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Research Article

New Relation-Theoretic Fixed Point Theorems in Revised Fuzzy Metric Spaces With an Application to Fractional Differential Equations

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Abstract. In this paper, we introduce the notion of revised fuzzy $\mathcal{R}$-ψ-contractive mappings and prove some relevant results on the existence and uniqueness of fixed points for this type of mappings in the setting of non-Archimedean revised fuzzy metric spaces. Several illustrative examples are also given to support our newly proven results. Furthermore, we apply our main results to prove the existence and uniqueness of a solution for Caputo fractional differential equations.

Keywords. Revised fuzzy metric space, Fixed point, Binary relation, $\mathcal{R}$-ψ-contractive mappings, Caputo fractional differential equation

Mathematics Subject Classification (2020). 46N20, 46S40, 37C25

1. Introduction and Preliminaries

In 1965, the knowledge of fuzzy sets was firstly developed by Zadeh [22], which is an effective tool for modeling indecision and elusiveness in many problems that arise in the field of technology and the concept of fuzzy metric space was introduced by Kramosil and Michálek [9] in 1975. George and Veeramani [3] defined Hausdorff topology on fuzzy metric space after a slight modification in the definition of fuzzy metric presented. One of the most interesting motivations is the fixed point theory established in fuzzy metric spaces, which was initiated by Grabiec [4], where a fuzzy metric version of the Banach contraction principle was presented.

Relation-theoretic fixed point theory, on the other hand, is a relatively young branch of fixed point theory. Turinici [21] began this route, and it has become a highly busy field with the publication of excellent results by Ran and Reurings [17], and Nieto and Lopez [14,15], who gave a new version of the Banach contraction principle with an ordered binary relation. The authors gave various interesting applications to boundary value problems and matrix equations in [10], which strongly supported their fixed point conclusions. Following that, a slew of fixed point theorems were developed, each with its own set of binary relation definitions (e.g., [2,8,16,19] and a slew of others).

Šostak [20] improved on George and Veeramani's notion of revised fuzzy metric spaces, which they introduced in 2018. Grigorenko [6] developed the class of fuzzy (pseudo) metric spaces in 2020 to exploit this notion in topology. Muraliraj and Thangathamizh [11,12] and Muraliraj *et al.* [13] went on to develop fuzzy mapping and come up with a fixed point result for it. Many broad topological ideas and discoveries were then applied to fuzzy topological space.

2. Main Results

We start our main section with the following lemma which will be useful in the proof of our main results.

Definition 2.1 ([20])**.** A *revised fuzzy metric space* is can be an ordered triple $(X,\mu,\oplus)$ such X is a nonempty set, $\oplus$ is a continuous t-conorm and μ is a revised fuzzy set on $X\times X\times(0,\infty)\to[0,1]$ satisfies the subsequent conditions: $\forall\ x,y,z\in X$ and $s,t>0$

(RGV1) $\mu(x,y,t)<1,\ \forall\ t>0$,

(RGV2) $\mu(x,y,t)=0$ if and only if $x=y,\ t>0$,

(RGV3) $\mu(x,y,t)=\mu(y,x,t)$,

(RGV4) $\mu(x,z,t+s)=\mu(x,y,t)\oplus\mu(y,z,s)$,

(RGV5) $\mu(x,y,-):(0,\infty)\to[0,1)$ is continuous.

Then μ is called a revised fuzzy metric on X.

Definition 2.2 ([11])**.** Let $(X,\mu,\oplus)$ be a revised fuzzy metric space,

(i) A sequence $\{x_n\}$ in X is said to be convergent towards a point $x\in X$ if
$$\lim_{n\to\infty}\mu(x,y,t)=0,\quad \text{for all } t>0.$$

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence, if for all $0<\epsilon<1$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that $\mu(x_n,x_m,t)<\epsilon$ for each $n,m=n_0$.

(iii) A revised fuzzy metric space in which each Cauchy sequence is converges is said to be complete.

(iv) A revised fuzzy metric space in which each sequence has a converging subsequence is called compact.

Lemma 2.1 ([11])**.** *Let* $(X,\mu,\oplus)$ *be a revised fuzzy metric space. For each* $u,v\in X$, $\mu(u,v,-)$ *is non-increasing function.*

Definition 2.3. Let $(X,\mu,\oplus)$ be a revised fuzzy metric space, is said to be complete if each Cauchy sequence in X is convergent.

Definition 2.4. Let $(X,\mu,\oplus)$ be a revised fuzzy metric space, is said to be compact if each sequence contains a convergent subsequence.

Lemma 2.2. *Let $(X,\mu,\oplus)$ be a revised fuzzy metric space X, Let $\{y_n\}$ be a sequence. If there is a positive number $k<1$ such that $\mu(y_{n+2},y_{n+1},kt)\le\mu(y_{n+1},y_n,t)$, $t>0$, $n\in N$ then $\{y_n\}$ could be a Cauchy sequence in X.*

Lemma 2.3. *If for two points x, y in X is a positive number $k<1$, $\mu(x,y,kt)\le\mu(x,y,t)$, then $x=y$.*

Remark 2.1. Since $\oplus$ is continuous, it follows from (RGV4) that the sequence limit whith in the revised fuzzy metric space is uniquely determined.

Lemma 2.4. *Let $f:X\to X$ and $\mathcal{R}$ a transitive binary relation which is f-closed. Assume that there exists $x_0\in X$ such that $x_0\mathcal{R}fx_0$ and define $\{x_n\}$ in X by $x_n=fx_{n-1}$, for all $n\in\mathbb{N}_0$. Then*

$$x_m\mathcal{R}x_n, \quad \text{for all } m,n\in\mathbb{N}_0 \text{ with } m<n. \tag{2.1}$$

Proof. As there exists $x_0\in X$ such that $x_0\mathcal{R}x_0$ and $x_n=fx_{n-1}$, then $x_0\mathcal{R}x_1$. As $\mathcal{R}$ is f-closed and $x_0\mathcal{R}x_1$, we deduce that $x_1\mathcal{R}x_2$. By continuing this process, we find $x_n\mathcal{R}x_{n+1}$ for all $n\in\mathbb{N}_0$. Suppose that $m<n$, so $x_m\mathcal{R}x_{m+1}$ and $x_{m+1}\mathcal{R}x_{m+2}$. Due to the transitivity of $\mathcal{R}$, we find $x_m\mathcal{R}x_{m+2}$. Similarly, as $x_m\mathcal{R}x_{m+2}$ and $x_{m+2}\mathcal{R}x_{m+3}$, we find $x_m\mathcal{R}x_{m+3}$. By continuing this process, we obtain $x_m\mathcal{R}x_n$ for all $m,n\in\mathbb{N}_0$ with $m<n$. □

