Communications in Mathematics and Applications
Vol. 13, No. 2, pp. 647–659, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i2.1768

http://www.rgnpublications.com

Research Article

Some Fixed Point Theorems in Extended Cone *b*-Metric Spaces

Abhishikta Das and T. Bag*

Department of Mathematics, Siksha-Bhavana, Visva-Bharati, Santiniketan 731235, Birbhum, West Bengal, India
***Corresponding author:** tarapadavb@gmail.com

Received: December 19, 2021 **Accepted:** May 1, 2022

Abstract. In this paper, the notion of extended cone b-metric space is introduced, established the structure of the open ball and defined the notion of convergence of a sequence. Finally, restructured the Banach and Kannan contraction theorems without the normality condition in this new setting.

Keywords. Cone, Cone metric, Cone metric space, Extended cone b-metric space, Contraction mapping

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

Advanced metric fixed point theory is based on the contraction mapping principle in different types of generalized metric spaces. Among those generalized metric spaces one is b-metric space, introduced by Bakhtin [3], and Czerwik [5]. Bakhtin [3] generalized the famous Banach contraction principle in b-metric space. Following the idea of b-metric, Kamran *et al.* [9] developed the idea of extended b-metric. They redefined the inequality (b3)(Definition 1) [10] replacing the constant ($s \geq 1$) by a function $\theta : X \times X \to [1,\infty)$. Later, Aydi *et al.* [2] extended the function θ from $X \times X$ to $X \times X \times X$ and introduced a new setting. In 2004, replacing the set of non-negative real numbers with an ordered real Banach space, Guang *et al.* [6] developed cone metric space. By using the ideas of b-metric and cone metric, Hussain *et al.* [7] formulated cone b-metric space. They also developed some topological properties and some results on KKM mappings.

Following the concept of extended b-metric, in this paper, we introduce the idea of extended cone b-metric and study its structure. Finally, without the normality condition, we have generalized the Banach [4], and Kannan [10] contraction principle in the view of an extended cone b-metric space. Furthermore, we have justified our results with proper examples.

The organization of the paper is as follows. Section 2 provides some preliminary results which are used to study the main results of this paper. Extended cone b-metric spaces are introduced in Section 3. In Section 4, Banach and Kannan contraction type theorems are established.

2. Preliminaries

To remind the readers, we picked up some basic definitions and results which are given below.

Let us start with the definition of b-metric.

Definition 2.1 ([5]). Let X be a nonempty set and $s \geq 1$ be a given real number. A function $B : X \times X \to \mathbb{R}_{\geq 0}$ is called a b-metric if for all $x,y,z \in X$ it satisfies the following conditions:

(b1) $B(x,y) = 0$ if and only if $x = y$,

(b2) $B(x,y) = B(y,x)$,

(b3) $B(x,z) \leq s[B(x,y) + B(y,z)]$.

The pair (X,B) is called a b-metric space.

Definition 2.2 ([2]). Let X be a nonempty set and $\theta : X \times X \times X \to [1,\infty)$ be a function. Suppose $d : X \times X \to [0,\infty)$ be a function which satisfies

(i) $d(x,y) > 0$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(iii) $d(x,y) \leq \theta(x,y,z)(d(x,z) + d(z,y))$ for all $x,y,z \in X$.

Then d is called an extended b-metric on X and the pair (X,d) is called an extended b-metric space.

Definition 2.3 ([6]). Let E be a real Banach space and $P \subset E$. P is called a cone if and only if

(i) P is closed, nonempty, and $P \neq \{\theta\}$.

(ii) $a,b \in \mathbb{R}$, $a,b \geq 0$, $x,y \in P \implies ax + by \in P$.

(iii) $P \cap (-P) = \{\theta\}$

In a cone $P \subset E$, a partial ordering $\preceq$ is considered by $x \preceq y$ if and only if $y - x \in P$ and $x \prec y$ indicates that $x \prec y$ but $x \neq y$, while $x \lll y$ indicates the interior of P, in short int$P$.

Definition 2.4 ([6]). A cone P in a real Banach space E is called normal if there is number $M > 0$ such that for all $x,y \in E$,

$$\theta \preceq x \preceq y \implies \|x\| \leq M\|y\|.$$

The least positive number satisfying above is called the normal constant of P.

In the following always P is a cone in the real Banach E with non-empty interior and $\preceq$ is the partial ordering with respect to P.

Definition 2.5 ([6])**.** Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies

(i) $d(x,y) \succ \theta$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(iii) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and the pair (X,d) is called a cone metric space.

Definition 2.6 ([6])**.** Consider a sequence $\{x_n\}$ in a cone metric space (X,d) and P be a normal cone in E with normal constant M. Then

(i) $\{x_n\}$ converges to x if for every $c \in E$ with $c \gg \theta$, $\exists\ N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n,x) \ll c$. Denoted by $\lim_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is said to be Cauchy if for any $c \in E$ with $c \gg \theta$, $\exists\ N \in \mathbb{N}$ such that for all $n,m \geq N$, $d(x_n,x_m) \ll c$.

(iii) (X,d) is said to be a complete cone metric space if every Cauchy sequence is convergent in X.

Definition 2.7 ([12])**.** Let (X,d) be a cone metric space and $B \subseteq X$.

(i) A point $b \in B$ is called an interior point of B whenever there exist $p \gg \theta$ such that $B(b,p) \subseteq B$ where $B(b,p) = \{y \in X : d(b,y) \ll p\}$.

(ii) B is called open if each element of B is an interior point of B.

(iii) A point $b \in B$ is called an limit point of B whenever for every $p \gg \theta$, $B(b,p) \cap (B \setminus \{x\}) \neq \phi$.

(iv) B is called closed if each limit point of B belongs to B.

