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Research Article

Some Results on r-Row-Regular Circulant Partial Hadamard Matrices of Order $(k \times 2k)$

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Abstract. This paper provides some new results on r-row-regular circular partial Hadamard matrices of order $(k \times 2k)$, and also discusses the possible linear relationship between r and k. Furthermore, a method of constructing such a matrix is given.

Keywords. Hadamard matrix, Circulant matrix, Partial Hadamard matrix, Orthogonal design

Mathematics Subject Classification (2020). 05B15; 05B20; 05B30

1. Introduction

A square matrix H of order n and with entries ± 1 is called a Hadamard matrix if $HH' = nI_n$. Hadamard matrices are a class of square matrices first described by James Sylvester (1814-1897) in 1867. He called them anallagmatic pavement [4]. Later on 1893, French mathematician Jacques Hadamard published a paper investigating the values of determinants of square matrices with entries restricted to the set $\{-1, 1\}$ [4]. He found that the determinants of these matrices have a maximum value. It can be easily shown that if H is a Hadamard matrix of order n, then $n = 1, 2$, or $4\ s$ for some positive integer s. A long-standing conjecture in combinatorics states that a Hadamard matrix of order n exists for every $n \equiv 0 \pmod 4$. Despite the work of many researchers, the conjecture is far from being resolved [4, 6, 7, 9].

A rectangular matrix H of order $(k \times n)$ with entries ± 1 is called a partial Hadamard matrix if $HH' = nI_k$. Following are some examples of partial Hadamard matrices:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \quad [1,4,9]$$

A partial Hadamard matrix H is called circulant partial Hadamard matrix if it is a rectangular circulant matrix. Some examples of circulant partial Hadamard matrices are listed below:

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

Circulant partial Hadamard matrices are row-regular matrices. A circulant partial Hadamard matrix of order $(k \times n)$ and with row sum r is denoted by $r - H(k \times n)$.

The interesting and fascinating fact about a circulant partial Hadamard matrix $r - H(k \times 2k)$ is that we do not really know how n is related to the maximum value of k.

Specifically, when the maximum value of k will be n in a circulant partial Hadamard matrix $r - H(k \times n)$, we get a circulant Hadamard matrix of order n.

Known circulant Hadamard matrices are only of order 1 or 4. In 1963, H. J. Ryser [8] conjectured that H will be a circulant Hadamard matrix if and only if H is of order 1 or 4. This conjecture inspires researchers to investigate circulant partial Hadamard matrices because it is still open to be resolved [1–3].

In 2013, Craigen *et al.* [1] have obtained remarkable results along with some research problem.

In circulant partial Hadamard matrix $r - H(k \times n)$, the following results are holds:

(i) $r = n \Rightarrow k = 1$.

(ii) $r \le \frac{n}{2}$.

(iii) $\frac{n}{\sqrt{5}} < r \le \frac{n}{2} \Rightarrow k = 4$.

(iv) $k = n \Rightarrow n = r^2$.

(v) $r = 0 \Rightarrow k < \frac{n}{2}$.

(vi) $r\sqrt{k} \le n$.

(vii) The sum of column sums of H is rk.

(viii) The sum of squares of column sums of H is kn.

(ix) $r\sqrt{k} = n \Rightarrow H$ are concatenations of $\frac{n}{k}$ copies of circulant Hadamard matrices each of order k.

In a circulant partial Hadamard matrix $r - H(k \times n)$, the hardest part is finding the relationship between the variables r, k and n. For $r = 2$, there exist a circulant partial Hadamard matrix $2 - H(k \times 2k)$ in which $(k-1)$ is an odd prime power. But for $r = 4$, we have not identified any infinite class of the circulant partial Hadamard matrices of the type $2 - H(k \times 2k)$. The only known circulant partial Hadamard matrices of this type are $4 - H(4 \times 8)$, $4 - H(6 \times 12)$ and $4 - H(14 \times 28)$ [1].

In this article, some new properties of circulant partial Hadamard matrices of the form $r-H(k\times 2k)$ have been obtained together with a construction method of the type $4-H(k\times 2k)$.