Next, we introduce the notion of revised fuzzy $\mathcal{R}$-ψ-contractive mapping as follows:

Definition 2.5. Let $(X,\mu,\oplus)$ be a non-Archimedean revised fuzzy metric space, $\mathcal{R}$ a binary relation on X and $f:X\to X$. We say that f is a revised fuzzy $\mathcal{R}$-ψ-contractive mapping if there exists $\psi\in\Psi$ such that (for all $x,y\in X$ and all $t>0$ with $x\mathcal{R}y$)

$$\mu(x,y,t)<1,\ \ \max\{\mu(x,y,t),\min\{\mu(fx,x,t),\mu(y,fy,t)\}\}\ge\psi(\mu(fx,fy,t)). \tag{2.2}$$

The following is an example of a revised fuzzy $\mathcal{R}$-ψ-contractive mapping.

Example 2.1. Let $X=[0,\infty)$ and let $\oplus$ be the product t-conorm given by $t\oplus s=t+s-ts$ for all $t,s\in[0,1]$. Define $\mu:\ X^2\times[0,\infty)\to[0,1]$ for all $x,y\in X$ by

$$\mu(x,y,t)=\begin{cases}0, & \text{if } t=0,\\ \frac{t}{1+t}|x-y|, & \text{if } t\neq 0.\end{cases}$$

Define $f:X\to X$, $\psi:[0,1]\to[0,1]$, and $\mathcal{R}$ on X by

$$fx=\begin{cases}e^{-x} & \text{if } x\in[0,2),\\ \frac{x+3}{2} & \text{if } x\in[2,5],\\ e^{-x}+6 & \text{if } x\in[5,\infty).\end{cases}\qquad \psi(t)=t^3,\ x\mathcal{R}y\leftrightarrow x,y\in[2,5],\ x\le y.$$

Then f is a revised fuzzy $\mathcal{R}$-ψ-contractive mapping as we will prove later on.

Now, we are equipped to state and prove our first main result as under.

Theorem 2.1. *Let $(X,\mu,\oplus)$ be a non-Archimedean revised fuzzy metric space equipped with a binary relation $\mathcal{R}$ and $f : X \to X$. Assume that X is an $\mathcal{R}$-complete and f is a revised fuzzy $\mathcal{R}$-ψ-contractive mapping such that:*

(i) *there exists x_0 in X such that $x_0 \mathcal{R} f x_0$ and $\mu(x_0, fx_0, t) < 1$ for all $t > 0$;*

(ii) *R is transitive and f-closed;*

(iii) *one of the following holds:*

  (a) *f is continuous, or*

  (b) *$\mathcal{R}$ is μ-self-closed.*

Then f has a fixed point in X.

Proof. From (i), there exists $x_0 \in X$ such that $x_0 \mathcal{R} f x_0$ and $\mu(x_0, fx_0, t) < 1$ for all $t > 0$. Define a sequence $\{x_n\}$ in X by $fx_n = x_{n+1}$, for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$, for some $n \in \mathbb{N}_0$, then x_n is a fixed point of f. Assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}_0$.

As $\mu(x_0, fx_0, t) = \mu(x_0, x_1, t) < 1$ for all $t > 0$, and in view of Lemma 2.4 and (2.2), we obtain

$$\begin{aligned} & \max\{\mu(x_0,x_1,t), \min\{\mu(fx_0,x_0,t), \mu(x_1,fx_1,t)\}\} \geq \psi(\mu(fx_0,fx_1,t)) \\ \implies & \max\{\mu(x_0,x_1,t), \min\{\mu(x_1,x_0,t), \mu(x_1,x_2,t)\}\} \geq \psi(\mu(x_1,x_2,t)) \\ \implies & 1 > \mu(x_0,x_1,t) \geq \psi(\mu(x_1,x_2,t)) \geq \mu(x_1,x_2,t). \end{aligned} \tag{2.3}$$

If there is some $t_0 > 0$ such that $\mu(x_1,x_2,t_0) = 1$, then $\psi(\mu(x_1,x_2,t_0)) = 1$. This implies that $\mu(x_1,x_2,t_0) = 1$, (due to condition (C) of the definition of ψ) which contradicts (2.3).

Roldán-López-de-Hierro [18] defined a comparison function $\psi : [0,1] \to [0,1]$ which satisfies:

(A) ψ is non-increasing and right continuous;

(B) $\psi(t) < t$ for all $t \in (0,1)$;

(C) $\psi(0) = 0$.

Let Ψ denotes the family of all such functions y.

For example, $\psi(t) = t^2$ for all $t \in (0,1)$. Notice that, using the previous definition, the condition $y(1) = 1$ is not necessarily true.

Therefore, $\mu(x_1,x_2,t_0) < 1$ for all $t > 0$. Continuing with the same scenario, we deduce that for all $n \in \mathbb{N}_0$ and all $t > 0$,

$$1 > \mu(x_n,x_{n+1},t) \geq \psi(\mu(x_n,x_{n+1},t)) \geq \mu(x_{n-1},x_n,t) > 0,$$

for all $n \in \mathbb{N}_0$ and all $t > 0$, which implies that the sequence $\{\mu(x_n,x_{n+1},t)\}$ is non-increasing sequence and bounded above. Hence, there exists $0 \leq \delta(t) < 1$ for all $t > 0$, such that

$$\lim_{n\to\infty} \mu(x_n,x_{n+1},t) = \delta(t).$$

Now, we show that $\delta(t) = 0$ for all $t > 0$. If there is $t_0 > 0$ such that $\delta(t_0) < 1$ then

$$1 > \delta(t_0) \geq \mu(x_n,x_{n+1},t_0) \geq \psi(\mu(x_n,x_{n+1},t_0)) \geq \mu(x_{n-1},x_n,t_0) > 0, \tag{2.4}$$

hence, $0 < \delta(t_0) < 1$. As ψ is right-continuous and $\{\mu(x_n,x_{n+1},t)\}$ is non-increasing sequence of positive numbers, letting $n \to \infty$ in (2.4) we obtain $\psi(\delta(t_0)) = \delta(t_0)$, a contradiction ($\delta(t_0) \in (0,1)$).

