is a generalization of G-metric space. Further, Dung *et al.* [7] appeared that this assertion was inaccurate. However, S-metric space and G-metric space are independent of each other.

2. Preliminaries

We start by reviewing a few fundamental definitions and come about for S-metric spaces that will be required within the sequel.

Definition 2.1 ([19])**.** Let X be a non-empty set. An S-metric on X is a mapping $S : X \times X \times X \to \mathbb{R}^+$ which satisfies the following condition:

(S_1) $S(u,v,w) = 0$ if and only if $u = v = w = 0$;

(S_2) $S(u,v,w) \le S(u,u,a) + S(v,v,a) + S(w,w,a)$, for all $u,v,w,a \in X$.

The pair (X,S) is called an S-metric space.

Example 2.2 ([19])**.** Let $X = \mathbb{R}$. Then $S(u,v,w)$ is an S-metric on $\mathbb{R}$ given by $S(u,v,w) = |u-w| + |v-w|$, which is known as usual S-metric space on X.

Lemma 2.3 ([19])**.** *If (X,S) is an S-metric space on a non-empty set X, then (X,S) satisfy the symmetric condition, that is $S(u,u,v) = S(v,v,u)$, for all $u,v \in X$.*

Definition 2.4 ([8])**.** A sequence $\{u_n\}$ in (X,S) is said to be convergent to some point $u \in X$, if $S(u_n,u_n,u) \to 0$ as $n \to \infty$.

Definition 2.5 ([11])**.** A sequence $\{u_n\}$ in (X,S) is said to be Cauchy sequence if $S(u_n,u_n,u_m) \to 0$ as $n,m \to \infty$.

Definition 2.6 ([11])**.** An S-metric space (X,S) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 2.7 ([18])**.** *Let (X,S) be an S-metric space. If $u_n \to u$ and $v_n \to v$ then $S(u_n,u_n,v_n) \to S(u,u,v)$.*

Lemma 2.8 ([19])**.** *Let (X,S) be an S-metric space and $\{u_n\}$ is a convergent sequence in X. Then $\lim_{n\to\infty} u_n$ is unique.*

Lemma 2.9 ([19])**.** *If $\{u_n\}$ is a sequence of elements from S-metric space (X,S) satisfying the following property $S(u_n,u_n,u_{n+1}) \le \lambda S(u_{n-1},u_{n-1},u_n)$, for each $\lambda \in [0,1)$ where $n \in \mathbb{N}$, then $\{u_n\}$ is a Cauchy sequence.*

Definition 2.10 ([12]). A function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if the following property is satisfied:

(κ_1) $\kappa(0) = 0$,

(κ_2) κ is monotonically non-decreasing function,

(κ_3) κ is a continuous function.

By K we denote the set of all altering distance functions.

Definition 2.11 ([12]). Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying the following conditions:

(θ_1) θ is Lebesgue integrable function on each compact subset of $\mathbb{R}^+$.

(θ_2) θ is nonnegative.

(θ_3) $\int_0^u \theta(t)dt > 0$, for each $u > 0$.

We denote the set of all functions satisfying the conditions of Definition 2.11 by Ω.

We obtain a function $\kappa \in K$ by using a mapping $\theta \in \Omega$, which can be shown in the following lemma:

Lemma 2.12 ([15], [1]). *Let $\theta \in \Omega$. Define $\kappa_0 : \mathbb{R}^+ \to \mathbb{R}^+$ by*

$$\kappa_0(p) = \int_0^p \theta(t)dt, \quad \text{for all } t \in \mathbb{R}^+.$$

Then $\kappa_0 \in K$.

Lemma 2.13 ([9], [6]). *Let (X,S) be an S-metric space. If $\{u_n\}$ is not a Cauchy sequence in X then prove the following:*

(i) $\lim_{c\to\infty} S(u_{m(c)}, u_{m(c)}, u_{n(c)}) = \varepsilon$,

(ii) $\lim_{c\to\infty} S(u_{m(c)+1}, u_{m(c)+1}, u_{n(c)}) = \varepsilon$,

(iii) $\lim_{c\to\infty} S(u_{m(c)-1}, u_{m(c)-1}, u_{n(c)-1}) = \varepsilon$,

(iv) $\lim_{c\to\infty} S(u_{m(c)}, u_{m(c)}, u_{n(c)+1}) = \varepsilon$,

(v) $\lim_{c\to\infty} S(u_{m(c)+1}, u_{m(c)+1}, u_{n(c)+1}) = \varepsilon$,

(vi) $\lim_{c\to\infty} S(u_{m(c)+1}, u_{m(c)+1}, u_{n(c)+2}) = \varepsilon$.

In next section, we begin with the generalization of fixed point results via altering distance function in S-metric space which was proved by Rakotch [17]. Furthermore, we prove some fixed point theorems in S-metric space which are extension of results provided by Das and Gupta [4] as well as results given by Kumar *et al*. [13]. Also, we will use the P property initiated by Jeong and Rhoades in [10] on some contractive conditions.

3. Main Results

Theorem 3.1. *Let $\kappa \in K$ and (X,S) be a complete S-metric space. If T be a self mapping on X satisfying the condition:*

$$\kappa(S(Tu,Tu,Tv)) \le \lambda\kappa(S(u,u,v)), \tag{3.1}$$

for some $0 < \lambda < 1$ *and all* $u, v \in X$. *Then* $w \in X$ *is a unique fixed point of* T *such that for each* $u \in X$, $\lim_{n\to\infty} T^n u = w$.

Proof. Let $u_0 \in X$ be any arbitrary point. We define a sequence $\{u_n\}$ as $u_{n+1} = Tu_n = T^n u_0$, for all $n \geq 1$. It follows from (3.1) that

$$\kappa(S(u_n,u_n,u_{n+1})) \leq \lambda\kappa(S(u_{n-1},u_{n-1},u_n)). \tag{3.2}$$

Since κ is non decreasing function and $\lambda \in (0,1)$, then $\{S(u_n,u_n,u_{n+1})\}_n$ is a non-increasing sequence, thus there exists some $r \geq 0$ such that $\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = r$.

Now, we show that $r = 0$. Suppose that $r > 0$, then by taking limit $n \to \infty$ in (3.1) and we conclude that

$$0 \leq \kappa(r) \leq \lim_{n\to\infty} \sup \kappa(S(u_n,u_n,u_{n+1})) \leq \lim_{n\to\infty} \sup[\lambda\kappa(S(u_{n-1},u_{n-1},u_n))] < \kappa(r),$$

which is contradiction. Thus $r = 0$, that is

$$\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = 0. \tag{3.3}$$

Now, we prove that $\{u_n\}$ is a Cauchy sequence in X. If $\{u_n\}$ is not Cauchy, then for any $\varepsilon > 0$, there exists integers $m(c)$ and $n(c)$ where $c > 0$ with $m(c) > n(c) > c$ such that $S(u_{m(c)},u_{m(c)},u_{n(c)+1}) \geq \varepsilon$ and $S(u_{m(c)-1},u_{m(c)-1},u_{n(c)}) < \varepsilon$.

Thus, from Lemma 2.13, we have

$$\varepsilon = \lim_{c\to\infty} S(u_{m(c)},u_{m(c)},u_{n(c)+1}) = \lim_{c\to\infty} S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+2}), \tag{3.4}$$

therefore, from (3.1) and (3.2) we deduce that

$$\kappa(S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+2})) \leq \lambda\kappa(S(u_{m(c)},u_{m(c)},u_{n(c)+1})).$$

Taking limit $c \to \infty$ and using (3.1) and (3.4) we conclude that

$$0 \leq \kappa(\varepsilon) = \lim_{c\to\infty} \lambda \sup \kappa(S(u_{m(c)},u_{m(c)},u_{n(c)+1})) < \kappa(\varepsilon),$$

which is contradiction. Thus, sequence $\{u_n\}$ is a Cauchy sequence. As (X,S) is complete then there exists $w \in X$ such that $\lim_{n\to\infty} T^n u = w$.