Recently, Lin *et al.* [5] shown that the circulant partial Hadamard matrices can be used in *functional Magnetic Resonance Imaging* (fMRI).

2. Some Results on Circulant Partial Hadamard Matrices $r-H(k\times 2k)$

In this section, we will look at the possibilities of some linear relationship between r and k in the circulating partial Hadamard matrix $r-H(k\times 2k)$. The ideas have been put forward through the following theorems:

Theorem 2.1. *Existence of a circulant Hadamard matrix of order k implies existence of $2\sqrt{k}-H(k\times 2k)$.*

Proof. If M is circulant Hadamard matrix of order k, then $M=\sqrt{k}-H(k\times k)$, and therefore $(M\mid M)$ will be a circulant partial Hadamard matrix $2\sqrt{k}-H(k\times 2k)$. □

Illustration 2.1. *Since we have*

$$2-H(4\times 4)=\begin{pmatrix} 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1\\ -1 & 1 & 1 & 1\end{pmatrix},$$

therefore, we can find a circulant partial Hadamard matrix $4-H(4\times 8)$ given by

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1\\ -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1\end{pmatrix}.$$

Theorem 2.2. *Let $(\alpha_1,\alpha_2,\dots,\alpha_k,\alpha_{k+1},\dots,\alpha_{2k})$ be the 1st row of circulant Hadamard matrix $r-H(k\times 2k)$ and let c_i be the sum of ith column of H, for all $i=1,2,\dots,2k$. Let $x=(\alpha_1,\alpha_2,\dots,\alpha_k)$, $y=(\alpha_{k+1},\alpha_{k+2},\dots,\alpha_{2k})$, $c=(c_1,c_2,\dots,c_k)$ and $d=(c_{k+1},c_{k+2},\dots,c_{2k})$. Then*

(i) $x\cdot y=\frac{r^2}{2}-k$.

(ii) $r^2\le 4k$.

(iii) $r=k\Rightarrow k\le 4$.

(iv) *If r divide k and $rt=k$ for some integers t then $k\le 4t^2$.*

(v) $(x+y)\cdot(c+d)=r^2$.

Proof. Let M be the square circulant matrix of $2k$ with the first row $(\alpha_1,\alpha_2,\dots,\alpha_{2k})$. Then, the product of 1st and $(k+1)$th rows of M is r^2-2k, which implies $2x\cdot y=r^2-2k$ or $x\cdot y=\frac{r^2}{2}-k$. This proves (i).

Next, $x\cdot y\le k\Rightarrow\frac{r^2}{2}-k\le k$ or $r^2\le 4k$. This proves (ii).

$r=k\Rightarrow k^2\le 4k\Rightarrow k\le 4$ and $r=\frac{k}{t}\Rightarrow k\le 4t^2$. The conditions (iii) and (iv) are done.

We have $(x,y)\cdot(c,d)=2k$ and $(y,x)\cdot(c,d)=r^2-2k\Rightarrow(x+y,y+x)\cdot(c,d)=r^2$ or $(x+y)\cdot(c+d)=r^2$. This competes the proof of Theorem 2.2. □

Theorem 2.3. *Let* (x, y) *be the* 1*st row of* $r-H(k\times 2k)$, *where* x *and* y *are given by*

$$x=(\alpha_1,\alpha_2,\dots,\alpha_k) \text{ and } y=(\alpha_{k+1},\alpha_{k+2},\dots,\alpha_{2k}).$$

If u *and* v *are the numbers of places at which* x *and* y *agree with* $+1$ *and* -1, *respectively. Then*

$$u=\frac{r(r+2)}{8} \text{ and } v=\frac{r(r-2)}{8}.$$

Proof. Since $x\cdot y=\frac{r^2}{2}-k$, therefore

$$2(u+v)-k=\frac{r^2}{2}-k$$

$$\Rightarrow\quad u+v=\frac{r^2}{4} \tag{2.1}$$

Again

$$r=2u-2v$$

$$\Rightarrow\quad u-v=\frac{r}{2} \tag{2.2}$$

On solving (2.1) and (2.2), we get the required result. □

Notes 2.1. (1) *If* $r=0$ *or* 2, *then* $v=0$. *That is for the circulant partial Hadamard matrix* $r-H(k\times 2k)$ *in which* $r=0$ *or* 2, x *and* y *do not agree with* -1 *at any place.*