Therefore, $\delta(t)=0$ for all $t>0$. That is,

$$\lim_{n\to\infty}\mu(x_n,x_{n+1},t)=0. \tag{2.5}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in $(X,\mu,\oplus)$. If on the contrary, $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon\in(0,1)$ and some $t_0>0$ such that, for all $k\in\mathbb{N}_0$, there exist $m(k),n(k)\in\mathbb{N}_0$ such that $m(k)\geq n(k)\geq k$ satisfies

$$\mu(x_{m(k)},x_{n(k)},t_0)\geq\varepsilon,\ \mu(x_{m(k)-1},x_{n(k)},t_0)<\varepsilon,\quad \text{for all } k\in\mathbb{N}_0.$$

As $(X,\mu,\oplus)$ is non-Archimedean, we have for all $k\in\mathbb{N}_0$,

$$\begin{aligned}\varepsilon&\leq\mu(x_{m(k)},x_{n(k)},t_0)\\ &\leq\mu(x_{m(k)},x_{m(k)-1},t_0)\oplus\mu(x_{m(k)-1},x_{n(k)},t_0)\\ &<\mu(x_{m(k)},x_{m(k)-1},t_0)\oplus\varepsilon.\end{aligned}$$

Letting $k\to\infty$, and using that $\oplus$ is continuous, and (2.5) we can conclude that

$$\lim_{n\to\infty}\mu(x_{m(k)},x_{n(k)},t_0)=\varepsilon. \tag{2.6}$$

Additionally, as $(X,\mu,\oplus)$ is non-Archimedean, we have (for all $k\in\mathbb{N}_0$)

$$\mu(x_{m(k)-1},x_{n(k)-1},t_0)\leq\mu(x_{m(k)-1},x_{n(k)},t_0)\oplus\mu(x_{n(k)},x_{n(k)-1},t_0)<\varepsilon\oplus\mu(x_{n(k)},x_{n(k)-1},t_0),$$

and

$$\mu(x_{m(k)},x_{n(k)},t_0)\leq\mu(x_{m(k)},x_{m(k)-1},t_0)\oplus\mu(x_{m(k)-1},x_{n(k)-1},t_0)\oplus\mu(x_{n(k)},x_{n(k)-1},t_0).$$

Taking $k\to\infty$, in the above inequalities and using (2.5) and (2.6), we find

$$\lim_{k\to\infty}\mu(x_{m(k)-1},x_{n(k)-1},t_0)=\varepsilon. \tag{2.7}$$

That is,$\mu\left(x_{m(k)-1},x_{n(k)-1},t_0\right)<1$ whenever k is large enough. Now, using (2.2) and Lemma 2.4, we have, (for all k)

$$\begin{aligned}&\max\{\mu(x_{m(k)-1},x_{n(k)-1},t_0),\min\{\mu(fx_{m(k)-1},x_{m(k)-1},t_0),\mu(x_{n(k)-1},fx_{n(k)-1},t_0)\}\}\\ &\quad\geq\psi(\mu(fx_{m(k)-1},f\,x_{n(k)-1},t_0)).\end{aligned}$$

Hence,

$$\begin{aligned}&\max\{\mu(x_{m(k)-1},x_{n(k)-1},t_0),\min\{\mu(x_{m(k)},x_{m(k)-1},t_0),\mu(x_{n(k)-1},x_{n(k)},t_0)\}\}\\ &\quad\geq\psi(\mu(x_{m(k)},x_{n(k)},t_0)).\end{aligned}$$

Letting $k\to\infty$, and using (2.5)-(2.7) and the fact that y is left-continuous we deduce that

$$1>\max\{\varepsilon,\max\{0,0\}\}\geq\psi(\varepsilon)\Longrightarrow\varepsilon\geq\psi(\varepsilon)>\varepsilon,$$

a contradiction. Hence, $\{x_n\}$ must be a Cauchy sequence in $(X,\mu,\oplus)$. Now, we have $\{x_n\}$, an $\mathcal{R}$-Cauchy sequence, and $(X,\mu,\oplus)$, an $\mathcal{R}$-complete, so there exists $x\in X$ such that $x_n\to n$.

Now, if f is continuous, then taking the limit as $n\to\infty$ on the both sides of $x_{n+1}=fx_n$, $n\in\mathbb{N}_0$, we obtain $x=fx$.

Otherwise, if R is μ-self-closed, then there exists a subsequence $\{x_{n(k)}\}\subseteq\{x_n\}$ such that $x_{n(k)}\mathcal{R}x$ for all $k\in\mathbb{N}_0$. We claim that $x=f(x)$. As $\lim\limits_{k\to\infty}x_{n(k)}=x$ we have $\lim\limits_{k\to\infty}\mu(x_{n(k)},x,t)=0$ for all $t>0$. Then $\mu(x_{n(k)},x,t)<1$ when k is large enough for all $t>0$ and as $x_{n(k)}\mathcal{R}x$, from condition

(2.2) we find

$$\max\{\mu(x_{n(k)},x,t),\min\{\mu(fx_{n(k)},x_{n(k)},t),\mu(x,fx,t)\}\} \geq \psi(\mu(fx_{n(k)},fx,t)).$$

Thus,

$$\max\{\mu(x_{n(k)},x,t),\min\{\mu(x_{n(k)+1},x_{n(k)},t),\mu(x,fx,t)\}\} \geq \psi(\mu(x_{n(k)+1},fx,t)).$$

Letting $k \to \infty$, and using (2.5), $\lim_{k\to\infty} \mu(x_{n(k)},x,t) = 0$, we find

$$0 = \max\{0,\min\{0,\mu(x,fx,t\}\} \geq \lim_{k\to\infty} \psi(\mu(x_{n(k)+1},fx,t)).$$

This means that

$$\lim_{k\to\infty} \psi(\mu(x_{n(k)+1},fx,t)) = 0.$$

Hence, from Remark 2.1 and the continuity of μ, we obtain

$$\lim_{k\to\infty} \mu(x_{n(k)+1},fx,t) = 0.$$

Thus, $\lim_{k\to\infty} x_{n(k)+1} = fx$. The uniqueness of the limit gives that $fx = x$. This completes the proof. □

Next, we provide the following uniqueness theorem.

