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Research Article

n-Tuple Fixed Point Theorem in Bi-Complete *F*-quasi Metric Space

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Abstract. In the present paper, two theorems are discussed, one is fixed point theorem and another is n-tuple fixed point theorem for a new contractive condition in bi-complete F-quasi metric space. Also, an example is given to validate the result.

Keywords. Quasi metric space, n-tuple fixed point, Bi-complete

Mathematics Subject Classification (2020). 54H25, 47H10

1. Introduction

One of the most significant topic in functional analysis is fixed point theory. The whole fixed point theory based on very influential theorem Banach Contraction Principle [4]. Since then many researchers worked on it and develop results in different spaces like Metric space, Hilbert space, normed space, G-metric space, b-metric space, cone metric space etc. Further, fixed point theorems are extended in quasi-metric space.

Definition 1.1 ([16]). The function $q : X \times X \to [0,\infty)$ is a quasi-metric if it satisfies

(i) $q(a,b) = 0 \Leftrightarrow a = b$, for all $a,b \in X$.

(ii) $q(a,b) \le q(a,c) + q(c,b)$, for all $a,b,c \in X$.

The pair $(X,q)$is called quasi-metric space.

The study of fixed point theorems on quasi-metric space added by Aydi *et al*. [3], Bilgili *et al*. [6], Shatanawi *et al*. [14,15], and Alegre *et al*. [1].

Later, F-quasi metric space is defined as under

Definition 1.2 ([11])**.** (X,δ_q) is named as F-quasi metric space and δ_q is named as F-quasi metric, if a function $\delta_q : X \times X \to [0,\infty)$, a constant $B \in [0,+\infty)$ and a $f \in F$,so that

(δ_1) $\delta_q(x_1,x_2) = 0 \Leftrightarrow x_1 = x_2 \ \forall \ x_1,x_2 \in X$,

(δ_2) $\delta_q(x_1,x_2) > 0 \Rightarrow f(\delta_q(x_1,x_2)) \le f\left(\sum_{i=1}^{N-1} \delta_q(v_i,v_{i+1})\right) + B$.

For every $N \in \mathbb{N}$ with $N \ge 2$, $\forall \ x_1,x_2 \in X$ and for all $(v_i)_{i=1}^{N} \subset X$ with $(v_1,v_N) = (x_1,x_2)$.

The notion of coupled fixed point was initiated by Guo and Laxmikantham [10]. Berinde and Borcut [5] extended this theory to triple fixed point. Karapınar *et al*. [11,12] proved quadruple fixed point theorem in partial order metric space. This theory is further stretched for n-tuple fixed point theorems by Ertürk and Karakaya [7,8]. The n-dimensional theory is very useful in many engineering problems.

Definition 1.3 ([7])**.** Assume Xbe a non-empty set and let

$$F : \prod_{i=1}^{n} X^i \to X,$$

then $(x^1,x^2,\dots,x^n) \in \prod_{i=1}^{n} X^i$ is termed as n-tuple fixed point if

$$\begin{aligned} x^1 &= F(x^1,x^2,\dots,x^n) \\ x^2 &= F(x^2,x^3,\dots,x^n,x^1) \\ x^3 &= F(x^3,x^4,\dots,x^n,x^1,x^2) \\ &\vdots \\ x^n &= F(x^n,x^1,\dots,x^{n-1}). \end{aligned}$$

The purpose of this paper is to establish the results on n-tuple fixed point which will be generalization of above results. We have proved n-tuple fixed point theorem in F-quasi metric space for a new contractive condition.

The following definitions are required to discuss to understand this paper.

Definition 1.4 ([2])**.** A function $f : (0,\infty) \to R$ is called non-decreasing function if $f(x_1) \le f(x_2)$, for all $x_1,x_2 \in [0,+\infty)$. Also f is said to be logarithmic-like when every positive sequence $\{x_n\}$satisfies

$$\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} f(x_n) = -\infty.$$

In the sequel F is set of functions f.