(2) *If* $r=4$, *then* $u=3$ *and* $v=1$. *That is for the circulant partial Hadamard matrix* $r-H(k\times 2k)$ *in which* $r=4$, x *and* y *agree with* $+1$ *and* -1 *at* 3 *and* 1 *places, respectively.*

Theorem 2.4. *If* $r=ak+b$ *for some positive rational numbers* a *and* b *in* $r-H(k\times 2k)$, *then*

(i) $ab\le 1$.

(ii) $ab=1\Rightarrow H$ *is concatenations of two circulant Hadamard matrices each of order* k.

(iii) $ab<1\Rightarrow k\le \frac{1}{a^2}(2\sqrt{1-ab}+2-ab)$.

Proof. We have $r^2\le 4k$ in $r-H(k\times 2k)$, therefore

$$(ak+b)^2\le 4k$$

$$\Rightarrow\quad \left(k-\frac{2-ab}{a^2}\right)^2\le\frac{4(1-ab)}{a^4} \tag{2.3}$$

Since

$$\left(k-\frac{2-ab}{a^2}\right)^2\ge 0$$

$$\Rightarrow\quad \frac{4(1-ab)}{a^4}\ge 0,$$

therefore $ab\le 1$. This proves (i).

Now, $ab=1\Rightarrow k=\frac{1}{a^2}=b^2$. In this case Schwartz inequality will became equality because of $k=b^2\Rightarrow r=2b$. Thus, H is concatenations of two circulant Hadamard matrices each of order k. This proves (ii). From (2.3) we can find bounds of k given by

$$-\frac{2\sqrt{1-ab}}{a^2}+\frac{2-ab}{a^2}\le k\le\frac{2\sqrt{1-ab}}{a^2}+\frac{2-ab}{a^2}.$$

Since

$$-\frac{2\sqrt{1-ab}}{a^2}+\frac{2-ab}{a^2}\le 0,$$

therefore

$$k\le \frac{2\sqrt{1-ab}}{a^2}+\frac{2-ab}{a^2}.$$

This proves (iii). □

Notes 2.2. (1) *If $b=0$, then $k\le \frac{4}{a^2}$.*

(2) *a and $b$can not be simultaneously zero, otherwise r will become zero.*

Theorem 2.5. *If $k=br-a$ for some positive integers a and b in circulant partial Hadamard matrix $r-H(k\times 2k)$, then*

(i) $b^2\ge a$.

(ii) $b^2=a\Rightarrow k$*is divisible by* 4.

(iii) $2(b-\sqrt{b^2-a})\le k\le 2(b+\sqrt{b^2-a})$.

Proof. $r^2\le 4k$ in $r-H(k\times 2k)$

$$\Rightarrow\quad r^2\le 4(-a+br)$$

$$\Rightarrow\quad (r-2b)^2\le 4(b^2-a) \tag{2.4}$$

$$(r-2b)^2\ge 0$$

$$\Rightarrow\quad b^2\ge a.$$

This proves (i).

Next, if $b^2=a$, then $r=2b$ and $k=b^2=a$. Since 2 divides k, therefore $k=b^2\Rightarrow b$ is an even number, and therefore k will be divisible by 4. This proves (ii).

From (2.4) we can obtain the following relation:

$$2(b-\sqrt{b^2-a})\le k\le 2(b+\sqrt{b^2-a}).$$

□

Notes 2.3. (1) *Particularly, if $a=0$, then $k\le 2b$.*

(2) *If $a=b$, then $r\ge 2$.*

3. Representation of $r-H(k\times 2k)$ in Terms of Four Matrices

We can express the circulant partial Hadamard matrix $r-H(k\times 2k)$ in terms of four block matrices A, B, C and D as follows:

$$H=\begin{pmatrix} A & B & C & D\\ D & A & B & C\end{pmatrix},$$

where each of the block matrices is of order $\frac{k}{2}$ and with entries ± 1.