Theorem 2.2. *In addition to the hypotheses of Theorem* 2.1*, if the following condition holds:*

(iv) *for all* $x, y \in Fix(f)$*, there exists* $z \in X$ *such that* $x\mathcal{R}z$ *and* $y\mathcal{R}z$*,* $\mu(x,z,t) < 1$ *and* $\mu(y,z,t) < 1$ *for all* $t > 0$*. Then the fixed point of* f *is unique.*

Proof. In view of Theorem 2.1, $Fix(f) \neq \phi$. Let $x, y \in Fix(f)$, by condition (iv) there exists $z \in X$ such that $x\mathcal{R}z$, $y\mathcal{R}z \in \mu(x,z,t) < 1$ and $\mu(y,z,t) < 1$ for all $t > 0$. Define $z_0 = z$ and $z_{n+1} = fz_n$ for all $n \geq 0$. We claim that $x = y$. As $x\mathcal{R}z_0$, $\mu(x,z_0,t) < 1$ for all $t > 0$, then from (2.2) we have

$$\begin{aligned}
& \max\{\mu(x,z_0,t),\min\{\mu(fx,x,t),\mu(z_0,fz_0,t)\}\} \geq \psi(\mu(fx,fz_0,t)) \\
\Longrightarrow\ & \max\{\mu(x,z_0,t),\min\{\mu(x,x,t),\mu(z_0,z_1,t)\}\} \geq \psi(\mu(x,z_1,t)) \\
\Longrightarrow\ & \max\{\mu(x,z_0,t),\min\{0,\mu(z_0,z_1,t)\}\} \geq \psi(\mu(x,z_1,t)) \\
\Longrightarrow\ & \max\{\mu(x,z_0,t),0\} \geq \psi(\mu(x,z_1,t)) \\
\Longrightarrow\ & 1 > \mu(x,z_0,t) \geq \psi(\mu(x,z_1,t)) \geq \mu(x,z_1,t).
\end{aligned}$$

By induction, we find $\mu(x,z_n,t) < 1$ for all $n \in \mathbb{N}_0$ and $t > 0$ and as $\mathcal{R}$ is f-closed, we conclude that (by induction), $x\mathcal{R}z_n$ for all $n \in \mathbb{N}_0$. Hence

$$\begin{aligned}
& \max\{\mu(x,z_n,t),\min\{\mu(fx,x,t),\mu(z_n,fz_n,t)\}\} \geq \psi(\mu(fx,fz_n,t)) \\
\Longrightarrow\ & \max\{\mu(x,z_n,t),\min\{\mu(x,x,t),\mu(z_n,z_{n+1},t)\}\} \geq \psi(\mu(x,z_{n+1},t)) \qquad (2.8) \\
\Longrightarrow\ & 1 > \mu(x,z_n,t) \geq \psi(\mu(x,z_{n+1},t)) \geq \mu(x,z_{n+1},t).
\end{aligned}$$

Thus, $\{\mu(x,z_n,t)\}$ is non-increasing and bounded above. Hence, there exists $0 \leq \gamma(t) < 1$ for all $t > 0$ such that $\lim_{n\to\infty} \mu(x,z_n,t) = \gamma(t)$. Letting $n \to \infty$ in (2.8), and as ψ is right-continuous, we find $\psi(\gamma(t)) = \gamma(t)$. Therefore, in view of Remark 2.1, we deduce that $\gamma(t) = 0$ for all $t > 0$. Thus,

$$\lim_{n\to\infty} \mu(x,z_n,t) = 0, \quad \text{for all } t > 0.$$

Similarly, we can show that

$$\lim_{n\to\infty} \mu(y,z_n,t)=0, \quad \text{for all } t>0.$$

As $(X,\mu,\oplus)$ is non-Archimedean, we find (for all $n\in\mathbb{N}_0$)

$$\mu(x,y,t)\le \mu(x,z_n,t)\oplus\mu(z_n,y,t).$$

Letting $n\to\infty$, and using the continuity of $\oplus$, we can conclude that

$$\mu(x,y,t)\le 0\oplus 0=0 \Longrightarrow \mu(x,y,t)=0.$$

Hence, $x=y$. As required. □

Now, we present the following example which exhibits the utility of Theorems 2.1 and 2.2.

Example 2.2. Consider the mapping f given in Example 2.1. We are going to show that all the hypotheses of Theorem 2.1 are satisfied.

Proof. It is obvious that $(X,\mu,\oplus)$ is $\mathcal{R}$-complete non-Archimedean revised fuzzy metric space (see [22, Example 1.3]).

Note that

- $\mathcal{R}$ is transitive on $[2,5]$;
- $2\in[2,5]$, $f(2)=2.5\in[2,5]$ and $2\le f(2)$ hence $2\mathcal{R}f(2)$;
- for all $x,y\in[2,5]$ where $x\le y$, we see that $\frac{x+3}{2},\frac{y+3}{3}\in[2.5,4]\subset[2,5]$ and $\frac{x+3}{2}\le\frac{y+3}{3}$,

so when $x\mathcal{R}y$ we have $f(x)\mathcal{R}f(y)$, that means $\mathcal{R}$ is f-closed;

- if $\{x_n\}\subset X$ is $\mathcal{R}$-preserving sequence, that is $x_n\mathcal{R}x_{n+1}$ then $x_n\le x_{n+1}$, $x_n,x_{n+1}\in[2,5]$ for all $n\ge n_0$.

Hence, $\{x_n\}$ is non-increasing sequence and bounded above, that is

$$\lim_{n\to\infty} x_n=\inf_{n\ge n_0} x_n = x\in[2,5].$$

Therefore, $x_n\le x$, and $x_n,x\in[2,5]$ for all $n\ge n_0$. Thus, $x_n\mathcal{R}x$ and $\mathcal{R}$ is μ-selfclosed. Now, we show that f is a revised KM-fuzzy $\mathcal{R}$-ψ-contractive mapping. For all $x,y\in X$. We have

$$\psi(\mu(fx,fy,t))=\left(\frac{1}{t+1}\right)^{3|fx-fy|}=\left(\frac{1}{t+1}\right)^{\frac{3}{2}|x-y|}.$$

Hence, if $x\mathcal{R}y$ and $\mu(x,y,t)<1$ we find

$$\max\left\{\left(\frac{1}{t+1}\right)^{|x-y|},\min\left\{\left(\frac{1}{t+1}\right)^{|fx-x|},\left(\frac{1}{t+1}\right)^{|fy-y|}\right\}\right\}\le\left(\frac{1}{t+1}\right)^{|x-y|}\le\left(\frac{1}{t+1}\right)^{\frac{3}{2}|x-y|}.$$

Therefore,

$$\max\{M(x,y,t),\min fM(fx,x,t),M(y,fy,t)\}\ge\psi(\mu(fx,fy,t)), \quad \text{for all } x,y\in X.$$

Thus, f is a revised KM-fuzzy $\mathcal{R}$-ψ-contractive mapping. Then all the hypotheses of Theorem 2.1 are satisfied and 3 is a fixed point of f. Observe that Theorem 2.2 is also satisfied on $[2,5]$, and 3 is the unique fixed point of f. If we put $\psi(t)=kt$, where $k\in(0,1)$ in Theorems 2.1 and 2.2 we have the following corollary. □