Definition 1.5 ([11])**.** Consider F-quasi metric space (X,δ_q). Then $\{x_n\}$ in X is known as right convergent sequence (left convergent sequence) to $x \in X$ if

$$\lim_{n\to\infty} \delta_q(x,x_n) = 0, \quad \lim_{n\to\infty} \delta_q(x_n,x) = 0.$$

Definition 1.6 ([11])**.** When a sequence $\{x_n\}$in X is both right and left convergent, then it is said to be bi-convergent sequence.

Definition 1.7 ([11])**.** Let $\{x_n\}$ be sequence in F-quasi metric space (X,δ_q). Then $\{x_n\}$ is a right Cauchy sequence (left Cauchy sequence) if

$$\lim_{n,m\to\infty}\delta_q(x_n,x_m)=0 \qquad \lim_{n,m\to\infty}\delta_q(x_m,x_n)=0.$$

$\{x_n\}$ is a called bi-complete Cauchy sequence if it is left and right both Cauchy sequences.

Definition 1.8 ([13])**.** The functions $J:X\times X\to X$ and $K:X\to X$ are commutative if

$$J(Kx,Ky)=K(J(x,y)), \quad \forall\ x,y\in X.$$

2. Main Result

Theorem 2.1. *Let (X,δ_q) be a bicomplete F-quasi Metric Space and $J,K:X\to X$ be two mappings which satisfy*

(2.1.1) *J and K are commutative,*

(2.1.2) *$J(X)\subset K(X)$, $K(X)$ is closed, $\forall\ x,y\in X$,*

(2.1.3) *$\phi(t)<t$, for $t>0$,*

(2.1.4) *$\delta_q(Jx,Jy)\le\phi(\delta_q(Kx,Ky))$, $\forall\ x,y\in X$.*

Then J and Khave a unique common fixed point.

Proof. Let $x_0\in X$. Then $x_1\in X$ such that $Jx_0=Kx_1$. By continuing this process, we can define the sequence $\{y_n\}$ in X such that

$$y_n=Jx_n=Kx_{n+1}, \quad \text{for } n=0,1,2,\dots.$$

Here J and Khave common coincidence point.
To prove the unique coincidence point, consider u_1 and v_1 are two distinct coincidence points of J and K. Then $\exists\ u_2,v_2$ such that

$$\delta(u_2,v_2)>0,\ Ju_1=Ku_1=u_2,\ Jv_1=Kv_1=v_2.$$

From (2.1.4)

$$\delta_q(u_2,v_2)=\delta_q(Ju_1,Jv_1)\le\phi(\delta_q(Ku_1,Kv_1))=\phi(\delta_q(u_2,v_2))<\delta_q(u_2,v_2).$$

This is a contradiction, thus J and K have a unique coincidence point.
Consider $0<\delta_q(Jx_0,Jx_1)<\varepsilon$, then

$$\begin{aligned}\delta_q(Jx_n,Jx_{n+1})&\le\phi(\delta_q(Kx_n,Kx_{n+1}))\\&<\delta_q(Kx_n,Kx_{n+1})\\&=\delta_q(Jx_{n-1},Jx_n)\\&\le\phi(\delta_q(Kx_{n-1},Kx_n))\\&<\delta_q(Kx_{n-1},Kx_n)\\&=\delta_q(Jx_{n-2},Jx_{n-1})\\&\vdots\\&=\delta_q(Jx_0,Jx_1)<\varepsilon.\end{aligned}$$

Now, consider $(m,n)\in N$, $m>n$, therefore

$$\sum_{i=n}^{m-1}\delta_q(Jx_i,Jx_{i+1})\leq(m-1-n)\delta_q(Jx_0,Jx_1)<(m-1-n)\varepsilon<\varepsilon$$