(v) If $x \in B$ is a limit point then there exists a sequence $\{x_n\}$ in B which converges to x.

Definition 2.8 ([13])**.** Let (X,d) be a cone metric space. A set $B \subseteq X$ is called bounded above if $\exists\ c \in E$ with $c \gg \theta$ such that $d(x,y) \ll c$, $\forall\ x,y \in B$ and is called bounded if $\delta(B) = \sup\{d(x,y) : x,y \in B\}$ exists in E.

Definition 2.9 ([8])**.** In a cone metric space (X,d) if for any sequence $\{x_n\}$ in X, there is a subsequence of $\{x_n\}$ which converges in X, then X is called a sequentially compact cone metric space.

Lemma 2.10 ([12])**.** *Let (X,d) be a cone metric space.*

(i) *For each $\theta \ll c_1$ and $c_2 \in P$ there is an element $\theta \ll d$ such that $c_1 \ll d$ and $c_2 \ll d$.*

(ii) *For each $\theta \ll c_1$ and $\theta \ll c_2$ there is an element $\theta \ll c$ such that $c \ll c_1$ and $c \ll c_2$.*

Next, we recollect the notion of a cone b-metric.

Definition 2.11 ([7])**.** Let X be a nonempty set and $s \geq 1$ be a constant. Suppose the mapping $d: X \times X \to E$ satisfies

(i) $d(x,y) \succ \theta$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(iii) $d(x,y) \preceq s(d(x,z)+d(z,y))$ for all $x,y,z \in X$.

Then d is called a cone b-metric on X and the pair (X,d) is called a cone b-metric space.

3. Extended Cone *b*-metric Space

In this section we introduce the idea of an extended cone b-metric space which generalized the notion of cone b-metric spaces.

Through out the section, P is a cone in the real Banach space E with non-empty interior and $\preceq$ is the partial ordering with respect to P.

Definition 3.1. Let X be a nonempty set and $\sigma : X \times X \times X \to [1,\infty)$ be a function. A mapping $d_\sigma : X \times X \to E$ which satisfies the following conditions:

(E1) $d_\sigma(x,y) \succ \theta$ for all $x,y \in X$ and $d_\sigma(x,y) = \theta$ if and only if $x = y$;

(E2) $d_\sigma(x,y) = d_\sigma(y,x)$ for all $x,y \in X$;

(E3) $d_\sigma(x,y) \preceq \sigma(x,y,z)(d_\sigma(x,z)+d_\sigma(z,y))$ for all $x,y,z \in X$.

is called an extended cone b-metric on X and the pair (X,d_σ) is said to be an extended cone b-metric space (in short ECb-MS).

Remark 3.2. If $\sigma(x,y,z) = s \geq 1$ then we obtain the definition of a cone b-metric space and for $\sigma(x,y,z) = 1$ it represents the cone metric space.

Example 3.3. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$ and $P = \{(x,y) \in \mathbb{R}^2 : x,y \geq 0\}$ be a cone in E and the partial ordering is: $(x_1,y_1) \preceq (x_2,y_2)$ if and only if $x_2 - x_1 \in P$ and $y_2 - y_1 \in P$.
Define $\sigma : X \times X \times X \to [1,\infty)$ by $\sigma(x,y,z) = 2 + |x| + |y| + |z|$, $\forall\ x,y,z \in X$. Then with the function $d_\sigma(x,y) = ((x-y)^2, \alpha(x-y)^2)$, $\forall\ x,y \in X$ where $\alpha > 0$, (X,d_σ) becomes an ECb-MS.

Example 3.4. Let $X = \mathbb{R}$ and $P = \mathbb{R}_{\geq 0}$ be a cone in $E = \mathbb{R}$. Here the partial ordering $\preceq$ with respect to P is defined as $x \preceq y$ if and only if $y - x \in P$. Define

$$d_\sigma(x,y) = \begin{cases} 0 & \text{if } x = y, \\ |x-y| & \text{if } x,y \in \mathbb{Q}, \\ 5 & \text{if one of } x,\ y \in \mathbb{Q}\setminus\{0\} \text{ and another is in } \mathbb{Q}^c, \\ 1 & \text{if one of } x,y \in \mathbb{Q}^c \text{ another is } 0, \\ 2 & \text{if } x,y \in \mathbb{Q}^c. \end{cases}$$

Clearly, (X,d_σ) is an extended cone b-MS where $\sigma : X \times X \times X \to [1,\infty)$ is defined by

$$\sigma(x,y,z) = \begin{cases} |x|+|y|+|z|+1 & \text{if } x,y \in \mathbb{Q};\ z \in \mathbb{Q}^c, \\ 10 & \text{otherwise.} \end{cases}$$

Note that condition (E1) and (E2) holds trivially. To verify the condition (E3) we have to consider the following cases:

(a)	$x,y\in\mathbb{Q}\setminus\{0\}$;	subcases: (i) $z\in\mathbb{Q}\setminus\{0\}$	(ii) $z\in\mathbb{Q}^c$	(iii) $z=0$
(b)	$x,y\in\mathbb{Q}^c$;	subcases: (i) $z\in\mathbb{Q}\setminus\{0\}$	(ii) $z\in\mathbb{Q}^c$	(iii) $z=0$
(c)	$x\in\mathbb{Q}\setminus\{0\}$, $y\in\mathbb{Q}^c$;	subcases: (i) $z\in\mathbb{Q}\setminus\{0\}$	(ii) $z\in\mathbb{Q}^c$	(iii) $z=0$
(d)	$x\in\mathbb{Q}\setminus\{0\}$, $y=0$;	subcases: (i) $z\in\mathbb{Q}\setminus\{0\}$	(ii) $z\in\mathbb{Q}^c$	(iii) $z=0$
(e)	$x\in\mathbb{Q}^c$, $y=0$;	subcases: (i) $z\in\mathbb{Q}\setminus\{0\}$	(ii) $z\in\mathbb{Q}^c$	(iii) $z=0$

In all these cases (E3) holds. Thus (X,d_σ) is an extended cone b-MS. But if we choose $x\in\mathbb{Q}\setminus\{0\}$, $y=0$ and $z\in\mathbb{Q}^c$, then $d_\sigma(x,y)=|x|$ and $d_\sigma(x,z)+d_\sigma(z,y)=5+1=6$. So, we can not find a constant $s\geq 1$ for which the inequality (E3) satisfies and hence (X,d_σ) is not a cone b-metric space.