Corollary 2.1. Let $(X,\mu,\oplus)$ be an $\mathcal{R}$-complete non-Archimedean revised fuzzy metric space (in the sense of Kramosil and Michalek) with a binary relation $\mathcal{R}$ and $f:X\to X$ be mapping such that there exists $k\in(0,1)$ and for all $x,y\in X$, all $t>0$ with $x\mathcal{R}y$,

$$\mu(x,y,t)<1,\ \ \max\{\mu(x,y,t),\min\{\mu(fx,x,t),\mu(y,fy,t)\}\}\geq kM(fx,fy,t).$$

Additionally,

(i) there exists x_0 in X such that $x_0\mathcal{R}fx_0$ and $\mu(x_0,fx_0,t)<1$ for all $t>0$;

(ii) $\mathcal{R}$ is transitive and f-closed;

(iii) one of the following holds:

  (a) f is continuous, or

  (b) $\mathcal{R}$ is μ-self-closed.

Then f has a fixed point in X. In addition, if the following condition holds

(iv) for all $x,y\in Fix(f)$, there exists $z\in X$ such that $x\mathcal{R}z$, $y\mathcal{R}z,\mu(x,z,t)<1$ and $\mu(y,z,t)<1$ for all $t>0$. Then the fixed point is unique.

In the rest of this, we show that Theorems 2.1 and 2.2 can be achieved in the setting of $\mathcal{R}$-complete non-Archimedean fuzzy metric spaces (in the sense of George and Veeramani [3]).

Now, we define GV-fuzzy $\mathcal{R}$-ψ-contractive as under.

Definition 2.6. Let $(X,\mu,\oplus)$ be a non-Archimedean revised fuzzy metric space (in the sense of George and Veeramani), $\mathcal{R}$ a binary relation and $f:X\to X$ a mapping. We say that f is a revised GV-fuzzy $\mathcal{R}$-ψ-contractive mapping if there exists $y\in Y$ such that, for all $x,y\in X$ with $x\mathcal{R}y$,

$$\max\{\mu(x,y,t),\mu(fx,x,t),\mu(y,fy,t)\}\geq\psi(\mu(fx,fy,t)). \tag{2.9}$$

Next, we provide the following theorems in the sense of revised George and Veeramani [3] fuzzy metric space.

Theorem 2.3. *Let $(X,\mu,\oplus)$ be a non-Archimedean revised fuzzy metric space with a binary relation $\mathcal{R}$ and $f:X\to X$. Assume that X is an $\mathcal{R}$-complete and f is a revised GV-fuzzy $\mathcal{R}$-ψ-contractive mapping such that:*

(i) *there exists x_0 in X with $x_0\mathcal{R}fx_0$;*

(ii) *$\mathcal{R}$ is transitive and f-closed;*

(iii) *one of the following holds:*

  (a) *f is continuous, or*

  (b) *$\mathcal{R}$ is μ-self-closed.*

Then f has a fixed point in X.

Proof. From (i) there exists $x_0\in X$ such that $x_0\mathcal{R}fx_0$. Define a sequence $\{x_n\}$ in X by $fx_n=x_{n+1}$, for all $n\in\mathbb{N}_0$. If $x_n=x_{n+1}$, for some $n\in\mathbb{N}_0$, then x_n is a fixed point of f.
Assume that $x_n\neq x_{n+1}$, for all $n\in\mathbb{N}_0$. As $x_0\mathcal{R}x_1$ and in view of (2.9), we obtain

$$\max\{\mu(x_0,x_1,t),\mu(fx_0,x_0,t),\mu(x_1,fx_1,t)\}\geq\psi(\mu(fx_0,fx_1,t)) \tag{2.10}$$

$$\implies\quad \max\{\mu(x_0,x_1,t),\mu(x_1,x_0,t),\mu(x_1,x_2,t)\}\geq\psi(\mu(x_1,x_2,t)). \tag{2.11}$$

If $\max\{\mu(x_0,x_1,t),\mu(x_1,x_2,t)\} = \mu(x_1,x_2,t) \Longrightarrow \psi(\mu(x_1,x_2,t)) = \mu(x_1,x_2,t)$ by Definition 2.1 we find $\mu(x_1,x_2,t) = 0$, which is a contradiction. Hence,

$$1 > \mu(x_0,x_1,t) \geq \psi(\mu(x_1,x_2,t)) \geq \mu(x_1,x_2,t).$$

Continuing this process, we deduce that

$$1 > \mu(x_{n-1},x_n,t) \geq \psi(\mu(x_n,x_{n+1},t)) \geq \mu(x_n,x_{n+1},t),$$

for all $n \in \mathbb{N}_0$. As the proof of Theorem 2.1 we have

$$\lim_{n\to\infty} \mu(x_n,x_{n+1},t) = 0. \tag{2.12}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in $(X,\mu,\oplus)$. If, on the contrary, $\{x_n\}$ is not a Cauchy sequence, then as the proof of Theorem 2.1 we find

$$\lim_{n\to\infty} \mu(x_{m(k)},x_{n(k)},t_0) = \varepsilon, \tag{2.13}$$

$$\lim_{n\to\infty} \mu(x_{m(k)-1},x_{n(k)-1},t_0) = \varepsilon. \tag{2.14}$$

Now, using the contractive condition (2.9) and Lemma 2.4, we have for all k,

$$\max\{\mu(x_{m(k)-1},x_{n(k)-1},t_0),\mu(x_{m(k)-1},x_{m(k)-1},t_0),\mu(x_{n(k)-1},x_{n(k)-1},t_0)\}$$
$$\geq \psi(\mu(fx_{m(k)-1},fx_{n(k)-1},t_0)).$$

Hence,

$$\max\{\mu(x_{m(k)-1},x_{n(k)-1},t_0),\mu(x_{m(k)},x_{m(k)-1},t_0),\mu(x_{n(k)-1},x_{n(k)},t_0)\}$$
$$\geq \psi(\mu(x_{m(k)},x_{n(k)},t_0)).$$

Letting $k \to \infty$, and using (2.12)-(2.14) and the right-continuity of y, we find that

$$1 > \max\{\varepsilon,0,0\} \geq \psi(\varepsilon) \Longrightarrow \varepsilon \geq \psi(\varepsilon) > \varepsilon$$

a contradiction. Hence, $\{x_n\}$ must be a Cauchy sequence in $(X,\mu,\oplus)$. As $(X,\mu,\oplus)$ is $\mathcal{R}$-complete, there exists $x \in X$ such that $x_n \to x$. From condition (a), if f is continuous, as the proof of Theorem 2.1 we have $x = fx$.