Now, consider $f(B)\in F\times(0,\infty)$, so that (δ_2) is satisfied

$$(2.1.5)\quad f\left(\sum_{i=n}^{m-1}\delta_q(Jx_i,Jx_{i+1})\right)\leq f(\varepsilon)<f(\varepsilon)-B,\ \forall\ n\in N$$

For $m>n\geq N$, using (δ_2) and (2.1.5)

$$\delta_q(Jx_n,Jx_m)>0\Rightarrow\delta_q(Jx_n,Jx_m)\leq f\left(\sum_{i=n}^{m-1}\delta_q(Jx_i,Jx_{i+1})\right)\leq f(\varepsilon)$$

Hence $\delta_q(Jx_n,Jx_m)<\varepsilon$

$\{y_n\}=\{Jx_n\}$ is a right Cauchy sequence.

Similarly, consider the pair (x_{i+1},x_i), then by above process, one can prove that $\{y_n\}$ is also left Cauchy sequence and thus a Cauchy sequence.
Since (X,δ_q) is bi-complete metric space, therefore $\{y_n\}$ is convergent to $z\in X$.
Now, $\{Jx_n\}=\{Kx_{n+1}\}\subset K(X)$.
Therefore, we have $\lim_{n\to\infty}\delta_q(Kx_n,Kz)=0$, because K is closed.
Now, to prove that J and K have z as coincidence point, assume that $\delta_q(Jz,Kz)>0$.
Then

$$\begin{aligned}f(\delta_q(Jz,Kz))&\leq f(\delta_q(Jz,Jx_n)+\delta_q(Jx_n,Kz))+B\\&\leq f(\phi(\delta_q(Kz,Kx_n))+\delta_q(Kx_{n+1},Kz))+B.\end{aligned}$$

As $n\to\infty$

$$\lim_{n\to\infty}f(\phi(\delta_q(Kz,Kx_n))+\delta_q(Kx_{n+1},Kz))+B\to-\infty.$$

which is a contradiction, therefore $\delta_q(Jz,Kz)=0\Rightarrow Jz=Kz$.
Therefore, z is coincidence point of J and K.
Therefore, $J(z)=K(z)=w$.
Now, $K(w)=K(K(z))=J(F(z))=J(K(z))=J(w)$.
Hence w is another coincidence point of J and K, but J and K have unique coincidence point.
Therefore,

$$z=w.$$

Therefore,

$$J(z)=K(z)=z.$$

Therefore, J and K have a single fixed point in common. □

Lemma 2.1 ([9]). *Consider a F-quasi-metric space (X,δ_q). Then, the following assertions hold:*

(i) *(X^n,δ_q) is a F-quasi-metric space with*

$$\Delta_q((u_1,u_2,\dots,u_n),(v_1,v_2,\dots,v_n))=\max(\delta_q(u_1,v_1),\delta_q(u_2,v_2),\dots,\delta_q(u_n,v_n))$$

for $u_1,u_2,\dots,u_n$, $v_1,v_2,\dots,v_n\in X$.

(ii) *The mapping* $a : X^n \to X$ *and* $b : X \to X$ *contain an n tuple common fixed point iff the mapping* $A : X^n \to X^n$ *and* $B : X^n \to X^n$ *defined by*

$$A(u_1,u_2,\ldots,u_n) = (a(u_1,u_2,\ldots,u_n), a(u_2,\ldots,u_n,u_1),\ldots,a(u_n,u_1,u_2,\ldots,u_{n-1})),$$
$$B(u_1,u_2,\ldots,u_n) = (bu_1,bu_2,\ldots,bu_n),$$

possess a common fixed point in X^n.

(iii) (X,δ_q) *is bi-complete iff* (X^n,Δ_q) *is bi-complete.*

Theorem 2.2. *Let* (X,δ_q)*be a bi-complete* F*-quasi metric space. Also, assume* $a : X^n \to X$, $b : X \to X$ *be two mappings which satisfy*

(2.1.6) $a(X^n) \subset b(X)$, $b(X)$ *is closed,* $\forall\ x,y \in X$

(2.1.7) $\delta_q(a(x_1,x_2,\ldots,x_n),a(y_1,y_2,\ldots,y_n)) \le \phi\left\{\frac{1}{n}(\delta_q(bx_1,by_1)+\delta_q(bx_2,by_2)+\ldots+\delta_q(bx_n,by_n))\right\}$

for all $(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n) \in X^n$.