Remark 3.5. From Example 3.4 it is very clear that extended cone b-MS is larger space than cone b-metric space.

First, we are interested to define open and closed balls in extended cone b-metric spaces.

Definition 3.6. Let us choose $x\in X$ and for some $p\gg\theta$, define $B(b,p)=\{y\in X: d_\sigma(b,y)\ll p\}$ and $B[b,p]=\{y\in X: d_\sigma(b,y)\preceq p\}$ and called them the open ball and closed ball, respectively.

Next, we define the notion of convergence in extended cone b-metric spaces.

Definition 3.7. Let (X,d_σ) be an extended cone b-MS and E be a real Banach space with a cone P. Then

(i) $\{x_n\}\subset X$ converges to x if for every $c\in E$ with $c\gg\theta$, $\exists\ N\in\mathbb{N}$ such that for all $n\geq N$, $d_\sigma(x_n,x)\ll c$. We denote it by $\lim_{n\to\infty} x_n=x$.

(ii) $\{x_n\}\subset X$ is said to be Cauchy if for any $c\in E$ with $c\gg\theta$, $\exists\ N\in\mathbb{N}$ such that for all $n,m\geq N$, $d_\sigma(x_n,x_m)\ll c$.

(iii) (X,d_σ) is said to be a complete cone metric space if every Cauchy sequence in X converges to some point in X.

Proposition 3.8. *Let (X,d_σ) be an extended cone b-MS and P be a normal cone in E with normal constant M. Then the following results are hold*

(i) *$\{x_n\}\subset X$ converges to x if and only if $d_\sigma(x_n,x)\to\theta$ as $n\to\infty$.*

(ii) *$\{x_n\}\subset X$ is Cauchy if and only if $d_\sigma(x_n,x_m)\to\theta$ as $n,m\to\infty$.*

(iii) *Every convergent sequence is bounded.*

Proof. Proof of (i) and (ii) are same as the proof of Lemma 1 and Lemma 4 of [6].
We only prove (iii). For, let $\{x_n\}\subset X$ converges to x. Then for any $c\in E$ with $c\gg\theta$, $\exists\ N\in\mathbb{N}$ such that $\forall\ n\geq N$, $d_\sigma(x_n,x)\ll c$. Again,

$$d_\sigma(x_i,x)\ll\alpha,\quad \forall\ i=1,2,3,\dots(N-1),$$

for some $\alpha\in E$.
So by Lemma 2.10, $\exists\ d\gg\theta$ such that $d_\sigma(x_n,x)\ll d$, $\forall\ n\in\mathbb{N}$. Hence, $\{x_n\}$ is bounded. □

Definition 3.9. In an ECb-MS (X,d_σ), $A\subseteq X$ is said to be closed if for any sequence $\{x_n\}\subseteq A$ whenever $x_n\to x$ implies $x\in A$.

Lemma 3.10. *Let (X,d_σ) be an ECb-MS and d_σ be continuous with respect to one variable, then for each $a\in X$ and $r\gg\theta$, $B[a,r]$ is closed.*

Proof. Let $\{x_n\}\subset B[a,r]$ such that $x_n\to x\in X$ as $n\to\infty$. We have $d_\sigma(x_n,a)\preceq r$, $\forall\ n\in\mathbb{N}$. Now,

$$d_\sigma(x,a)=\lim_{n\to\infty}d_\sigma(x_n,a)\preceq r$$
$$\implies\quad d_\sigma(x,a)\preceq r$$

Hence $x\in B[a,r]$. □

Definition 3.11. (i) A point $b\in B$ is called an interior point of B whenever there exist $p\gg\theta$ such that $B(b,p)\subseteq B$.

(ii) A point $b\in B$ is called a limit point of B whenever for every $p\gg\theta$, $B(b,p)\cap(B\setminus\{x\})\neq\phi$.

Lemma 3.12. (i) *A set B is open if and only if $X\setminus B$ is closed.*

(ii) *B is called closed set if and only if each limit point of B belongs to B.*

Remark 3.13. In an extended cone b-MS, an open ball is not an open set.

Example 3.14. We consider the extended cone b-metric defined in Example 3.4 and take

$$B(e,3)=\{x\in\mathbb{R}:d_\sigma(e,x)<3\}=\{0\}\cup\mathbb{Q}^c$$

and so

$$(B(e,3))^c=\mathbb{Q}\setminus\{0\}.$$

Let $x_n=\frac{1}{n}$, $\forall\ n\in\mathbb{N}$. So,

$$\{x_n\}\subset\mathbb{Q}.$$

Now,

$$d_\sigma(x_n,0)=d_\sigma\left(\frac{1}{n},0\right)=\frac{1}{n}\to 0\ \text{ as } n\to\infty.$$

Hence, $\{x_n\}$ converges to 0 but $0\notin(B(e,3))^c$.
Thus $(B(e,3))^c$ is not closed which implies $B(e,3)$ is not open.