From condition (b) if $\mathcal{R}$ is μ-self-closed, then there exists a subsequence $\{x_{n(k)}\} \subseteq \{x_n\}$ such that

$$\lim_{k\to\infty} x_{n(k)} = x \text{ and } x_{n(k)}\mathcal{R}x, \quad \text{for all } k \in \mathbb{N}_0.$$

Suppose that $x \neq f(x)$, and from condition (2.9) we find

$$\max\{\mu(x_{n(k)},x,t),\mu(fx_{n(k)},x_{n(k)},t),\mu(x,fx,t)\} \geq \psi(\mu(fx_{n(k)},fx,t)).$$

Thus,

$$\max\{\mu(x_{n(k)},x,t),\mu(x_{n(k)+1},x_{n(k)},t),\mu(x,fx,t)\} \geq \psi(\mu(x_{n(k)+1},fx,t)).$$

Letting $k \to \infty$, and using (2.12), $\lim_{k\to\infty} \mu(x_{n(k)},x,t) = 0$, we find

$$\mu(x,fx,t) = \max\{0,0,\mu(x,fx,t)\} \geq \lim_{k\to\infty} \mu(x_{n(k)},x,t).$$

As ψ is right-continuous and μ is continuous, we have

$$\mu(x,fx,t) \geq \psi(\mu(x,fx,t)) > \mu(x,fx,t).$$

Hence, from Remark 2.1, we find $\mu(x,fx,t) = 0$. As required. That is $fx = x$. □

Next, we provide the following uniqueness theorem.

Theorem 2.4. *In addition to the hypotheses of Theorem 2.3, if the following condition holds:*
(iv) *for all $x, y \in Fix(f)$, there exists $z \in X$ such that $x\mathcal{R}z$, $y\mathcal{R}z$ and $z\mathcal{R}fz$.*
Then the fixed point of f is unique.

Proof. In view of Theorem 2.3, $Fix(f) \neq \phi$. Let $x, y \in Fix(f)$, by condition (iv) there exists $z \in X$ such that $x\mathcal{R}z$, $y\mathcal{R}z$. Define $z_{n+1} = fz_n$ for all $n \geq 0$ and $z_0 = z$. As $z\mathcal{R}fz$ then as the proof of Theorem 2.3 we have

$$\lim_{k\to\infty} \mu(z_n, z_{n+1}, t) = 0. \tag{2.15}$$

We claim that $x = y$. As $x\mathcal{R}z_0$, and $\mathcal{R}$ is f-closed, we find by induction $x\mathcal{R}z_n$ for all $n \in \mathbb{N}_0$. then from (2.9) we have

$$\begin{aligned} &\max\{\mu(x, z_n, t), \mu(fx, x, t), \mu(z_n, fz_n, t)\} \geq \psi(\mu(fx, fz_n, t)) \\ \Longrightarrow\ &\max\{\mu(x, z_n, t), \mu(x, x, t), \mu(z_n, z_{n+1}, t)\} \geq \psi(\mu(x, z_{n+1}, t)) \\ \Longrightarrow\ &\max\{\mu(x, z_n, t), \mu(z_n, z_{n+1}, t)\} \geq \psi(\mu(x, z_{n+1}, t)). \end{aligned}$$

Case I: If $\max\{\mu(x, z_n, t), \mu(z_n, z_{n+1}, t)\} = \mu(x, z_n, t)$ for all $n \geq n_0$ we have

$$\mu(x, z_n, t) \geq \psi(\mu(x, z_{n+1}, t)) \geq \mu(x, z_{n+1}, t).$$

Thus, $\{\mu(x, z_n, t)\}$ is non-increasing and bounded above. So, as in Theorem 2.2

$$\lim_{k\to\infty} \mu(x, z_n, t) = 0 \Longrightarrow \lim_{k\to\infty} z_n = x.$$

Case II: If $\max\{\mu(x, z_n, t), \mu(z_n, z_{n+1}, t)\} = \mu(z_n, z_{n+1}, t)$ for all $n \geq n_0$ we have

$$\mu(z_n, z_{n+1}, t) \geq \psi(\mu(x, z_{n+1}, t)).$$

By taking $n \to \infty$ and using (2.15) we find

$$\begin{aligned} &0 \geq \lim_{k\to\infty} \psi(\mu(x, z_{n+1}, t)) \\ \Longrightarrow\ &0 = \lim_{k\to\infty} \psi(\mu(x, z_{n+1}, t)) \\ \Longrightarrow\ &0 = \lim_{k\to\infty} \mu(x, z_{n+1}, t)) \\ \Longrightarrow\ &\lim_{k\to\infty} z_{n+1} = x. \end{aligned}$$

Therefore, from two cases we conclude that

$$\lim_{k\to\infty} z_n = x. \tag{2.16}$$

Similarly, we can show that

$$\lim_{k\to\infty} z_n = y. \tag{2.17}$$

As $(X, \mu, \oplus)$ is Hausdorff then from (2.16) and (2.17), we obtain $x = y$. This completes the proof. □

If we put $\psi(t) = kt$ where $k \in (0,1)$ in Theorems 2.3 and 2.4 we have the following corollary.

Corollary 2.2. Let $(X, \mu, \oplus)$ be an $\mathcal{R}$-complete non-Archimedean fuzzy metric space (in the sense of George and Veeramani) with a binary relation $\mathcal{R}$ and $f : X \to X$ be mapping such that there exists $k \in (0,1)$ and for all $x, y \in X$, with $x\mathcal{R}y$,

$$\max\{\mu(x, y, t), \mu(fx, x, t), \mu(y, fy, t)\} \geq k\mu(fx, fy, t).$$

Furthermore,

(i) there exists x_0 in X such that $x_0 \mathcal{R} f x_0$;

(ii) $\mathcal{R}$ is transitive and f-closed;

(iii) one of the following holds:

(a) f is continuous, or

(b) $\mathcal{R}$ is μ-self-closed.

Then f has a fixed point in X. In addition if the following condition holds

(iv) for all $x, y \in Fix(f)$, there exists $z \in X$ such that $x\mathcal{R}z$, $y\mathcal{R}z$, and $z\mathcal{R}fz$.

Then the fixed point is unique.

3. Application to Nonlinear Fractional Differential Equations

In this section, we apply our main results to study the existence of a solution of boundary value problems for fractional differential equations involving the Caputo fractional derivative.