(2.1.8) $\Delta_q : X^n \times X^n \to [0,\infty)$ *be defined by* (2.1.1) *and* (2.1.3) *of Theorem* 2.1*).*

Then a and b have a common n-tuple fixed point in X^n.

Proof. Assume $A : X^n \to X^n$ by $A(x_1,x_2,\ldots,x_n) = (a(x_1,x_2,\ldots,x_n), a(x_2,\ldots,x_n,x_1),\ldots,a(x_n,x_1,x_2,\ldots,x_{n-1}))$.

Also assume, $B : X^n \to X^n$ defined by

$$B(x_1,x_2,\ldots,x_n) = (bx_1,bx_2,\ldots,bx_n)$$

Using Lemma 2.1 (X^n,Δ_q) is bi-complete F-quasi metric space.

Also, $(x_1,x_2,\ldots,x_n) \in X^n$ is a common n-tuple fixed point of a and b iff A and B have a common fixed point.

Now,

$$\begin{aligned}\Delta_q(A(x_1,x_2,\ldots,x_n),A(y_1,y_2,\ldots,y_n)) &= \Delta_q(a(x_1,x_2,\ldots,x_n),a(x_2,\ldots,x_n,x_1),\ldots,a(x_n,x_1,x_2,\ldots,x_{n-1}),\\ &\qquad a(y_1,y_2,\ldots,y_n),a(y_2,\ldots,y_n,y_1),\ldots,a(y_n,y_1,y_2,\ldots,y_{n-1}))\\ &= \max\{\delta_q(a(x_1,x_2,\ldots,x_n),a(y_1,y_2,\ldots,y_n)),\\ &\qquad \delta_q(a(x_2,x_3\ldots,x_n,x_1),a(y_2,\ldots,y_n,y_1)),\ldots,\\ &\qquad \delta_q(a(x_n,x_1,x_2,\ldots,x_{n-1}),a(y_n,y_1,y_2,\ldots,y_{n-1}))\}.\end{aligned}$$

Consider

$$\begin{aligned}\Delta_q(A(x_1,x_2,\ldots,x_n),A(y_1,y_2,\ldots,y_n)) &= \delta_q(a(x_1,x_2,\ldots,x_n),a(y_1,y_2,\ldots,y_n)\\ &\le \phi\left\{\frac{1}{n}(\delta_q(bx_1,by_1)+\delta_q(bx_2,by_2)+\ldots+\delta_q(bx_n,by_n))\right\}\\ &\le \max\{(\delta_q(bx_1,by_1),\delta_q(bx_2,by_2),\ldots,\delta_q(bx_n,by_n))\}\\ &= \Delta_q(B(x_1,x_2,\ldots,x_n),B(y_1,y_2,\ldots,y_n))\end{aligned}$$

or

$$\begin{aligned}\Delta_q(A(x_1,x_2,\ldots,x_n),A(y_1,y_2,\ldots,y_n)) &= \delta_q(a(x_2,\ldots,x_n,x_1),a(y_2,\ldots,y_n,y_1)\\ &\le \phi\left\{\frac{1}{n}(\delta_q(bx_2,by_2)+\ldots+\delta_q(bx_n,by_n)+\delta_q(bx_1,by_1))\right\}\end{aligned}$$

$$\leq \frac{1}{n}(\delta_q(bx_2,by_2)+\ldots+\delta_q(bx_n,by_n)+\delta_q(bx_1,by_1))$$
$$\leq \max\{(\delta_q(bx_2,by_2),\ldots,\delta_q(bx_n,by_n),\delta_q(bx_1,by_1))\}$$
$$=\Delta_q(B(x_2,x_3,\ldots,x_n,x_1),B(y_2,y_3,\ldots,y_n,y_1)).$$