Now, we are going to prove that $Tw = w$. By using inequalities (3.1) we have

$$0 \leq \kappa(S(u_{n+1},u_{n+1},Tw)) \leq \lambda\kappa(S(u_n,u_n,w)) < \kappa(S(u_n,u_n,w)),$$

where $\kappa(S(u_n,u_n,w)) \to 0$ as $n \to \infty$. That is, $\lim_{n\to\infty} \kappa(S(u_{n+1},u_{n+1},Tw)) = 0$.

Since $\kappa \in K$ we have that $\lim_{n\to\infty} S(u_{n+1},u_{n+1},Tw) = 0$. Consequently,

$$S(w,w,Tw) \leq 2S(w,w,u_{n+1}) + S(u_{n+1},u_{n+1},Tw) \to 0, \text{ as } n \to \infty,$$

which means that $Tw = w$.

At last, we claim that T has a unique fixed point. If not so then there exists another fixed point $w' \in X$ such that

$$0 \leq \kappa(S(w',w',w)) = \kappa(S(Tw',Tw',Tw)) \leq \lambda\kappa(S(w',w',w)) < \kappa(S(w',w',w)),$$

which is contradiction. Thus, we have $w = w'$ and hence T has a unique fixed point. □

Theorem 3.2. *Let T be a self mapping on a complete S-metric space (X,S) that satisfies the following condition:*

$$\kappa(S(Tu,Tu,Tv)) \le \zeta(S(u,u,v))\kappa(S(u,u,v)), \tag{3.5}$$

where $\kappa \in K$ and $\zeta : \mathbb{R}^+ \to [0,1)$ with

$$\limsup_{x\to t} \zeta(x) < 1, \quad \text{for all } t > 0. \tag{3.6}$$

Then $w \in X$ is a unique fixed point of T such that for each $u \in X$, $\lim_{n\to\infty} T^n u = w$.

Proof. Let $u_0 \in X$ be any arbitrary point. We define the recurrence relation as $u_{n+1} = Tu_n = T^n u_0$. It follows from (3.5) and (3.6) that

$$\kappa(S(u_n,u_n,u_{n+1})) \le \zeta(S(u_{n-1},u_{n-1},u_n))\kappa(S(u_{n-1},u_{n-1},u_n)) < \kappa(S(u_{n-1},u_{n-1},u_n)). \tag{3.7}$$

Since κ is an increasing function, then $\{S(u_n,u_n,u_{n+1})\}_n$ is a non-increasing sequence, thus there exists some $r \ge 0$ such that $\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = r$.
Now, we show that $r = 0$. If $r > 0$, then taking limit $n \to \infty$ in (3.7) and using (3.6) we obtain

$$\begin{aligned}
0 \le \kappa(r) &= \limsup_{n\to\infty} \kappa(S(u_n,u_n,u_{n+1})) \\
&\le \limsup_{n\to\infty} [\zeta(S(u_{n-1},u_{n-1},u_n))\kappa(S(u_{n-1},u_{n-1},u_n))] \\
&\le \limsup_{n\to\infty} \zeta(S(u_{n-1},u_{n-1},u_n)) \limsup_{n\to\infty} \kappa(S(u_{n-1},u_{n-1},u_n)) \\
&\le \limsup_{x\to r} \zeta(x)\kappa(r) < \kappa(r).
\end{aligned}$$

Thus, we get contradiction which implies that $r = 0$, that is

$$\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = 0.$$

Now, we prove that $\{u_n\}$ is a Cauchy sequence in X. If $\{u_n\}$ is not Cauchy, then for any $\varepsilon > 0$, there exists integers $m(c)$ and $n(c)$ where $c > 0$ with $m(c) > n(c) > c$ such that $S(u_{m(c)},u_{m(c)},u_{n(c)+1}) \ge \varepsilon$ and $S(u_{m(c)-1},u_{m(c)-1},u_{n(c)}) < \varepsilon$. Thus, from Lemma 2.13 we have

$$\varepsilon = \lim_{c\to\infty} S(u_{m(c)},u_{m(c)},u_{n(c)+1}) = \lim_{c\to\infty} S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+2}). \tag{3.8}$$

Therefore, from (3.5) and (3.6) we deduce that

$$\begin{aligned}
\kappa(S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+2})) &\le \zeta(S(u_{m(c)},u_{m(c)},u_{n(c)+1}))\kappa(S(u_{m(c)},u_{m(c)},u_{n(c)+1})) \\
&< \kappa(S(u_{m(c)},u_{m(c)},u_{n(c)+1})).
\end{aligned} \tag{3.9}$$

Taking limit $c \to \infty$ in (3.9) and using (3.6) and (3.8) we obtain

$$\begin{aligned}
0 \le \kappa(\varepsilon) &= \limsup_{c\to\infty} \kappa(S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+2})) \\
&\le \limsup_{c\to\infty} \zeta(S(u_{m(c)},u_{m(c)},u_{n(c)+1})) \lim_{c\to\infty} \kappa(S(u_{m(c)},u_{m(c)},u_{n(c)+1})) < \kappa(\varepsilon),
\end{aligned}$$

which is contradiction. Thus, $\{u_n\}$ is a Cauchy sequence. Also, as (X,S) is complete therefore there exists $w \in X$ such that $\lim_{n\to\infty} T^n u = w$.
Now, we are going to prove that $Tw = w$.

By using inequalities (3.5) and (3.6) we get

$$0 \leq \kappa(S(u_{n+1},u_{n+1},Tw)) \leq \zeta(S(u_n,u_n,w))\kappa(S(u_n,u_n,w)) < \kappa(S(u_n,u_n,w)),$$

where $\kappa(S(u_n,u_n,w)) \to 0$ as $n \to \infty$.
That is,

$$\lim_{n\to\infty} \kappa(S(u_{n+1},u_{n+1},Tw)) = 0.$$

Since $\kappa \in K$, we have that $\lim_{n\to\infty} S(u_{n+1},u_{n+1},Tw) = 0$. Consequently,

$$S(w,w,Tw) \leq 2S(w,w,u_{n+1}) + S(u_{n+1},u_{n+1},Tw) \to 0, \quad \text{as } n \to \infty,$$

which means that $Tw = w$.
Now, we claim that T has a unique fixed point.
Let us assume that T has another fixed point $w' \in X$ then,

$$\begin{aligned}0 \leq \kappa(S(w',w',w)) &= \kappa(S(Tw',Tw',Tw)) \\ &\leq \zeta(S(w',w',w))\kappa(S(w',w',w)) \\ &< \kappa(S(w',w',w)),\end{aligned}$$

which is contradiction.
That is, $w = w'$ and hence T has a unique fixed point. □

Corollary 3.3. *Let $T: X \to X$ be a mapping on a complete S-metric space (X,S) which satisfying the following inequality*

$$\int_0^{\kappa(S(Tu,Tu,Tv))} \theta(t)dt \leq \zeta(S(u,u,v)) \int_0^{\kappa(S(u,u,v))} \theta(t)dt, \tag{3.10}$$

where $\kappa \in K$, $\theta \in \Omega$ and $\zeta: \mathbb{R}^+ \to [0,1)$ with $\limsup_{x\to t} \zeta(x) < 1$, for all $t > 0$.
Then T has a unique fixed point $w \in X$ such that for each $u \in X$, $\lim_{n\to\infty} T^n u = w$.

Proof. We define $\kappa_0 : \mathbb{R}^+ \to \mathbb{R}^+$ by $\kappa_0(x) = \int_0^x \theta(t)dt$ for $\theta \in \Omega$, then $\kappa_0 \in K$ and so inequality (3.10) becomes

$$\kappa_0(\kappa(S(Tu,Tu,Tv))) \leq \zeta(S(u,u,v))\kappa_0(\kappa(S(u,u,v))).$$

Further, we can write it as

$$\kappa_1(S(Tu,Tu,Tv)) \leq \zeta(S(u,u,v))\kappa_1(S(u,u,v)),$$

where $\kappa_1 = \kappa_0 \circ \kappa \in K$. Hence, from Theorem 3.2 we deduce that the fixed point we obtain is unique. □

Example 3.4. Let $X = [0,1]$ and (X,S) be a complete S-metric space defined by

$$S(u,v,w) = |u-w| + |v-w|, \quad \text{for every } u,v,w \in X.$$

Let a mapping $T: X \to X$ be defined as $Tu = \frac{1}{25}u^2 e^{u^{-2}}$, for each $u \in X$.
We define a function $\zeta: \mathbb{R}^+ \to [0,1)$ by $\zeta(x) = \frac{1}{25}$, for all $x \in \mathbb{R}^+$ and a non-decreasing altering distance function be defined as $\kappa(t) = 2t$, for all $t \in \mathbb{R}^+$.