Lemma 3.15. *Let (X,d_σ) be an extended cone b-MS and P be a cone in the real Banach space E. Further, assume $\sigma:X\times X\times X\to[1,\infty)$ is bounded. Then*

(i) *Any convergent sequence has unique limit.*

(ii) *Every convergent sequence is Cauchy.*

Proof. (i) Let $\{x_n\}\subseteq X$ converges to x and y. Since σ is bounded, so $\exists\ K>0$ such that $\forall$ $x,y,z\in X$, $\sigma(x,y,z)<K$. Then $\forall\ n$,

$$d_\sigma(x,y)\preceq\sigma(x,y,x_n)(d_\sigma(x,x_n)+d_\sigma(x_n,y))$$
$$\implies\quad d_\sigma(x,y)\preceq K(d_\sigma(x,x_n)+d_\sigma(x_n,y))$$

For any given $c \gg \theta$ in E, $\exists\ N_1$ and $N_2 \in \mathbb{N}$ such that $d_\sigma(x,x_n) \ll \frac{c}{2(K+1)}$, $\forall\ n \geq N_1$ and $d_\sigma(x_n,y) \ll \frac{c}{2(K+1)}$, $\forall\ n \geq N_2$.

Let $N = \max\{N_1,N_2\}$. Then $\forall\ n \geq N$, $d_\sigma(x,y) \ll c$.

Since $c \gg \theta$ arbitrary, so $d_\sigma(x,y) \ll \frac{c}{n}$, $\forall\ n \geq 1$. This implies $\frac{c}{n} - d_\sigma(x,y) \in P$. Hence as limit $n \to \infty$, we get $-d_\sigma(x,y) \in P$. Thus we obtain $d_\sigma(x,y) = \theta$ that is $x = y$.

(ii) Since σ is bounded, so $\exists\ K > 0$ such that $\sigma(x_n,x_m,y) < K$, for any sequence $\{x_n\}$ in X and for any $y \in X$.

Suppose that $\{x_n\}$ be a sequence in X converges to x. Then for any given $c \gg \theta$ in E, $\exists\ N \in \mathbb{N}$ such that $d_\sigma(x,x_n) \ll \frac{c}{2(K+1)}$, $\forall\ n \geq N$. Then for $m \geq n \geq \mathbb{N}$,

$$\begin{aligned} d_\sigma(x_n,x_m) &\preceq \sigma(x_n,x_m,x)(d_\sigma(x_n,x) + d_\sigma(x_m,x)) \\ &\prec K\left[\frac{c}{2(K+1)} + \frac{c}{2(K+1)}\right] \\ &\prec c. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. □

Remark 3.16. The boundedness of σ is necessary for the Lemma 3.15.

In the next two examples we have taken $E = \mathbb{R}$ as the real Banach space under the partial ordering $x \preceq y$ if and only if $y - x \in P$ where $P = \mathbb{R}_{\geq 0}$.

Example 3.17. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{\sqrt{2}, \sqrt{3}\} = A \cup B$. Define d_σ on $X \times X$ by

$$d_\sigma(x,y) = \begin{cases} 0 & \text{if } x = y, \\ x & \text{if } x \in A,\ y \in B, \\ y & \text{if } x \in B,\ y \in A, \\ 1 & \text{if } x,y \in A, \\ 2 & \text{if } x,y \in B. \end{cases}$$

Define $\sigma : X \times X \times X \to [1,\infty)$ by

$$\sigma(x,y,z) = \begin{cases} \frac{2}{x+y} + 1 & \text{if } x,\ y \in A,\ z \in B, \\ \frac{8}{z} + 1 & \text{if } x,\ y \in B,\ z \in A, \\ \frac{4x}{1+z} + 1 & \text{if } x \in A,\ y \in B,\ z \in A, \\ \frac{4y}{1+z} + 1 & \text{if } x \in B,\ y \in A,\ z \in A, \\ 10 & \text{otherwise.} \end{cases}$$

Clearly, (X,d_σ) is an extended cone b-MS.
Here the function σ is unbounded. If we choose the sequence $\{x_n\} = \{\frac{1}{n}\}$ then $d_\sigma(x_n,\sqrt{2}) = \frac{1}{n} \to 0$ and $d_\sigma(x_n,\sqrt{3}) = \frac{1}{n} \to 0$ as $n \to \infty$ implies $\{x_n\}$ converges to both $\sqrt{2}$ and $\sqrt{3}$.

Example 3.18. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and define d_σ on $X \times X$ by $d_\sigma(x,y) = |x-y|$, $\forall\ x,y \in X$ and $\sigma : X \times X \times X \to [1,\infty)$ by

$$\sigma(x,y,z) = \begin{cases} 1 + \frac{1}{z} & \text{if } x,y \in X,\ z \in X \setminus \{0\}, \\ 1 & \text{if } x,y \in X,\ z = 0. \end{cases}$$

Clearly, (X, d_σ) is an extended cone b-MS.
Here σ is unbounded. If we choose the sequence $\{x_n\} = \{\frac{1}{n}\}$ then $d_\sigma(x_n, 0) = \frac{1}{n} \to 0$ as $n \to \infty$ implies $\{x_n\}$ converges to 0. To check the uniqueness of the converging point, let $y \in X$ such that $\{x_n\}$ converges to y. Then,

$$d_\sigma(0,y) \le \sigma\left(0,y,\frac{1}{n}\right)\left(d_\sigma\left(0,\frac{1}{n}\right)+d_\sigma\left(\frac{1}{n},y\right)\right)$$
$$=(1+n)\left(d_\sigma\left(0,\frac{1}{n}\right)+d_\sigma\left(\frac{1}{n},y\right)\right) \nrightarrow 0 \text{ as } n\to\infty$$

So, we can not conclude about the uniqueness.