Proceeding in a similar way, one can prove

$$\Delta_q(A(x_1,x_2,\ldots,x_n),A(y_1,y_2,\ldots,y_n))=\Delta_q(B(x_2,\ldots,x_n,x_1),B(y_2,\ldots,y_n,y_1)).$$

Thus by Lemma 2.1, a and b have a common n-tuple fixed point in X^n. □

Example 2.1. Let $X=[0,1]$. Define $\delta_q : X\times X\to[0,\infty)$ by

$$\delta_q(x,y)=\begin{cases}0, & x=y,\\ |y|+|x-y|, & \text{otherwise.}\end{cases}$$

δ_q is bi-complete F-quasi metric with $f(t)=\ln t$ and $B=0$.
Consider $a : X^n\to X$ by $a(x_1,x_2,\ldots,x_n)=\dfrac{x_1+x_2+\ldots+x_n}{n}$.
Define $b : X\to X$ by $b(x)=\frac{x}{n}$.
Then a and b are commutative.
Also

$$\begin{aligned}&\delta_q(a(x_1,x_2,\ldots,x_n),a(y_1,y_2,\ldots,y_n)\\ &\quad=|a(y_1,y_2,\ldots,y_n)|+|a(x_1,x_2,\ldots,x_n)-a(y_1,y_2,\ldots,y_n)|\\ &\quad=\frac{1}{n}[|y_1+y_2+\ldots+y_n|+|(x_1-y_1)+(x_2-y_2)+\ldots+(x_n-y_n)|]\\ &\quad=|by_1+by_2+\ldots+by_n|+|(bx_1-by_1)+(bx_2-by_2)+\ldots+(bx_n-by_n)|\\ &\quad=\left|\delta_q(bx_1,by_1)+\delta_q(bx_2,by_2)+\ldots+\delta_q(bx_n,by_n)\right|\\ &\quad\leq\phi\left\{\frac{1}{n}(\delta_q(bx_1,by_1)+\delta_q(bx_2,by_2)+\ldots+\delta_q(bx_n,by_n))\right\}.\end{aligned}$$

All conditions of Theorem 2.2 are met, so a and $b$have common n-tuple fixed point.

Corollary 2.1. *Let* (X,δ_q)*be a bi-complete F-quasi metric space. Consider $J,K : X\to X$ be two arbitrary mappings which satisfy* (2.1.1), (2.1.2), (2.1.3) *and the condition below:*
(2.1.9) $\delta_q(Jx,Jy)\leq k\delta_q(Kx,Ky)$, $\forall\ x,y\in X$, $k\in(0,1)$.
Then J and K have a unique common fixed point.

Proof. Consider $\phi(t)=kt$ in Theorem 2.1, we get the result. □

Corollary 2.2. *Let* (X,δ_q) *be a bi-complete F-quasi metric space. Also, assume $a : X^n\to X$, $b : X\to X$ be two arbitrary mappings which satisfy* (2.1.3), (2.1.6), (2.1.8) *of* Theorem 2.1 *and*
(2.1.10) $\delta_q(a(x_1,x_2,\ldots,x_n),a(y_1,y_2,\ldots,y_n))\leq\dfrac{k}{n}\{(\delta_q(bx_1,by_1)+\delta_q(bx_2,by_2)+\ldots+\delta_q(bx_n,by_n))\}$.
Then a and b have an n-tuple fixed point in common.

Proof. Consider $\phi(t)=kt$ in Theorem 2.1, we get the result. □

Remark 2.1. Corollary 2.1 and Corollary 2.2 are theorems proved by Ghasab *et al*. [9].

Remark 2.2. If $n = 4$ in Theorem 2.2, we will get quadruple fixed point.

Remark 2.3. If $n = 3$ in Theorem 2.2, we will get tripled fixed point.