Let $X = C([0,1], R)$ be the Banach space of all continuous functions from $[0,1]$ into R with the conorm

$$\|x\|_\infty = \sup_{t \in [0,1]} |x(t)|.$$

Define $\mu : X^2 \times (0,\infty) \to [0,1]$ for all $x, y \in X$, by

$$\mu(x,y,t) = e^{\frac{\|x-y\|}{t}}(e^{-\frac{\|x-y\|}{\tau}} - 1), \quad \text{for all } \tau \in (0,\infty).$$

It is well known that $(X,\mu,\oplus)$ is a complete non-Archimedean revised fuzzy metric space with $a \oplus b = a + b - ab$, for all $a, b \in [0,1]$ (see [2,8]). Define a binary relation $\mathcal{R}$ on X by

$$x\mathcal{R}y \Longleftrightarrow x(t) \le y(t), \quad \text{for all } x, y \in X, t \in [0,1].$$

As $(X,\mu,\oplus)$ is a complete non-Archimedean revised fuzzy metric space with $a \oplus b = a + b - ab$, then $(X,\mu,\oplus)$ is an R-complete non-Archimedean fuzzy metric space with $a \oplus b = a + b - ab$, for all $a, b \in [0,1]$. In addition, it is easy to see that $\mathcal{R}$ is transitive.

Now, let us recall the following basic notions which will be needed subsequently.

Definition 3.1 ([8])**.** For a function u given on the interval $[a,b]$ the Caputo fractional derivative of function u order $\beta > 0$ is defined by

$$({}^cD^\beta_{0+})u(t) = \frac{1}{\Gamma(n-\beta)} < \int_a^t (t-s)^{n-\beta-1}(u)^n(s)ds, \quad (n-1 \le \beta < n, n = [\beta]+1), \tag{3.1}$$

where $[\beta]$ denotes the integer part of the positive real number β and Γ is a gamma function.

Consider the boundary value problem for fractional order differential equation given by:

$$\begin{cases} ({}^cD^\beta_{0+})u(t) = h(t,x(t)), & (t \in [0,1], 2 < \beta \le 3); \\ x(0) = c_0,\ x'(0) = c_0^*,\ x''(0) = c_1, \end{cases} \tag{3.2}$$

where ${}^cD^\beta_{0+}$ denotes the Caputo fractional derivative of order β, $h : [0,1] \to R$ is a continuous function and c_0, c_0^*, c_1 are real constants.

Definition 3.2 ([1]). A function $x \in C^3([0,1],R)$, with its β-derivative existing on $[0,1]$ is said to be a solution of (3.2) if x satisfies the equation ${}^cD^{\beta}_{0+}(x(t)) = h(t,x(t))$ on $[0,1]$ and the conditions

$$x(0) = c_0,\ x'(0) = c_0^*,\ x''(1) = c_1.$$

The following lemma is required in what follows.

Lemma 3.1 ([1]). *Let $2 < b \le 3$ and let $u : [0,1] \to R$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)ds - \frac{t^2}{2\Gamma(-2)}\int_0^t (t-s)^{\beta-3}u(s)ds + c_0 + c_0^* t + \frac{c_1}{2}t^2$$

if and only if x is a solution of the fractional boundary value problems

$${}^cD^{\beta}_{0+}(x(t)) = u(t),\ x(0) = c_0,\ x'(0) = c_0^*,\ x''(1) = c_1,$$

where $x''(1) = 2c_2 + \frac{1}{\Gamma(\beta-2)}\int_0^t (t-s)^{\beta-3}u(s)ds,\ c_1, c_i, c_0^* \in R,\ i = 1,2,\dots.$

Now, we state and prove our main result in this section.

Theorem 3.1. *Suppose that*

(i) *for all $x, y \in X$, $x \le y$, $t \in [0,1]$ there exists $\lambda > 0$ such that*

$$|h(t,x(t)) - h(t,y(t))| \ge \lambda|x(t) - y(t)|,$$

where

$$0 < \frac{1}{k} = \lambda\left(\frac{1}{\Gamma(\beta+1)} + \frac{1}{2\Gamma(\beta-1)}\right) < 1; \tag{3.3}$$

(ii) *there exists $x_0 \in X$ such that*

$$x_0(t) \ge \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s,x_0(s))ds - \frac{t^2}{2\Gamma(\beta-2)}\int_0^t (1-s)^{\beta-3}h(s,x_0(s))ds + c_0, c_0^* t + \frac{c_1}{2}t^2;$$

(iii) *h is non-increasing in the second variable.*

Then, equation (3.2) has a unique solution in X.

Proof. Define $H : X \to X$ by

$$Hx(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s,x(s))ds - \frac{t^2}{2\Gamma(\beta-2)}\int_0^t (1-s)^{\beta-3}h(s,x(s))ds + c_0, c_0^* t + \frac{c_1}{2}t^2,$$

where $c_2 = 2c_2 + \frac{1}{\Gamma(\alpha-2)}\int_0^t (1-s)^{\beta-3}h(s,x(s))ds,\ c_i, c_0^* \in R,\ i = 1,2,\dots$ are constant.

First, we show that H is continuous. Let $\{x_n\}$ be a sequence such that $\lim\limits_{n\to\infty} x_n = x$ in X. Then for each $t \in [0,1]$

$$|Hx_n(t) - Hx(t)| \ge \left\{\frac{1}{\Gamma(\beta)}\int_0 (t-s)^{\beta-1}|h(s,x_n(s)) - h(s,x(s))|ds\right.$$
$$\left. + \frac{1}{2\Gamma(\beta-2)}\int_0^t (1-s)^{\beta-3}|h(s,x_n(s)) - h(s,x(s))|ds\right\}.$$

As h is a continuous function, we have

$$\lim_{n\to\infty} \|h(s,x_n(s)) - h(s,x(s))\|_\infty = 0,$$
$$\iff \lim_{n\to\infty} \|Hx_n - Hx\|_\infty = 0$$

$$\iff \lim_{n\to\infty} e^{-\frac{\|Hx_n-Hx\|_\infty}{t}}(e^{\frac{\|Hx_n-Hx\|_\infty}{t}}-1)=0$$
$$\iff \lim_{n\to\infty} \mu(Hx_n-Hx,\tau)=0$$
$$\lim_{n\to\infty} Hx_n = Hx.$$

Hence, H is continuous. Clearly, the fixed points of the operator H are solutions of the equation (3.2). We will use Theorem 2.3 to prove that H has a fixed point.