Then for all $u,v \in X$ we have

$$\begin{aligned}\kappa(S(Tu,Tu,Tv)) &= \kappa(2|Tu-Tv|)\\ &= 4|Tu-Tv|\\ &= 4\left|\frac{1}{25}u^2e^{u^{-2}} - \frac{1}{25}v^2e^{v^{-2}}\right|\\ &\le \frac{4}{25}|u-v|\\ &\le \frac{2}{25}S(u,u,v) = \frac{1}{25}\kappa(S(u,u,v))\\ &\le \zeta(S(u,u,v))\kappa(S(u,u,v)).\end{aligned}$$

Therefore, Theorem 3.2 is applicable to T. Also, T has a fixed point at $u=0$.

Theorem 3.5. *Let T be a self mapping on a complete S-metric space (X,S) that satisfies the following inequality:*

$$\kappa(S(Tu,Tu,Tv)) \le \zeta(S(u,u,v))\kappa(S(u,u,Tu)) + \eta(S(u,u,v))\kappa(S(v,v,Tv)), \tag{3.11}$$

where $\kappa \in K$ and $\zeta,\eta : \mathbb{R}^+ \to [0,1)$ with

$$\left.\begin{array}{l}\zeta(t)+\eta(t) < 1 \text{ for all } t \in \mathbb{R}^+,\ \lim_{x\to 0^+}\eta(x) < 1\\ \limsup_{x\to t^+}\frac{\zeta(x)}{1-\eta(x)} < 1 \text{ for all } t > 0\end{array}\right\} \tag{3.12}$$

Then $w \in X$ is a unique fixed point of T such that for each $u \in X$, $\lim_{n\to\infty} T^n u = w$.

Proof. Let $u_0 \in X$ be any arbitrary point and sequence $\{u_n\}$ be define by the relation $u_{n+1} = Tu_n = T^n u_0$. It follows from (3.11) that

$$\begin{aligned}\kappa(S(u_n,u_n,u_{n+1})) \le\ &\zeta(S(u_{n-1},u_{n-1},u_n))\kappa(S(u_{n-1},u_{n-1},u_n))\\ &+\eta(S(u_{n-1},u_{n-1},u_n))\kappa(S(u_n,u_n,u_{n+1})),\end{aligned}$$

which implies that

$$\kappa(S(u_n,u_n,u_{n+1})) \le \frac{\zeta(S(u_{n-1},u_{n-1},u_n))}{1-\eta(S(u_{n-1},u_{n-1},u_n))}\kappa(S(u_{n-1},u_{n-1},u_n)).$$

From (3.12) we obtain

$$\kappa(S(u_n,u_n,u_{n+1})) \le \kappa(S(u_{n-1},u_{n-1},u_n)).$$

In Theorem 3.2 we have already proved that $\{S(u_n,u_n,u_{n+1})\}_n$ is a decreasing sequence and it converges to 0, which implies that

$$\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = 0. \tag{3.13}$$

Now, we will prove that sequence $\{u_n\}$ is a Cauchy sequence in X. If $\{u_n\}$ is not Cauchy, then for any $\varepsilon > 0$, there exists integers $m(c)$ and $n(c)$ where $c > 0$ with $m(c) > n(c) > c$ such that $S(u_{m(c)},u_{m(c)},u_{n(c)}) \ge \varepsilon$ and $S(u_{m(c)-1},u_{m(c)-1},u_{n(c)}) < \varepsilon$. Thus, from Lemma 2.13 we have

$$\lim_{c\to\infty} S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+2}) = \varepsilon.$$

Also, from (3.11) we get

$$\begin{aligned}\kappa(\varepsilon) &= \lim_{c\to\infty}\sup\kappa(S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+2}))\\ &\le \lim_{c\to\infty}\sup[\zeta(S(u_{m(c)},u_{m(c)},u_{n(c)+1}))\kappa(S(u_{m(c)},u_{m(c)},u_{m(c)+1}))]\\ &\quad+\lim_{c\to\infty}\sup[\eta(S(u_{m(c)},u_{m(c)},u_{n(c)+1}))\kappa(S(u_{n(c)+1},u_{n(c)+1},u_{n(c)+2}))]\\ &\le \lim_{c\to\infty}\sup\zeta(S(u_{m(c)},u_{m(c)},u_{n(c)+1}))\lim_{c\to\infty}\sup\kappa(S(u_{m(c)},u_{m(c)},u_{m(c)+1}))\\ &\quad+\lim_{c\to\infty}\sup\eta(S(u_{m(c)},u_{m(c)},u_{n(c)+1}))\lim_{c\to\infty}\sup\kappa(S(u_{n(c)+1},u_{n(c)+1},u_{n(c)+2}))\\ &< \lim_{c\to\infty}\sup\kappa(S(u_{m(c)},u_{m(c)},u_{m(c)+1}))+\lim_{c\to\infty}\sup\kappa(S(u_{n(c)+1},u_{n(c)+1},u_{n(c)+2}))=0,\end{aligned}$$

which is contradiction. Hence, sequence $\{u_n\}$ is Cauchy sequence. Also, as (X,S) is a complete S-metric space therefore there exists $w\in X$ such that $\lim_{n\to\infty}T^n u = w$.
Now, we claim that $Tw = w$, that is $S(w,w,Tw) = 0$.
Suppose that $S(w,w,Tw)\ne 0$, then from (3.11) and (3.12) we have

$$\begin{aligned}0<\kappa(S(w,w,Tw)) &= \lim_{n\to\infty}\sup\kappa(S(T^n u,T^n u,Tu))\\ &\le \lim_{n\to\infty}\sup[\zeta(S(T^{n-1}u,T^{n-1}u,w))\kappa(S(T^{n-1}u,T^{n-1}u,T^n u))]\\ &\quad+\lim_{n\to\infty}\sup[\eta(S(T^{n-1}u,T^{n-1}u,w))\kappa(S(w,w,Tw))]\\ &= \lim_{u\to 0^+}\sup[\eta(x)\kappa(S(w,w,Tw))]<\kappa(S(w,w,Tw)),\end{aligned}$$

which is impossible. Thus, we conclude that $S(w,w,Tw)=0 \Longrightarrow w = Tw$.
Finally, we claim that T has a unique fixed point.
Let us assume that T has fixed point w' with $w'\ne w$ and $Tw' = w'$ then from (3.11) we have

$$\begin{aligned}0<\kappa(S(w,w,w')) &= \kappa(S(Tw,Tw,Tw'))\\ &\le \zeta(S(w,w,w'))\kappa(S(w,w,Tw))+\eta(S(w,w,w'))\kappa(S(w',w',Tw'))\\ &=0,\end{aligned}$$

which is contradiction. Thus, we conclude that $w = w'$ which implies that T has a unique fixed point. □

Corollary 3.6. *Let $T: X\to X$ be a mapping on a complete S-metric space (X,S) which satisfying the following inequality*

$$\int_0^{\kappa(S(Tu,Tu,Tv))}\theta(t)dt \le \zeta(S(u,u,v))\int_0^{\kappa(S(u,u,Tv))}\theta(t)dt+\eta(S(x,x,y))\int_0^{\kappa(S(v,v,Tv))}\theta(t)dt, \tag{3.14}$$

where $\kappa\in K$, $\theta\in\Omega$ and $\zeta,\eta:\mathbb{R}^+\to[0,1)$ with

$$\left.\begin{array}{l}\zeta(t)+\eta(t)<1 \text{ for all } t\in\mathbb{R}^+,\ \lim\limits_{u\to 0^+}\eta(x)<1\\ \lim\limits_{x\to t^+}\sup\frac{\zeta(x)}{1-\eta(x)}<1 \text{ for all } \ t>0\end{array}\right\}$$

Then T has a unique fixed point $w\in X$ such that for each $u\in X$, $\lim_{n\to\infty}T^n u = w$.