Since $HH'=2kI_k$, therefore

$$\begin{pmatrix} A & B & C & D\\ D & A & B & C\end{pmatrix}\begin{pmatrix} A' & D'\\ B' & A'\\ C' & B'\\ D' & C'\end{pmatrix}=2kI_k$$

$$\Rightarrow \quad \begin{pmatrix} AA'+BB'+CC'+DD' & (AB'+BC'+CD'+DA')' \\ AB'+BC'+CD'+DA' & AA'+BB'+CC'+DD' \end{pmatrix} = 2kI_k$$

$$\Rightarrow \quad AA'+BB'+CC'+DD' = 2kI_{\frac{k}{2}} \text{ and } AB'+BC'+CD'+DA' = 0$$

Since row sum in circulant partial Hadamard matrix $r-H(k\times 2k)$ is r, therefore $(A+B+C+D)J_{\frac{k}{2}} = rJ_{\frac{k}{2}}$. We summarize the result:

Theorem 3.1. *The existence of a circulant partial Hadamard matrix $r-H(k\times 2k)$ is equivalent to the existence four square matrices A, B, C and D each of order $\frac{k}{2}$ and with entries ± 1 such that*

(i) $(A\ \ B\ \ C\ \ D)$ *is a circulant.*

(ii) $AA'+BB'+CC'+DD' = 2kI_{\frac{k}{2}}$.

(iii) $AB'+BC'+CD'+DA' = 0$.

(iv) $(A+B+C+D)J_{\frac{k}{2}} = rJ_{\frac{k}{2}}$.

4. Constructions of Circulant Partial Hadamard Matrix $4-H(14\times 28)$

This circulant Hadamard matrix is already a known circulant partial Hadamard matrix. The matrix has been obtained using computer search [1]. Here we forward a method based on Theorem 3.1 for the construction.

We select four square matrices A, B, C and D each of order 7 given below:

$$A = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is easy to verify that all the conditions given in Theorem 3.1 are satisfied by these four-square matrices A, B, C and D, and therefore, $H = \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}$ will be a circulant partial Hadamard matrix $4-H(14\times 28)$.

5. Conclusion, Thoughts, Discussion, Questions and Conjecture

In the circulant partial Hadamard matrix $r-H(k\times n)$, the basic question is, what will be the maximum value of k? For given values of r and n. In this paper, we have considered a special type of circulant partial Hadamard matrices of the form $r-H(k\times 2k)$. In the present work we have studied the possibilities of linear relationship between r and k in $r-H(k\times 2k)$. Results are new with innovative ideas. The results may strengthen the theory on circulant partial Hadamard matrices and may be able to give new insights to scholars working in this field. One of the issues with circulant partial Hadamard matrices of the $r-H(k\times 2k)$ type is the lack of theory and insufficient related data. The known circulant partial Hadamard matrices of this types are $2-H(k\times 2k)$ and $4-H(k\times 2k)$ only. The existence of $2-H(k\times 2k)$ is subject to the existence of negacyclic matrix $W(k,k-1)$ [1]. To date, no class of $4-H(k\times 2k)$ has been detected. Our results can serve as a bridge to fill these gaps.

The forward method for constructing circular partial Hadamard matrices of the form $r-H(k\times 2k)$ is the first analytical method for constructing such circulant partial Hadamard matrices. The proposed analytical method may be useful for generating more data in this range of circulant partial Hadamard matrices. Overall, the work presented will promote a theoretical and empirical approach to the enhancement of knowledge in the relevant field.

The curiosity about the nature of k in the circular partial Hadamard matrix $4-H(k\times 2k)$ still linger in our minds.

With this we list the possibilities on the nature of k in $4-H(k\times 2k)$:

(i) $k-1$ is a prime power.

(ii) $k-1$ is an odd prime power.

(iii) $\frac{k}{2}$ is a prime power.

(iv) $\frac{k}{2}$ is an odd prime number.

Based on our work, insights and ideas, we have made the following conjecture:

Conjecture 1. *Existence of* $4-H(k\times 2k)$ *implies that* $\frac{k}{2}$ *is an odd prime power for* $k>4$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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