Definition 3.19. The function $\sigma : X \times X \times X \to [1,\infty)$ is said to be continuous if for any sequence $\{(x_n, y_n, z_n)\}$, $\sigma(x_n, y_n, z_n) \to \sigma(x, y, z)$ as $n \to \infty$ whenever $x_n \to x$, $y_n \to y$, $z_n \to z$ as $n \to \infty$.

Definition 3.20. An extended cone b-metric d_σ is said to be continuous if for any sequence $\{(x_n, y_n)\} \in X \times X$, $d_\sigma(x_n, y_n) \to d_\sigma(x, y)$ in E whenever $x_n \to x$, $y_n \to y$ in X.

Remark 3.21. Hussain *et al.* in their paper [8] had shown that a b-metric function $d(x,y)$ for $s \ge 1$ need not to be jointly continuous with respect to both variables and so is an extended b-metric space.
From Example 3.4 we can show that an extended cone b-metric space is not continuous.
For, let $x_n = \frac{1}{n}$, $\forall\ n \in \mathbb{N}$. So, $\{x_n\} \subset \mathbb{Q}$. Now $d_\sigma(x_n, 0) = d_\sigma(\frac{1}{n}, 0) = \frac{1}{n} \to 0$ as $n \to \infty$. Hence $\{x_n\}$ converges to 0. But $d_\sigma(x_n, \sqrt{2}) = d_\sigma(\frac{1}{n}, \sqrt{2}) = 5 \nrightarrow 1 = d_\sigma(0, \sqrt{2})$. So, d_σ is not continuous.

Proposition 3.22. *If d_σ is continuous concerning the first variable then it is continuous in the second variable and vice versa.*

Proof. First, assume that d_σ is continuous with respect to the first variable. Suppose $\{y_n\} \subset X$ such that $y_n \to y$ as $n \to \infty$.
For each $x \in X$, then we have

$$\lim_{n\to\infty} d_\sigma(x, y_n) = \lim_{n\to\infty} d_\sigma(y_n, x) = d_\sigma(y, x) = d_\sigma(x, y). \qquad \square$$

Next, we prove some simple propositions.

Proposition 3.23. *Let (X, d_σ) be an extended cone b-MS having a cone P in E and σ be continuous function on $X \times X \times X$. Suppose that $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively, then we have*

$$\frac{1}{\sigma(x,y,x)\sigma(x,y,y)} d_\sigma(x,y) \le \lim_{n\to\infty}\inf d_\sigma(x_n,y_n) \le \lim_{n\to\infty}\sup d_\sigma(x_n,y_n) \le \sigma(x,y,x)\sigma(x,y,y) d_\sigma(x,y).$$

In particular, if $x = y$ then $\lim_{n\to\infty} d_\sigma(x_n, y_n) = \theta$.
Moreover, for each $z \in X$, we have

$$\frac{1}{\sigma(x,z,x)} d_\sigma(x,z) \le \lim_{n\to\infty}\inf d_\sigma(x_n,z) \le \lim_{n\to\infty}\sup d_\sigma(x_n,z) \le \sigma(x,z,x) d_\sigma(x,z).$$

Proof. By (E3) we have

$$d_\sigma(x,y) \le \sigma(x,y,x_n)(d_\sigma(x,x_n) + d_\sigma(x_n,y))$$

$$\implies \quad d_\sigma(x,y) \preceq \sigma(x,y,x_n)d_\sigma(x,x_n) + \sigma(x,y,x_n)\sigma(x_n,y,y_n)(d_\sigma(x_n,y_n) + d_\sigma(y_n,y)) \tag{3.1}$$

and

$$d_\sigma(x_n,y_n) \preceq \sigma(x_n,y_n,x)d_\sigma(x_n,x) + \sigma(x_n,y_n,x)\sigma(x,y_n,y)(d_\sigma(x,y) + d_\sigma(y,y_n)). \tag{3.2}$$

Taking the lower limit as $n \to \infty$ in (3.1) and the upper limit as $n \to \infty$ in (3.2) we obtain the first desired result.
Similarly, using again (E3) the last assertion follows. □

Remark 3.24. If we take $\sigma(x,y,z) = s$, a constant, then for $s \geq 1$ we obtained [4, Lemma 2.1].

Proposition 3.25. *Let (X,d_σ) be an extended cone b-MS having a cone P in E and σ be a bounded and continuous function on $X \times X \times X$. Suppose that $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} d_\sigma(x_n,y_n) = \theta$ whenever $\lim_{n\to\infty} x_n = x$ for some $x \in X$, then $\lim_{n\to\infty} y_n = x$.*

Proof. By the triangle inequality in an extended cone b-MS, we have

$$d_\sigma(y_n,x) \preceq \sigma(y_n,x,x_n)(d_\sigma(y_n,x_n) + d_\sigma(x_n,x)).$$

Now by taking limit $n \to \infty$ from the above inequality, we get

$$\lim_{n\to\infty} d_\sigma(y_n,x) \preceq \lim_{n\to\infty} \sigma(y_n,x,x_n)(d_\sigma(x_n,y_n) + d_\sigma(x_n,x)) = \theta.$$

Hence $\lim_{n\to\infty} y_n = x$. □

Proposition 3.26. *If d_σ is continuous with respect to one variable and σ is bounded then for each pair $x,y \in X$, $\exists$ two disjoint open sets U and V containing x and y, respectively.*

Proof. Suppose x and y be two distinct points in X and say $d_\sigma(x,y) = c$, for some c in P. Since σ is bounded, so $\exists\ K > 0$ such that $\sigma(x,y,z) < K$, $\forall\ x,y,z \in X$. Now consider the open balls $B\big(x, \frac{c}{2(K+1)}\big)$ and $B\big(y, \frac{c}{2(K+1)}\big)$. To show that they are disjoint assume that $z \in B\big(x, \frac{c}{2(K+1)}\big) \cap B\big(y, \frac{c}{2(K+1)}\big)$. Now,

$$\begin{aligned} d_\sigma(x,y) &\preceq \sigma(x,y,z)[d_\sigma(x,z) + d_\sigma(z,y)] \\ &\prec K\left[\frac{c}{2(K+1)} + \frac{c}{2(K+1)}\right] \\ &\prec c. \end{aligned}$$

Hence a contradiction. □

4. Fixed Point Theorems for Some Contractive Mappings

In this section, we establish well known Banach type and Kannan type contraction principles in these new settings without the normality condition.