Proof. We define $\kappa_0 : \mathbb{R}^+ \to \mathbb{R}^+$ by $\kappa_0(x) = \int_0^x \theta(t)dt$ for $\theta \in \Omega$, then $\kappa_0 \in K$ and from condition (3.14) we have

$$\kappa_0(\kappa(S(Tu,Tu,Tv))) \le \zeta(S(u,u,v))\kappa_0(\kappa(S(u,u,Tu))) + \eta(S(u,u,v))\kappa_0(\kappa(S(v,v,Tv))).$$

Setting $\kappa_1 = \kappa_0 \circ \kappa \in K$, from Theorem 3.5 we deduce that T possess a unique fixed point. □

Corollary 3.7. *Let (X,S) be a complete S-metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$\int_0^{\kappa(S(Tu,Tu,Tv))} \theta(t)dt \le \zeta(S(u,u,v))\left(\int_0^{\kappa(S(u,u,Tu))} \theta(t)dt + \int_0^{\kappa(S(v,v,Tv))} \theta(t)dt\right),$$

where $\kappa \in K$, $\theta \in \Omega$ and $\zeta,\eta : \mathbb{R}^+ \to \left(0,\frac{1}{2}\right]$ is a function with

$$\lim_{x\to t^+} \sup \frac{\zeta(x)}{1-\zeta(x)} < 1, \quad \textit{for all } t > 0.$$

Then T has a unique fixed point $w \in X$.

Proof. Proof follows from Corollary 3.6. □

Theorem 3.8. *Let (X,S) be a complete S-metric space and let $T : X \to X$ be a continuous mapping. We denote*

$$M(u,u,v) = \max\left\{\frac{S(u,u,Tu)\, S(v,v,Tv)}{S(u,u,v)}, S(u,u,v)\right\}, \tag{3.15}$$

for all $u,v \in X$, $u \neq v$. Suppose that T satisfies the following condition:

$$\kappa(S(Tu,Tu,Tv)) \le \zeta(S(u,u,v))\kappa(M(u,u,v)), \tag{3.16}$$

for all $u,v \in X$, $\kappa \in K$ and $\zeta : \mathbb{R} \to [0,1)$ is a mapping with

$$\lim_{x\to t} \sup \zeta(x), \textit{ for all } t > 0. \tag{3.17}$$

Then T possess a fixed point $w \in X$ which is unique and $\lim_{n\to\infty} T^n u = w$ for all $u \in X$.

Proof. Let $u_0 \in X$ be any arbitrary point, we define recurrence relation as $u_{n+1} = Tu_n = T^n u_0$, $n \ge 1$. It follows from (3.16) that

$$\begin{aligned}\kappa(S(u_n,u_n,u_{n+1})) &= \kappa(S(Tu_{n-1},Tu_{n-1},Tu_n)) \\ &\le \zeta(S(u_{n-1},u_{n-1},u_n))\kappa(M(u_{n-1},u_{n-1},u_n)).\end{aligned}$$

By using (3.15) we get,

$$\begin{aligned}M(u_{n-1},u,u_n) &= \max\left\{\frac{S(u_{n-1},u_{n-1},Tu_{n-1})S(u_n,u_n,Tu_n)}{S(u_{n-1},u_{n-1},u_n)}, S(u_{n-1},u_{n-1},u_n)\right\} \\ &= \max\left\{\frac{S(u_{n-1},u_{n-1},u_n)S(u_n,u_n,u_{n+1})}{S(u_{n-1},u_{n-1},u_n)}, S(u_{n-1},u_{n-1},u_n)\right\} \\ &= \max\{S(u_n,u_n,u_{n+1}), S(u_{n-1},u_{n-1},u_n)\}.\end{aligned}$$

Thus, we obtain

$$\kappa(S(u_n,u_n,u_{n+1})) \le \zeta(S(u_{n-1},u_{n-1},u_n))\kappa(\max\{S(u_n,u_n,u_{n+1}),S(u_{n-1},u_{n-1},u_n)\}).$$

Since κ is monotonically non decreasing and from (3.17) we have

$$\kappa(S(u_n,u_n,u_{n+1})) < \kappa(\max\{S(u_n,u_n,u_{n+1}),S(u_{n-1},u_{n-1},u_n)\}) = \kappa(S(u_{n-1},u_{n-1},u_n)).$$

Then it follows that $\{S(u_n,u_n,u_{n+1})\}_n$ is non increasing sequence. Consequently, there exists $r \ge 0$ such that $\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = r$. Suppose that $r > 0$, then

$$0 \le \kappa(r) \le \kappa(S(u_n,u_n,u_{n+1})) < \kappa(S(u_{n-1},u_{n-1},u_n)).$$

Taking limit $n \to \infty$ above inequality yields $\kappa(r) < \kappa(r)$ which is contradiction. Therefore $r = 0$. So, we get

$$\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = 0. \tag{3.18}$$

Now, we prove that $\{u_n\}$ is a Cauchy sequence in X. If $\{u_n\}$ is not Cauchy, then for any $\varepsilon > 0$, there exists integers $m(c)$ and $n(c)$ where $c > 0$ with $m(c) > n(c) > c$ such that $S(u_{m(c)},u_{m(c)},u_{n(c)}) \ge \varepsilon$ and $S(u_{m(c)-1},u_{m(c)-1},u_{n(c)}) < \varepsilon$. From Lemma 2.13 we have

$$\lim_{c\to\infty} S(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1}) = \varepsilon. \tag{3.19}$$

Now from inequality (3.16) we have

$$\begin{aligned}\kappa(\varepsilon) &\le \kappa(S(u_{m(c)},u_{m(c)},u_{n(c)}))\\ &= \kappa(S(Tu_{m(c)-1},Tu_{m(c)-1},Tu_{n(c)-1}))\\ &\le \zeta(S(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1}))\kappa(M(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})).\end{aligned}$$

Also, from (3.15) we have

$$\begin{aligned}&M(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})\\ &\quad = \max\left\{\frac{S(u_{m(c)-1},u_{m(c)-1},u_{m(c)})S(u_{n(c)-1},u_{n(c)-1},u_{n(c)})}{M(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})}, M(u_{m(c)-1},u_{m(c)-1},u_{n(c)-1})\right\}.\end{aligned}$$

Now by taking upper limit as $c \to \infty$ and using (3.18) and (3.19), we have

$$\kappa(\varepsilon) \le \kappa \max\{0,\varepsilon\} = \kappa(\varepsilon),$$

which is contradiction. Hence, $\{u_n\}$ is a Cauchy sequence in (X,S). Thus, there exists $w \in X$ such that $\lim_{n\to\infty} T^n u = w$.

Now we claim that T has a fixed point.

$$0 \le \kappa(S(u_{n+1},u_{n+1},Tw)) \le \zeta(S(u_n,u_n,w))\kappa(M(u_n,u_n,w))$$

where $M(u_n,u_n,w) = \max\left\{\frac{S(u_n,u_n,u_{n+1})S(w,w,Tw)}{M(u_n,u_n,w)}, M(u_n,u_n,w)\right\}$.

Taking limit $n \to \infty$ in above inequality, we obtain

$$\kappa(S(w,w,Tw)) = 0 \Longrightarrow S(w,w,Tw) = 0 \ \text{ as } \ \kappa \in K.$$

Hence, we say that $Tw = w$, which means that T has a fixed point. Now we are going to prove that this fixed point so obtain is unique. So let w' be another fixed point of T then $w' = Tw'$ and

$w' \neq w$. Also,

$$\kappa(S(w,w,w')) = \kappa(S(Tw,Tw,Tw')) \leq \zeta(S(w,w,w'))\kappa(M(w,w,w')),$$

where $M(w,w,w') = \max\left\{\frac{S(w,w,Tw)\; S(w',w',Tw')}{S(w,w,w')}, S(w,w,w')\right\} = \max\{0, S(w,w,w')\}$.
Thus, we get $\kappa(S(w,w,w')) < \kappa S(w,w,w')$, which is contradiction.
Hence, T has a unique fixed point w in X. □

Next, we generalize the results of Das and Gupta [4] in S-metric space which are as follow:

Theorem 3.9. *Let $T : X \to X$ be a mapping on a complete S-metric space (X,S) such that*

$$S(Tu,Tu,Tv) \leq \lambda S(u,u,v) + \eta \frac{S(v,v,Tv)[1+S(u,u,Tu)]}{1+S(u,u,v)}, \tag{3.20}$$

for all $u,v \in X$, $\lambda,\eta > 0$, $\lambda+\eta < 1$. Then T possess a fixed point $w \in X$ which is unique.