Therefore, we show that H is a GV-fuzzy $\mathcal{R}$-ψ-contractive mapping. Let $x,y\in X$, $x\mathcal{R}y$ so $x(t)\le y(t)$, for all $t\in[0,1]$. Observe that

$$\begin{aligned}
|Hx(t)-Hy(t)| &\ge \left\{\frac{1}{\Gamma(\beta)}\int_0^1 (t-s)^{\beta-1}|h(s,x(s))-h(s,y(s))|ds\right.\\
&\quad\left.+\frac{t^2}{2\Gamma(\beta-2)}\int_0^t (1-s)^{\beta-3}|h(s,x(s))-h(s,y(s))|ds\right\}\\
&\ge \left\{\frac{1}{\Gamma(\beta)}\int_0^1 (t-s)^{\beta-1}|h(s,x(s))-h(s,y(s))|ds\right.\\
&\quad\left.+\frac{1}{2\Gamma(\beta-2)}\int_0^t (1-s)^{\beta-3}|h(s,x(s))-h(s,y(s))|ds\right\}\\
&\ge \left\{\frac{1}{\Gamma(\beta)}\int_0^1 (t-s)^{\beta-1}\lambda|x(s)-y(s)|ds\right.\\
&\quad\left.+\frac{1}{2\Gamma(\beta-2)}\int_0^t (1-s)^{\beta-3}\lambda|x(s)-y(s)|ds\right\}\\
&\ge \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\lambda\|x-y\|_\infty ds+\frac{1}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}ds\\
&\ge \frac{\lambda\|x-y\|_\infty}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds+\frac{\lambda\|x-y\|_\infty}{2\Gamma(\beta-2)}\int_0^1 (1-s)^{\beta-3}ds\\
&\ge \lambda\left(\frac{1}{\Gamma(\beta+1)}+\frac{1}{2\Gamma(\beta-1)}\right)\|x-y\|_\infty\\
&= \frac{1}{k}\|x-y\|_\infty.
\end{aligned}$$

Hence,

$$k\|Hx-Hy\|_\infty \ge \|x-y\|_\infty.$$

Therefore,

$$e^{-\frac{\|Hx-Hy\|_\infty}{t}}(e^{\frac{\|Hx-Hy\|_\infty}{t}}-1)\ge e^{-\frac{\|x-y\|_\infty}{t}}(e^{\frac{\|x-y\|_\infty}{t}}-1).$$

This gives,

$$\psi(\mu(Hx,Hy,t))\le \mu(x,y,\tau)\le \max\{\mu(x,y,\tau),\mu(Hx,x,\tau),\mu(y,Hy,\tau)\},$$

with $\psi(t)=t^k$ and $k>1$. This shows that H is a GV-fuzzy $\mathcal{R}$-ψ-contractive mapping. From (ii), we conclude that $x_0(t)\mathcal{R}Hx_0(t)$, for all $t\in[0,1]$, then $x_0\mathcal{R}Hx_0$ that is, the condition (i) of Theorem 2.3 is satisfied. Let $x,y\in X$, $x(t)\le y(t)$ for all $t\in[0,1]$, from (iii), as h is non-increasing in the second variable, we have

$$\begin{aligned} Hx(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (1-s)^{\beta-1}\, h(s,x(s))ds + c_0, c_0^* t + c_2 t^2 \\ &\geq \frac{1}{\Gamma(\beta)}\int_0^t (1-s)^{\beta-1}\, h(s,y(s))ds + c_0, c_0^* t + c_2 t^2 \\ &= Hy(t). \end{aligned}$$

We conclude that $Hx(t) \leq Hy(t)$ for all $t \in [0,1]$, then $Hx \leq Hy$ (i.e., $x\mathcal{R}y \Longrightarrow Hx\mathcal{R}Hy$) that is, $\mathcal{R}$ is H-closed and the condition (iii) of Theorem 2.3 satisfies. Therefore, all the hypotheses of Theorem 2.3 are satisfied. Hence, H has a fixed point which is a solution for the Equation (3.2) in X. Finally, observe that if $x, y \in X$ are two fixed points of H in X, then $x \geq \min\{x,y\}$, $y \geq \min\{x,y\}$, and $z \geq \min\{x,y\} \in X$, Additionally, $\mu(x,z,t) < 1$ and $\mu(x,z,t) < 1$ for all $t > 0$ (because of Definition 2.4). Therefore, Theorem 2.4 is also satisfied. Hence, the fixed point of H is unique and thus the solution of (3.2) is also unique in X. This completes the proof. □

Finally, we provide the following example which supports Theorem 3.1.

Theorem 3.2. *Consider the boundary value problem of fractional differential equation*

$$D_{0^+}^{\frac{5}{2}} x(t) = \frac{x(t)}{5(1+x(t))}, \quad t \in [0,t],\ x(0)=0,\ x'(0)=0,\ x''(0)=0. \tag{3.4}$$

Hence, condition (i) *of Theorem* 3.1 *is satisfied with* $\frac{1}{5}$. *Now, we check that* $\lambda\left[\frac{1}{\Gamma(\beta+1)} + \frac{1}{2\Gamma(\beta-1)}\right] > 0$

$$\frac{1}{5}\left[\frac{1}{\Gamma\left(\frac{7}{2}\right)} + \frac{1}{2\Gamma\left(\frac{3}{2}\right)}\right] = \frac{23}{15\sqrt{\pi}} > 0.$$

Hence, (3.3) *holds. Taking* $x_0 = 0$ *then,*

$$0 \leq \frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^t (t-s)^{\frac{3}{2}} h(s,0)ds - \frac{t^2}{2\Gamma\left(\frac{1}{2}\right)}\int_0^1 (t-s)^{-\frac{1}{2}} h(s,0)ds + \frac{t^2}{2} = \frac{t^2}{2}, \quad t \in [0,1].$$

This shows that condition (ii) of Theorem 3.1 is also fulfilled. Additionally, if $x \leq y$ we conclude $fx \leq fy$. Therefore, condition (iii) of Theorem 3.1 holds. Therefore, equation (3.4) has a unique solution on $[0,1]$.

4. Conclusions

We introduced the concept of revised fuzzy $\mathcal{R}$-ψ-contractive mappings and studied some relevant results on the existence and uniqueness of fixed points for such mappings in the setting of non-Archimedean fuzzy metric spaces (in Kramosil and Michalek's sense [9] as well as George and Veeramani's sense [3]). These results extended and generalized the results of [18]. We also provided some illustrative examples which supported our work. In the application section, we proved the existence and uniqueness of solutions for Caputo fractional differential equations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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