Theorem 4.1. *Let (X,d_σ) be a complete extended cone b-MS and P be a cone in E. Suppose $\sigma: X \times X \times X \to [1,\infty)$ be a bounded functional and $T: X \to X$ satisfy the contractive condition*

$$d_\sigma(Tx,Ty) \preceq k d_\sigma(x,y), \tag{4.1}$$

where $a < k < 1$, for some $a > 0$. Moreover, if $\lim_{n,m\to\infty} \sigma(x_n,x_m,x_{n+1}) < \frac{1}{k}$, where $x_n = Tx_{n-1}$, $n = 1,2,\cdots$ be an iterative sequence for some $x_0 \in X$ then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$ arbitrary and consider the iterative sequence

$$x_1 = Tx_0,\, x_2 = Tx_1 = T^2x_0, \cdots, x_{n+1} = Tx_n = T^n x_0, \cdots.$$

Then successively applying inequality (4.1) we obtain,

$$d_\sigma(x_n, x_{n+1}) \preceq k^n d_\sigma(x_0, x_1), \quad \forall\, n \in \mathbb{N}.$$

Using the inequality (E3) we have,

$$\begin{aligned}
&d_\sigma(x_n,x_m)\\
&\quad\preceq \sigma(x_n,x_m,x_{n+1})(d_\sigma(x_n,x_{n+1})+d_\sigma(x_{n+1},x_m))\\
&\quad\preceq \sigma(x_n,x_m,x_{n+1})k^n d_\sigma(x_0,x_1)+\sigma(x_n,x_m,x_{n+1})\sigma(x_{n+1},x_m,x_{n+2})(d_\sigma(x_{n+1},x_{n+2})+d_\sigma(x_{n+2},x_m))\\
&\quad\preceq \sigma(x_n,x_m,x_{n+1})k^n d_\sigma(x_0,x_1)+\sigma(x_n,x_m,x_{n+1})\sigma(x_{n+1},x_m,x_{n+2})k^{n+1}d_\sigma(x_1,x_0)+\dots\\
&\qquad+\sigma(x_n,x_m,x_{n+1})\sigma(x_{n+1},x_m,x_{n+2})\dots\sigma(x_{m-2},x_m,x_{m-1})d_\sigma(x_{m-1},x_m)\\
&\quad\preceq \sigma(x_n,x_m,x_{n+1})k^n d_\sigma(x_0,x_1)+\sigma(x_n,x_m,x_{n+1})\sigma(x_{n+1},x_m,x_{n+2})k^{n+1}d_\sigma(x_1,x_0)+\dots\\
&\qquad\sigma(x_n,x_m,x_{n+1})\sigma(x_{n+1},x_m,x_{n+2})\dots\sigma(x_{m-2},x_m,x_{m-1})k^{m-1}d_\sigma(x_1,x_0)\\
&\quad\preceq [\sigma(x_n,x_m,x_{n+1})k^n+\sigma(x_n,x_m,x_{n+1})\sigma(x_{n+1},x_m,x_{n+2})k^{n+1}+\dots+\\
&\qquad\sigma(x_n,x_m,x_{n+1})\sigma(x_{n+1},x_m,x_{n+2})\dots\sigma(x_{m-2},x_m,x_{m-1})k^{m-1}]d_\sigma(x_1,x_0)\\
&\quad\preceq [\sigma(x_1,x_m,x_2)\dots\sigma(x_{n-1},x_m,x_n)\sigma(x_n,x_m,x_{n+1})k^n\\
&\qquad+\sigma(x_1,x_m,x_2)\dots\sigma(x_{n-1},x_m,x_n)\sigma(x_n,x_m,x_{n+1})\sigma(x_{n+1},x_m,x_{n+2})k^{n+1}+\dots\\
&\qquad+\sigma(x_1,x_m,x_2)\dots\sigma(x_{n-1},x_m,x_n)\sigma(x_n,x_m,x_{n+1})\sigma(x_{n+1},x_m,x_{n+2})\dots\\
&\qquad\sigma(x_{m-2},x_m,x_{m-1})k^{m-1}]d_\sigma(x_1,x_0)
\end{aligned}$$

Since $\lim_{n,m\to\infty} \sigma(x_n,x_m,x_{n+1})k < 1$, so that for each $m \in \mathbb{N}$ the series $\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \sigma(x_i,x_m,x_{i+1})$ converges by ratio test.

Let $S = \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \sigma(x_i,x_m,x_{i+1})$ and $S_n = \sum_{i=1}^{n} k^i \prod_{j=1}^{n} \sigma(x_j,x_m,x_{j+1})$. Then

$$d_\sigma(x_n,x_m) \preceq d_\sigma(x_1,x_0)[S_{m-1}-S_n].$$

Letting $n\to\infty$, $d_\sigma(x_n,x_m)\to\theta$. Hence we conclude that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X there exist $x \in X$ such that $x_n \to x$.