Proof. Suppose $u_0 \in X$ be any arbitrary point and $\{u_n\}$ be an iterative sequence defined as $u_{n+1} = Tu_n = T^n u_0$, for each $n \geq 1$. Then, from (3.20) we have

$$\begin{aligned} S(u_n,u_n,u_{n+1}) &= S(Tu_{n-1},Tu_{n-1},Tu_n) \\ &\leq \lambda S(u_{n-1},u_{n-1},u_n) + \eta \frac{S(u_n,u_n,u_{n+1})[1+S(u_{n-1},u_{n-1},u_n)]}{1+S(u_{n-1},u_{n-1},u_n)} \\ &\leq \lambda S(u_{n-1},u_{n-1},u_n) + \eta S(u_n,u_n,u_{n+1}), \\ (1-\eta)S(u_n,u_n,u_{n+1}) &\leq \lambda S(u_{n-1},u_{n-1},u_n), \\ S(u_n,u_n,u_{n+1}) &\leq \frac{\lambda}{(1-\eta)} S(u_{n-1},u_{n-1},u_n) \\ &\leq \left(\frac{\lambda}{1-\eta}\right)^2 S(u_{n-2},u_{n-2},u_{n-1}). \end{aligned}$$

Continuing this way, we obtain

$$S(u_n,u_n,u_{n+1}) \leq \left(\frac{\lambda}{1-\eta}\right)^n S(u_0,u_0,u_1).$$

$$S(u_n,u_n,u_{n+1}) \leq t\, S(u_0,u_0,u_1), \quad \text{where } t = \left(\frac{\lambda}{1-\eta}\right)^n < 1.$$

Thus, using Lemma 2.9 we say that $\{u_n\}$ is a Cauchy sequence on a complete S-metric space and thus there exists $w \in X$ such that $\lim_{n\to\infty} u_n = w$.
Now we claim that T possess a fixed point z.

$$\begin{aligned} S(w,w,Tw) &\leq 2\, S(w,w,u_n) + S(u_n,u_n,Tw) \\ &\leq 2\, S(w,w,u_n) + \lambda S(u_{n-1},u_{n-1},w) + \eta \frac{S(w,w,Tw)[1+S(u_{n-1},u_{n-1},u_n)]}{1+S(u_{n-1},u_{n-1},w)}. \end{aligned}$$

$$\left[1 - \eta\left(\frac{1+S(u_{n-1},u_{n-1},u_n)}{1+S(u_{n-1},u_{n-1},w)}\right)\right] S(w,w,Tw) \leq 2\, S(w,w,u_n) + \lambda S(u_{n-1},u_{n-1},w).$$

Now taking limit as $n \to \infty$ we get,

$$\lim_{n\to\infty} S(w,w,Tw) \leq \frac{2}{[1-\eta]} \lim_{n\to\infty} S(w,w,u_n) + \frac{\lambda}{[1-\eta]} \lim S(u_{n-1},u_{n-1},w)_{n\to\infty},$$

$$\lim_{n\to\infty} S(w,w,Tw) = 0 \Longrightarrow Tw = w,$$

which implies that T has a fixed point.
Now we claim that the fixed point we obtain is unique.
Let w' be another fixed point of T such that $Tw' = w'$ with $w' \neq w$.
Therefore, (3.20) implies that

$$\begin{aligned} S(w,w,w') &= S(Tw,Tw,Tw') \\ &\le \lambda(S(w,w,w')) + \eta\frac{S(w',w',Tw')[1+S(w,w,Tw)]}{1+S(w,w,w')} \\ &\le \lambda(S(w,w,w')), \end{aligned}$$

which is contradiction as $\lambda \in (0,1)$. Hence $w = w'$.
Thus, T has unique fixed point. □

Example 3.10. Let (X,S) be a complete S-metric space on $X = [0,1)$ defined by

$$S(u,v,w) = \frac{1}{2}(|u-w|+|v-w|), \quad \text{for all } u,v,w \in X.$$

Let a self-map T on X be defined by $Tu = \frac{u}{4}$. Then for all $u,v \in X$ and $\lambda = \frac{1}{4}, \eta = \frac{3}{5}$ we have

$$\begin{aligned} &S(Tu,Tv,Tv) = \left|\frac{u}{4} - \frac{v}{4}\right| = \frac{1}{4}|u-v|, \quad S(u,u,v) = |u-v|, \\ &S(v,v,Tv) = |v - Tv| = \left|v - \frac{v}{4}\right|, \quad S(u,u,Tu) = \left|u - \frac{u}{4}\right|. \end{aligned}$$

Therefore,

$$\begin{aligned} S(Tu,Tu,Tv) = \frac{1}{4}|u-v| &\le \frac{1}{4}|u-v| + \frac{3}{5}\frac{\left|v-\frac{v}{4}\right|\left[1+\left|u-\frac{u}{4}\right|\right]}{1+|u-v|} \\ &\le \lambda S(u,u,v) + \eta\frac{S(v,v,Tv)[1+S(u,u,Tu)]}{1+S(u,u,v)}. \end{aligned}$$

Hence, T satisfies all condition of Theorem 3.9 and has a fixed point at $u = 0$.
Here by considering the altering distance function we will generalize the Theorem 3.9.

Theorem 3.11. *Let $\kappa \in K$ and (X,S) be a complete S-metric space. If T be a self mapping on X satisfying the following condition:*

$$\kappa(S(Tu,Tu,Tv)) \le \lambda\kappa(S(u,u,v)) + \eta\kappa(M(u,u,v)), \tag{3.21}$$

for all $u,v \in X$, $\lambda,\eta > 0$, $\lambda+\eta < 1$ and $M(u,u,v) = S(v,v,Tv)\left[\frac{1+S(u,u,Tu)}{1+S(u,u,v)}\right]$. Then T possess a fixed point $w \in X$ which is unique and $\lim_{n\to\infty} T^n u = w$ for all $u \in X$.

Proof. Suppose $u_0 \in X$ be any arbitrary point and the sequence $\{u_n\}$ define by a relation $u_{n+1} = Tu_n = T^n u_0$, for each $n \ge 1$. Then, from (3.21)

$$\begin{aligned} \kappa(S(u_n,u_n,u_{n+1})) &= \kappa(S(Tu_{n-1},Tu_{n-1},Tu_n)) \\ &\le \lambda\kappa(S(u_{n-1},u_{n-1},u_n)) + \eta\kappa(M(u_{n-1},u_{n-1},u_n)), \end{aligned}$$

where, $M(u_{n-1},u_{n-1},u_n) = S(u_n,u_n,u_{n+1})\left[\frac{1+S(u_{n-1},u_{n-1},u_n)}{1+S(u_{n-1},u_{n-1},u_n)}\right] = S(u_n,u_n,u_{n+1})$.