Remark 2.4. If $n = 2$ in Theorem 2.2, we will get coupled fixed point.

Remark 2.5. If $n = 1$ in Theorem 2.2, we will get fixed point.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. Aleire and J. Marın, Modified w-distances on quasi-metric spaces and a fixed point theorem on complete quasi-metric spaces, *Topology and its Applications* **203** (2016), 32 – 41, DOI: 10.1016/j.topol.2015.12.073.

[2] O. Alqahtani, E. Karapınar and P. Shahi, Common fixed point results in function weighted metric spaces, *Journal of Inequalities and Applications* **2019** (2019), Article ID 164, DOI: 10.1186/s13660-019-2123-6.

[3] H. Ayadi, M. Jellali and E. Karapinar, On fixed point results for α-implicit contractions in quasi-metric spaces and consequences, *Nonlinear Analysis Modelling and Control* **21** (2016), 40 – 56, DOI: 10.15388/NA.2016.1.3.

[4] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae* **3** (1922), 133 – 181, URL: https://eudml.org/doc/213289.

[5] V. Berinde and M. Borcut, Triple fixed point theorems for contractive type mappings in partially ordered metric spaces, *Nonlinear Analysis: Theory, Methods & Applications* **74**(15) (2011), 4889 – 4897, DOI: 10.1016/j.na.2011.03.032.

[6] N. Bilgili, E. Karapınar and B. Samet, Generalized α-ψ contractive mapping in quasi-metric spaces and related fixed point theorems, *Journal of Inequalities and Applications* **2014** (2014), Article number: 36, DOI: 10.1186/1029-242X-2014-36.

[7] M. Ertürk and V. Karakaya, n-tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces, *Journal of Inequalities and Applications* **2013** (2013), Article number: 196, DOI: 10.1186/1029-242X-2013-196.

[8] M. Ertürk and V. Karakaya, Correction: 'n-tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces', *Journal of Inequalities and Applications* **2013** (2013), Article number: 368, DOI: 10.1186/1029-242X-2013-368.

[9] E.L. Ghasab, H. Majani, E. Karapinar and G.S. Rad, New fixed point results in F-quasi metric spaces and an Application, *Advances in Mathematical Physics* **2020** (2020), Article ID 9452350, DOI: 10.1155/2020/9452350.

[10] D. Guo and V. Laxmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Analysis: Theory, Methods & Applications* **11**(5) (1987), 623 – 632, DOI: 10.1016/0362-546X(87)90077-0.

[11] E. Karapınar and V. Berinde, Quadruple fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Banach Journal of Mathematical Analysis* **6**(1) (2012), 74 – 89, DOI: 10.15352/bjma/1337014666.

[12] E. Karapınar and N.V. Luong, Quadruple fixed point theorems for nonlinear contractions, *Computers & Mathematics with Applications* **64**(6) (2012), 1839 – 1848, DOI: 10.1016/j.camwa.2012.02.061.

[13] V. Lakshmikanthm and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis: Theory, Methods and Applications* **70**(12) (2009), 4341 – 4349, DOI: 10.1016/j.na.2008.09.020.

[14] W. Shatanawi, M.S. MD Noorani, H. Alsamir and A. Bataihah, Fixed and common fixed point theorems in partially ordered quasi-metric spaces, *Journal of Mathematics and Computer Science* **16** (2012), 516 – 528, URL: https://www.isr-publications.com/jmcs/1352/download-fixed-and-common-fixed-point-theorems-in-partially-ordered-quasi-metric-spaces.

[15] W. Shatanawi and A. Pitea, Some coupled fixed point theorems in quasi-partial metric spaces, *Fixed Point Theory and Applications* **2013** (2013), Article number: 153, DOI: 10.1186/1687-1812-2013-153.

[16] W.A. Wilson, On quasi-metric spaces, *American Journal of Mathematics* **53**(3) (1931), 675 – 684, DOI: 10.2307/2371174.