To show that x is a fixed point of T,

$$\begin{aligned}
d_\sigma(Tx,x) &\preceq \sigma(Tx,x,x_n)(d_\sigma(Tx,x_n)+d_\sigma(x_n,x))\\
&\preceq \sigma(Tx,x,x_n)(kd_\sigma(x,x_{n-1})+d_\sigma(x_n,x))\\
&\prec \sigma(Tx,x,x_n)(d_\sigma(x,x_{n-1})+d_\sigma(x_n,x)). \quad (\text{since, } k<1)
\end{aligned}$$

Since σ is bounded, there exist $M>0$ such that $\sigma(x,y,z)<M$, $\forall\, x,y,z\in X$. Again for any given $c \gg \theta$, $\exists\, N_1,N_2 \in \mathbb{N}$ such that

$$d_\sigma(x_{n-1},x) \ll \frac{c}{2(M+1)}, \quad \forall\,(n-1)\ge N_1$$

and

$$d_\sigma(x_n,x) \ll \frac{c}{2(M+1)}, \qquad \forall\, n \ge N_2.$$

Let $N = \max\{N_1+1, N_2\}$. Then $\forall\ n \geq N$,

$$d_\sigma(Tx,x) \prec c.$$

Thus $d_\sigma(Tx,x) \ll \frac{c}{m}$, $\forall\ m \geq 1$. This implies that $\frac{c}{m} - d_\sigma(Tx,x) \in P$, $\forall\ m \geq 1$. As $m \to \infty$, $\frac{c}{m} \to \theta$ and hence we get $-d_\sigma(Tx,x) \in P$. Thus $d_\sigma(Tx,x) = \theta$ that is $x = Tx$. For the uniqueness of the fixed point, let $x \neq y \in X$ such that $Ty = y$. Then

$$d_\sigma(Tx,Ty) \preceq k d_\sigma(x,y)$$

$$\Longrightarrow \quad d_\sigma(x,y) \preceq k d_\sigma(x,y)$$

Therefore, $d_\sigma(x,y) = \theta$.
This completes the proof. □

Remark 4.2. If $\theta(x,y,z) = 1$ then the above Theorem 4.1 reduces to the Banach type contraction in a cone metric space [6].

Example 4.3. Let us consider the non-normal cone $P = \{f \in E \mid f(t) > 0\}$ in $E = C^1[0,1]$, where $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and the partial ordering with respect to P is defined as

$$f \leq g \Longrightarrow g(t) - f(t) \in P, \quad \forall\ t \in [0,1].$$

Consider the extended cone b-MS (X,d_σ) where $X = [0,1]$ and $d_\sigma(x,y) = (x-y)^2 e^t$, $\forall\ x,y \in X$ and $\sigma(x,y,z) = 2+x+y+z$, $\forall\ x,y,z \in X$.
Define $T : X \to X$ by $Tx = \frac{x}{4}$, $\forall\ x \in X$. Then $d_\sigma(Tx,Ty) = \frac{(x-y)^2}{16} e^t < \frac{(x-y)^2}{4} e^t = k d_\sigma(x,y)$ where $k = \frac{1}{4}$ is taken and $0 < a < \frac{1}{4}$.
Note that for each $x \in X$, $T^n x = \frac{x}{4^n}$. Thus we obtain,

$$\lim_{n,m\to\infty} \theta(x_n,x_m,x_{n+1}) = \lim_{n,m\to\infty} \left(2 + \frac{x}{4^n} + \frac{x}{4^m} + \frac{x}{4^{n+1}}\right) = 2 < 4 = \frac{1}{k}.$$

Therefore, all the conditions of Theorem 4.1 are satisfied and so T has a unique fixed point in X. Here the fixed point is $x = 0$.

Theorem 4.4. *Let (X,d_σ) be a complete extended cone b-MS having a cone P in E and the function $\sigma : X \times X \times X \to [1,\infty)$ be bounded. Let T be a self mapping on X which satisfy the contractive condition*

$$d_\sigma(Tx,Ty) \preceq k[d_\sigma(Tx,x) + d_\sigma(Ty,y)], \quad \forall\ x,y \in X \tag{4.2}$$

for some constant $a < k < \frac{1}{2}$ where $a > 0$ be a scalar. Moreover, if σ bounded by $(\frac{1}{k} - 1)$ with $\lim_{n,m\to\infty} \sigma(x_n,x_m,x_{n+1}) < \frac{1-k}{k}$, where $x_n = Tx_{n-1}$, $n = 1,2,\cdots$ be an iterative sequence for $x_0 \in X$ then T has a unique fixed point in X.

Proof. For $x_0 \in X$, consider the iterative sequence $x_{n+1} = Tx_n$, $n = 0,1,2,\cdots$. Now applying inequality (4.2) we obtained

$$d_\sigma(x_n,x_{n+1}) \preceq \left(\frac{k}{1-k}\right) d_\sigma(x_n,x_{n-1}) = l d_\sigma(x_n,x_{n-1}) \preceq l^n d_\sigma(x_0,x_1), \quad \forall\ n \in \mathbb{N},\ 0 < l = \frac{k}{1-k} < 1.$$

From the proof of Theorem 4.1 we can conclude that $\{x_n\}$ is a Cauchy sequence in X and so it must converge to some point $x \in X$. Next, we show that x is a fixed point of T. For

$$\begin{aligned} d_\sigma(Tx,x) &\preceq \sigma(Tx,x,x_n)[d_\sigma(Tx,x_n)+d_\sigma(x_n,x)] \\ &\preceq \left(\frac{1}{k}-1\right)[k(d_\sigma(Tx,x)+d_\sigma(x_{n-1},x_n))+d_\sigma(x_n,x)] \\ &\preceq (1-k)d_\sigma(Tx,x)+(1-k)d_\sigma(x_n,x_{n-1})+\left(\frac{1}{k}-1\right)d_\sigma(x_n,x). \end{aligned}$$