Thus, we obtain

$$\kappa(S(u_n,u_n,u_{n+1})) \le \lambda\kappa(S(u_{n-1},u_{n-1},u_n)) + \eta\kappa(S(u_n,u_n,u_{n+1})),$$

$$\begin{aligned}\kappa(S(u_n,u_n,u_{n+1})) &\le \frac{\lambda}{1-\eta}\kappa(S(u_{n-1},u_{n-1},u_n))\\ &\le \left(\frac{\lambda}{1-\eta}\right)^2 \kappa(S(u_{n-2},u_{n-2},u_{n-1})).\end{aligned}$$

Continue this way we get

$$\kappa(S(u_n,u_n,u_{n+1})) \le \left(\frac{\lambda}{1-\eta}\right)^n \kappa(S(u_0,u_0,u_1)).$$

Since $\frac{\lambda}{1-\eta} \in (0,1)$, we obtain that $\lim_{n\to\infty} \kappa(S(u_n,u_n,u_{n+1})) = 0$.
From the fact that $\kappa \in K$, we have

$$\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = 0. \tag{3.22}$$

Now, we prove that $\{u_n\}$ is a Cauchy sequence in X. If $\{u_n\}$ is not Cauchy, then for any $\varepsilon > 0$, there exists integers $m(c)$ and $n(c)$ where $c > 0$ with $m(c) > n(c) > c$ such that $S(u_{m(c)},u_{m(c)},u_{n(c)}) \ge \varepsilon$ and $S(u_{m(c)-1},u_{m(c)-1},u_{n(c)}) < \varepsilon$. From Lemma 2.13 we have

$$\lim_{c\to\infty} S(u_{m(c)},u_{m(c)},u_{n(c)}) = \varepsilon \tag{3.23}$$

and

$$\lim_{c\to\infty} S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+1}) = \varepsilon. \tag{3.24}$$

For $u = u_{m(c)}$ and $v = u_{n(c)}$ from (3.21) we have,

$$\begin{aligned}\kappa(S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+1})) &= \kappa(S(Tu_{m(c)},Tu_{m(c)},u_{n(c)}))\\ &\le \lambda\kappa(S(u_{m(c)},u_{m(c)},u_{n(c)}))\\ &\quad + \eta\kappa\left(S(u_{n(c)},u_{n(c)},u_{n(c)+1})\left[\frac{1+S(u_{m(c)},u_{m(c)},u_{m(c)+1})}{1+S(u_{m(c)},u_{m(c)},u_{n(c)})}\right]\right).\end{aligned}$$

Using (3.22), (3.23) and (3.24) we conclude

$$\begin{aligned}\kappa(\varepsilon) &= \lim_{c\to\infty} \kappa[S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+1})]\\ &\le \lambda \lim_{c\to\infty} \kappa[S(u_{m(c)},u_{m(c)},u_{n(c)})]\\ &\le \lambda\kappa(\varepsilon).\end{aligned}$$

Thus, we get contradiction as $\lambda \in (0,1)$. Hence $\{u_n\}$ is a Cauchy sequence in (X,S), so there exists $w \in X$ such that $\lim_{n\to\infty} u_n = w$.
Now we claim that$w$ is a fixed point of T.

$$\begin{aligned}\kappa(S(u_{n+1},u_{n+1},Tw)) &= \kappa(S(Tu_n,Tu_n,Tw))\\ &\le \lambda\kappa(S(u_n,u_n,w)) + \eta\kappa\left(S(w,w,Tw)\left[\frac{1+S(u_n,u_n,Tu_n)}{1+S(u_n,u_n,w)}\right]\right),\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \kappa(S(u_{n+1},u_{n+1},Tw)) \le \eta\kappa(S(w,w,Tw)),$$
$$\kappa(S(w,w,Tw)) \le \eta\kappa(S(w,w,Tw)).$$

Since $\eta \in (0,1)$, then $\kappa(S(w,w,Tw)) = 0 \Longrightarrow S(w,w,Tw) = 0$ and thus, we obtain $w = Tw$, which means that w is a fixed point of T.
Next, we claim that the fixed point so obtain is unique.
Let z' be another fixed point of T such that $Tw' = w'$ with $w' \neq w$. Therefore, from (3.21) we have

$$\begin{aligned}\kappa(S(w,w,w')) &= \kappa(S(Tw,Tw,Tw'))\\ &\le \lambda\kappa(S(w,w,w')) + \eta\kappa\left(S(w',w',Tw')\left[\frac{1+S(w,w.Tw)}{1+S(w,w,w')}\right]\right).\\ &\le \lambda\kappa(S(w,w,w')),\end{aligned}$$

which implies that $\kappa(S(w,w,w')) = 0$. Since $\kappa \in K$ we obtain $S(w,w,w') = 0$.
Hence $w = w'$. Thus, T has unique fixed point. □

Corollary 3.12. *Let (X,S) be a S-metric space which is complete and T be a mapping on X into itself. We assume that for all $u,v \in X$,*

$$\int_0^{S(Tu,Tu,Tv)} \theta(t)dt \le \lambda \int_0^{S(u,u,v)} \theta(t)dt + \eta \int_0^{S(v,v,Tv)\left[\frac{1+S(u,u,Tu)}{1+S(u,u,v)}\right]} \theta(t)dt, \tag{3.25}$$

where $0 < \lambda + \eta < 1$ with $\lambda, \eta > 0$ and $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be Lebesgue integrable function summable on each compact subset of $[0,+\infty]$, non negative such that $\int_0^\varepsilon \theta(t)dt > 0$ for all $\varepsilon > 0$. Then T admits a fixed point $w \in X$ which is unique and for each $u \in X$, $\lim_{n\to\infty} T^n u = w$.

Proof. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be as in the hypothesis, we define $\kappa_0(t) = \int_0^t \theta(t)dt$, $t \in \mathbb{R}^+$. It is clear that $\kappa_0(0) = 0$. κ_0 is monotonically increasing and by hypothesis κ_0 is absolutely continuous, which implies κ_0 is continuous. Hence, $\kappa_0 \in K$. So, (3.25) becomes

$$\kappa_0(S(Tu,Tu,Tv)) \le \lambda\kappa(S(u,u,v)) + \eta\kappa\left(S(v,v,Tv)\left[\frac{1+S(u,u,Tu)}{1+S(u,u,v)}\right]\right).$$

Hence from Theorem 3.11 unique fixed point exist such that for each $u \in X$, $\lim_{n\to\infty} T^n u = w$. □

Theorem 3.13. *Let (X,S) be a complete S-metric space, we have $\kappa \in K$. Let $T : X \to X$ be a mapping satisfying the following condition:*

$$\kappa(S(Tu,Tu,Tv)) \le \lambda\kappa(S(u,u,v)) + \eta\kappa\left(\frac{S^2(u,u,Tu)+S(u,u,Tu)\,S(v,v,Tv)+S^2(v,v,Tv)}{1+S(u,u,Tu)+S(v,v,Tv)}\right), \tag{3.26}$$

for all $u,v \in X$, $\lambda,\eta > 0$, $\lambda + 2\eta < 1$.
Then T admits a fixed point $w \in X$ which is unique and for each $u \in X$, $\lim_{n\to\infty} T^n u = w$.

Proof. Let $u_0 \in X$ be an arbitrary point and the iterative sequence $\{u_n\}$ be defined as, for every

$$n \geq 0, \quad u_{n+1} = Tu = T^{n+1}u_0. \tag{3.27}$$

Now, from (3.26)

$$\begin{aligned}
&\kappa(S(u_n,u_n,u_{n+1})) \\
&\quad = \kappa(S(Tu_{n-1},Tu_{n-1},Tu_n)) \\
&\quad \leq \lambda\kappa(S(u_{n-1},u_{n-1},u_n)) \\
&\qquad + \eta\kappa\left(\frac{S^2(u_{n-1},u_{n-1},Tu_{n-1}) + S(u_{n-1},u_{n-1},Tu_{n-1})\,S(u_n,u_n,Tu_n) + S^2(u_n,u_n,Tu_n)}{1 + S(u_{n-1},u_{n-1},Tu_{n-1}) + S(u_n,u_n,Tu_n)}\right) \\
&\quad \leq \lambda\kappa(S(u_{n-1},u_{n-1},u_n)) \\
&\qquad + \eta\kappa\left(\frac{S^2(u_{n-1},u_{n-1},u_n) + S(u_{n-1},u_{n-1},u_n)\,S(u_n,u_n,u_{n+1}) + S^2(u_n,u_n,u_{n+1})}{1 + S(u_{n-1},u_{n-1},u_n) + S(u_n,u_n,u_{n+1})}\right) \\
&\quad \leq \lambda\kappa(S(u_{n-1},u_{n-1},u_n)) \\
&\qquad + \eta\kappa\left(\frac{(S(u_{n-1},u_{n-1},u_n) + S(u_n,u_n,u_{n+1}))^2 - S(u_{n-1},u_{n-1},u_n)\,S(u_n,u_n,u_{n+1})}{1 + S(u_{n-1},u_{n-1},u_n) + S(u_n,u_n,u_{n+1})}\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\kappa(S(u_n,u_n,u_{n+1})) &\leq \lambda\kappa(S(u_{n-1},u_{n-1},u_n)) + \eta\kappa(S(u_{n-1},u_{n-1},u_n) + S(u_n,u_n,u_{n+1}) \\
(1-\eta)\kappa(S(u_n,u_n,u_{n+1})) &\leq (\lambda+\eta)\kappa(S(u_{n-1},u_{n-1},u_n)) \\
\kappa(S(u_n,u_n,u_{n+1})) &\leq \left(\frac{\lambda+\eta}{1-\eta}\right)\kappa(S(u_{n-1},u_{n-1},u_n)) \\
&\leq \left(\frac{\lambda+\eta}{1-\eta}\right)^2 \kappa(S(u_{n-2},u_{n-2},u_{n-1})).
\end{aligned}$$

Continuing like this way we obtain

$$\kappa(S(u_n,u_n,u_{n+1})) \leq \left(\frac{\lambda+\eta}{1-\eta}\right)^n \kappa(S(u_{n-2},u_{n-2},u_{n-1})).$$

Since $\frac{\lambda+\eta}{1-\eta} < 1$, we get,

$$\lim_{n\to\infty} \kappa(S(u_n,u_n,u_{n+1})) = 0\,.$$

From the result given that $\kappa \in K$, we get

$$\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = 0. \tag{3.28}$$

Now, we prove that $\{u_n\}$ is a Cauchy sequence in X. If $\{u_n\}$ is not Cauchy, then for any $\varepsilon > 0$, there exists integers $m(c)$ and $n(c)$ where $c > 0$ with $m(c) > n(c) > c$ such that $S(u_{m(c)},u_{m(c)},u_{n(c)+1}) \geq \varepsilon$ and $S(u_{m(c)-1},u_{m(c)-1},u_{n(c)}) < \varepsilon$. Thus, from Lemma 2.13 we have

$$\lim_{c\to\infty} S(u_{m(c)},u_{m(c)},u_{n(c)}) = \varepsilon \text{ and} \tag{3.29}$$

$$\lim_{c\to\infty} S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+1}) = \varepsilon. \tag{3.30}$$

Now,

$$\begin{aligned}
&\kappa(S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+1}))\\
&\quad=\kappa(S(Tu_{m(c)},Tu_{m(c)},Tu_{n(c)}))\\
&\quad\le\lambda\kappa(S(u_{m(c)},u_{m(c)},u_{n(c)}))+\eta\kappa\left[\frac{\begin{pmatrix}S^2(u_{m(c)},u_{m(c)},u_{m(c)+1})\\ +S(u_{m(c)},u_{m(c)},u_{m(c)+1})\,S(u_{n(c)},u_{n(c)},u_{n(c)+1})\\ +S^2(u_{n(c)},u_{n(c)},u_{n(c)+1})\end{pmatrix}}{1+S(u_{m(c)},u_{m(c)},u_{m(c)+1})+\ S(u_{n(c)},u_{n(c)},u_{n(c)+1})}\right].
\end{aligned}$$

Using (3.27), (3.29) and (3.30) we have

$$\kappa(\varepsilon)=\lim_{c\to\infty}\kappa S(u_{m(c)+1},u_{m(c)+1},u_{n(c)+1})\le\lambda\kappa(S(u_{m(c)},u_{m(c)},u_{n(c)}))\le\lambda\kappa(\varepsilon).$$

Since $\lambda\in(0,1)$, we get a contradiction. Hence, $\{u_n\}$ is a Cauchy sequence in X. As (S,X) is a complete S-metric space thus there exist $w\in X$ such that $u_n\to w$ as $n\to\infty$.
Taking $u=u_n$ and $v=w$ in (3.26) we have

$$\begin{aligned}
&\kappa(S(Tu_n,Tu_n,Tw))\\
&\quad=\kappa(S(u_{n+1},u_{n+1},Tw))\\
&\quad\le\lambda\kappa(S(u_n,u_n,w))+\eta\kappa\left[\frac{S^2(u_n,u_n,Tu_n)+S(u_n,u_n,Tu_n)\,S(w,w,Tw)+S^2(w,w,Tw)}{1+S(u_n,u_n,Tu_n)+\ S(w,w,Tw)}\right]\\
&\quad\le\kappa(S(u_n,u_n,w))+\eta\kappa\left[\frac{S^2(u_n,u_n,u_{n+1})+S(u_n,u_n,u_{n+1})\,S(w,w,Tw)+S^2(w,w,Tw)}{1+S(u_n,u_n,u_{n+1})+\ S(w,w,Tw)}\right].
\end{aligned}$$

Taking limit $n\to\infty$ we obtain

$$\kappa(S(w,w,Tw))\le\eta\kappa(S(w,w,Tw)).$$

Since $\eta\in(0,1)$, then $\kappa(S(w,w,Tw))=0$, which implies that $S(w,w,Tw)=0$ as $\kappa\in K$.
Hence, T has a fixed point.
Now we claim the uniqueness of the fixed point.
Let T has another fixed point w' such that $w\ne w'$ and $w'=Tw'$, then by (3.26) we have

$$\begin{aligned}
\kappa(S(w,w,w'))&=\kappa(S(Tw,Tw,Tw'))\\
&\le\lambda\kappa(S(w,w,w'))+\eta\kappa\left[\frac{S^2(w,w,Tw)+S(w,w,Tw)\,S(w',w',Tw')+S^2(w',w',Tw')}{1+S(w,w,Tw)+S(w',w',Tw')}\right]\\
&\le\lambda\kappa(S(w,w,w')).
\end{aligned}$$

Hence, $\kappa(S(w,w,w'))=0$ and so $S(w,w,w')=0$. Thus, we get $w=w'$. □

Theorem 3.14. *Let $T:X\to X$ be a mapping on a complete S-metric space be (X,S). Then for each $x,y\in X$ we assume that*

$$\int_0^{\kappa(S(Tu,Tu,Tu))}\theta(t)dt\le\lambda\int_0^{\kappa(S(u,u,v))}\theta(t)dt+\eta\int_0^{\kappa(\frac{S^2(u,u,Tu)+S(u,,Tu)\ S(v,v,Tv)+S^2(v,v,Tv)}{1+S(u,u,Tu)+\ S(v,v,Tv)})}\theta(t)dt,$$

where $\lambda+2\eta<1$ and $\kappa:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue integrable mapping which is Summable on each compact subset of $[0,\infty)$*, non-negative such that $\int_0^{\epsilon}\theta(t)dt$, for all $\epsilon>0$.*

Then T possess a fixed point $w \in X$ which is unique such that $\lim_{n\to\infty} T^n u = u$, for each $u \in X$.

Proof. Let $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping define as

$$\kappa(r) = \int_0^r \theta(t)dt, \quad r \in \mathbb{R}^+.$$

It is clear that $\kappa(0) = 0$ and κ is monotonically increasing and K is absolutely continuous by hypothesis. Thus κ is continuous. Therefore, $\kappa \in K$. So, from (3.26) we have

$$\kappa(S(Tu,Tu,Tv)) \le \lambda\kappa(S(u,u,v)) + \eta\kappa\left(\frac{S^2(u,u,Tu)+S(u,u,Tu)\,S(v,v,Tv)+S^2(v,v,Tv)}{1+S(u,u,Tu)+\ S(v,v,Tv)}\right).$$

Hence from Theorem 3.13 we obtain that $w \in X$ is a unique fixed point and moreover for each $u \in X$, $\lim_{n\to\infty} T^n u = w$. □

Definition 3.15. Let T be a self-mapping of S-metric space (X,S) with a non empty fixed point set $F(T)$. Then T is said to satisfy the property P if $F(t) = F(T^n)$, for each $n \in \mathbb{N}$.

In this section we are going to prove that the mapping satisfying the contractive conditions (3.1), (3.20), (3.21) and (3.26) fulfill the property P.