Since $\{x_n\}$ converges to x, so for any $c \gg \theta$ in E, $\exists\ N_1,N_2 \in \mathbb{N}$ such that

$$(1-k)d_\sigma(x_n,x_{n-1}) \ll \frac{c}{2}, \quad \forall\ (n-1) \geq N_1$$

and

$$\left(\frac{1}{k}-1\right)d_\sigma(x_n,x) \ll \frac{c}{2}, \quad \forall\ n \geq N_2.$$

Let $N = \max\{N_1+1,N_2\}$. Then $\forall\ n \geq N$, $kd_\sigma(Tx,x) \preceq c$. So, we have $kd_\sigma(Tx,x) \preceq \frac{c}{m}$, $\forall\ m \in \mathbb{N}$. As $m \to \infty$, $\frac{c}{m} \to \theta$ and hence $-kd_\sigma(Tx,x) \in P$. Again $kd_\sigma(Tx,x) \in P$. Thus $d_\sigma(Tx,x) = \theta$ that is $x = Tx$. And the uniqueness of the fixed point easily follows from the contractive condition. □

Example 4.5. We consider the extended cone metric space of Example 3.4 and define $T_1(x) = \frac{x}{4}$, $\forall\ x \in X$ and $T_2(x) = \frac{1}{2}$ if $0 \leq x < 1$; $T_2(1) = \frac{1}{4}$. Clearly, T_1 is continuous but T_2 is not.
Now $\forall\ x,y \in X$, $d_\sigma(T_1x,T_1y) = \frac{(x-y)^2}{16}e^t$ and $d_\sigma(T_1x,x)+d_\sigma(y,T_1y) = \frac{9}{16}(x^2+y^2)e^t$. So,

$$\frac{(x-y)^2}{16}e^t \preceq \frac{1}{16}(x^2+y^2)e^t = \frac{1}{9}\cdot\frac{9}{16}(x^2+y^2)e^t \implies d_\sigma(T_1x,T_1y) \preceq k(d_\sigma(T_1x,x)+d_\sigma(T_1y,y)),$$

where $k = \frac{1}{9}$ and $0 < a < \frac{1}{9}$. Other conditions can be verified easily.
Again T_2 trivially satisfies all the conditions of above Theorem 4.4 for $k = \frac{1}{9}$ and hence both must have unique fixed point in X. For T_1 the fixed point is $x = 0$ and for T_2 which is $x = \frac{1}{2}$.

Acknowledgement

The author AD is thankful to University Grant Commission (UGC), New Delhi, India for awarding her senior research fellowship [Grant No.1221/(CSIRNETJUNE2019)]. Authors are thankful to the Department of Mathematics, Siksha-Bhavana, Visva-Bharati. We are also grateful to the referees for their valuable comments which are helped us to revise the paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. Aghajani, M. Abbas and J. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Mathematica Slovaca* **64**(4) (2014), 941 – 960, DOI: 10.2478/s12175-014-0250-6.

[2] H. Aydi, A. Felhi, T. Kamran, E. Karapinar and M. Usman, On nonlinear contractions in new extended b-metric spaces, *Applications and Applied Mathematics* **14** (2019), 537 – 547, URL: https://digitalcommons.pvamu.edu/aam/vol14/iss1/37.

[3] I.A. Bakhtin, The contraction mapping principle in almost metric spaces, *Functional Analysis* **30** (1989), 26 – 37.

[4] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae* **3** (1922), 133 – 181, DOI: 10.4064/fm-3-1-133-181.

[5] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti del Seminario Matematico e Fisico dell'Universita di Modena e Reggio Emilia* **46** (1998), 263 – 276.

[6] L.G. Huang and X. Zhang, Cone metric space and fixed point theorems of contractive mapping, *Journal of Mathematical Analysis and Applications* **322**(2) (2007), 1468 – 1476, DOI: 10.1016/j.jmaa.2005.03.087.

[7] N. Hussian and M.H. Shah, KKM mappings in cone b-metric spaces, *Computers & Mathematics with Applications* **62**(4) (2011), 1677 – 1684, DOI: 10.1016/j.camwa.2011.06.004.

[8] N. Hussain, V. Parvaneh, J.R. Roshan and Z. Kadelburg, Fixed points of cyclic weakly (ψ,ϕ,L,A,B)-contractive mappings in ordered b-metric spaces with applications, *Fixed Point Theory and Applications* **2013** (2013), Article number: 256, DOI: 10.1186/1687-1812-2013-256.

[9] T. Kamran, M. Samreen and A. Ul-Qurat, A generalization of b-metric space and some fixed point theorems, *Mathematics* **5**(2) (2017), 19, DOI: 10.3390/math5020019.

[10] R. Kannan, Some results on fixed points, *Bulletin of the Calcutta Mathematical Society* **60** (1968), 71 – 76, DOI: 10.1080/00029890.1969.12000228.

[11] G. Rano, T. Bag and S.K. Samanta, Fuzzy metric space and generating space of quasi-metric family, *Annals of Fuzzy Mathematics and Informatics* **11**(2) (2016), 183 – 195, URL: http://www.afmi.or.kr/papers/2016/Vol-11_No-02/PDF/AFMI-11-2(183-195)-H-150506R1.pdf.

[12] Sh. Rezapour, M. Derafshpour and R. Hamlbarani, A review on topological properties of cone metric spaces, in: *Proceedings of the International Conference on Analysis, Topology and Applications*, ATA'08, Virnjacka Banja, Serbia, May-June 2008, http://at.yorku.ca/c/a/w/q/04.htm.

[13] D. Turkoglu and M. Abuloha, Cone metric spaces and fixed point theorems in diametrically contractive mappings, *Acta Mathematica Sinica (English series)* **26** (2010), 489 – 496, DOI: 10.1007/s10114-010-8019-5.