Theorem 3.16. *Let $\kappa \in K$ and (X,S) be a S-metric space which is complete. If T be a self mapping on X satisfying the condition* (3.1) *of Theorem* 3.1*. Then $F(t) \neq \phi$ and T satisfies the property P.*

Proof. We say that T has a fixed point from Theorem 3.1, therefore $F(T^n) \neq \phi$ for each $n \in \mathbb{N}$. Fix $n > 1$ and we assume that $w \in F(T^n)$ we want to show that $w \in F(T)$.
Suppose that $w \neq Tw$, using (3.1),

$$\begin{aligned}\kappa(S(w,w,Tw)) &= \lambda(S(T^n w, T^n w, T^{n+1}w))\\ &\le \lambda\kappa(S(T^{n-1}w, T^{n-1}w, T^n w))\\ &\le \dots\\ &\le \lambda^n \kappa(S(w,w,Tw)).\end{aligned}$$

Since $\lambda \in (0,1)$, $\lim_{n\to\infty} \kappa(S(w,w,Tw)) = 0$. From the fact that $\kappa \in K$, we conclude that $S(w,w,Tw) = 0$ and so $w = Tw$. Thus $w \in F(T)$ and hence T possess a property P. □

Theorem 3.17. *Let (X,S) be a complete S-metric space and let T be a self mapping on X that satisfy the condition* (3.20) *of Theorem* 3.9*. Then $F(t) \neq \phi$ and T possess a property P.*

Proof. From Theorem 3.9, we have T has a fixed point i.e. $F(t) \neq \phi$.
Let $u \in F(t)$ then $T(u) = u$.
Therefore, $T^n(u) = T^{n-1}(T(u)) = T^{n-1}(u) = \cdots = u$, which implies that $F(T^n) \neq \phi$ for each $n \in \mathbb{N}$.
Now fix $n > 1$ and we assume that $w \in F(T^n)$ we want to show that $w \in F(T)$.
Suppose that $w \neq Tw$, using (3.20),

$$S(w,w,Tw) = S(T^n w, T^n w, T^{n+1}w)$$

$$\begin{aligned}
&\le \lambda(S(T^{n-1}w,T^{n-1}w,T^nw))\\
&\quad +\eta\left(S(T^nw,T^nw,T^{n+1}w)\left[\frac{1+S(T^{n-1}w,T^{n-1}w,T^nw)}{1+S(T^{n-1}w,T^{n-1}w,T^nw)}\right]\right)\\
&\le \lambda(S(T^{n-1}w,T^{n-1}w,T^nw))+\eta(S(T^nw,T^nw,T^{n+1}w)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
S(w,w,Tw)&=S(T^nw,T^nw,T^{n+1}w)\\
&\le \frac{\lambda}{1-\eta}S(T^{n-1}w,T^{n-1}w,T^nw)\\
&\le \left(\frac{\lambda}{1-\eta}\right)^2 S(T^{n-1}w,T^{n-1}w,T^nw).
\end{aligned}$$

Continuing this way, we obtain

$$S(w,w,Tw)\le \left(\frac{\lambda}{1-\eta}\right)^n S(w,w,Tw),$$

which is contradiction. Consequently, $w\in F(T)$ and T has a property P. □

Theorem 3.18. *Let $\kappa\in K$ and (X,S) be a S-metric space which is complete. If T be a self mapping on X satisfy the condition* (3.21)*. Then $F(t)\ne\phi$ and T satisfies the property P.*

Proof. We say that T has a fixed point from Theorem 3.11. Thus, $F(T^n)\ne\phi$ for every $n\in\mathbb{N}$. Fix $n>1$ and assume that $w\in F(T^n)$. We want to show that $w\in F(T)$.
Suppose that $w\ne Tw$, using (3.21),

$$\begin{aligned}
\kappa(S(w,w,Tw))&=\kappa(S(T^nw,T^nw,T^{n+1}w))\\
&\le \lambda\kappa(S(T^{n-1}w,T^{n-1}w,T^nw))\\
&\quad +\eta\kappa\left(S(T^nw,T^nw,T^{n+1}w)\left[\frac{1+S(T^{n-1}w,T^{n-1}w,T^nw)}{1+S(T^{n-1}w,T^{n-1}w,T^nw)}\right]\right)\\
&\le \lambda\kappa(S(T^{n-1}w,T^{n-1}w,T^nw))+\eta\kappa S(T^nw,T^nw,T^{n+1}w).
\end{aligned}$$

Hence,

$$\begin{aligned}
(1-\eta)\kappa(S(T^nw,T^nw,T^{n+1}w))&\le \lambda\kappa(S(T^{n-1}w,T^{n-1}w,T^nw))\\
K(S(w,w,Tw))&=\kappa(S(T^nw,T^nw,T^{n+1}w))\\
&\le \frac{\lambda}{(1-\eta)}\kappa(S(T^{n-1}w,T^{n-1}w,T^nw))\\
&\le \left(\frac{\lambda}{1-\eta}\right)^2\kappa(S(T^{n-2}w,T^{n-2}w,T^{n-1}w)).
\end{aligned}$$

Proceeding this process, we obtain

$$\kappa(S(w,w,Tw))\le \left(\frac{\lambda}{1-\eta}\right)^n \kappa(S(w,w,Tw)),$$

which is contradiction as $\lambda+\eta<1$, therefore $\kappa(S(w,w,Tw))=0$, since $\kappa\in K$, we conclude that $S(w,w,Tw)=0$, which implies that $w\in F(T)$ and hence T satisfy the property P. □

Theorem 3.19. *Let $\kappa \in K$ and (X,S) be a S-metric space which is complete. If T be a self mapping on X satisfying the contractive condition:*

$$\kappa(S(Tu,Tu,Tv)) \le \lambda\kappa(S(u,u,v)) + \eta\kappa\left(\frac{S^2(u,u,Tu)+S(u,u,Tu)\,S(v,v,Tv)+S^2(v,v,Tv)}{1+S(u,u,Tu)+S(v,v,Tv)}\right), \quad (3.31)$$

for all $u,v \in X$, $\lambda,\eta > 0$, $\lambda + 2\eta < 1$. Then $F(t) \ne \phi$ and t has a property P.

Proof. We say that T has a fixed point from Theorem 3.13. Thus, $F(T^n) \ne \phi$ for every $n \in \mathbb{N}$. Fix $n > 1$ and assume that $w \in F(T^n)$. We want to show that $w \in F(T)$. Suppose that $w \ne Tw$, then

$$\begin{aligned}
\kappa(S(w,w,Tw)) &= \kappa(S(T^n w,T^n w,T^{n+1}w)) \\
&\le \lambda\kappa(S(T^{n-1}w,T^{n-1}w,T^n w)) \\
&\quad + \eta\kappa\left[\frac{\begin{pmatrix} S^2(T^{n-1}w,T^{n-1}w,T^n w) \\ +S(T^{n-1}w,T^{n-1}w,T^n w)\,S(T^n w,T^n w,T^{n+1}w) \\ +S^2(T^n w,T^n w,T^{n+1}w)\end{pmatrix}}{1+S(T^{n-1}w,T^{n-1}w,T^n w)+S(T^n w,T^n w,T^{n+1}w)}\right] \\
&\le \lambda\kappa(S(T^{n-1}w,T^{n-1}w,T^n w)) + \eta\kappa[S(T^{n-1}w,T^{n-1}w,T^n w) \\
&\quad + S(T^n w,T^n w,T^{n+1}w)].
\end{aligned}$$

Hence,

$$\begin{aligned}
\kappa(S(w,w,Tw)) &= \kappa(S(T^n w,T^n w,T^{n+1}w)) \\
&\le \left(\frac{\lambda+\eta}{1-\eta}\right)\kappa(S(T^{n-1}w,T^{n-1}w,T^n w)) \\
&\le \left(\frac{\lambda+\eta}{1-\eta}\right)^2 \kappa(S(T^{n-2}w,T^{n-2}w,T^{n-1}w)).
\end{aligned}$$

Continue like this way we obtain

$$\kappa(S(w,w,Tw)) \le \left(\frac{\lambda+\eta}{1-\eta}\right)^n \kappa(S(w,w,Tw)).$$

This is a contradiction. Hence, $\kappa(S(w,w,Tw)) = 0$, since $\kappa \in K$. This conclude that $S(w,w,Tw) = 0$ and therefore $w \in F(t)$ and t has a property P. □

4. Conclusion

In this attempt, by using the Altering distance function we proved some fixed point results with different contraction conditions in S-metric space. Our results are generalization and extension of many existing results in the literature. Finally, we explore the property P on this contractive conditions.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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