Communications in Mathematics and Applications
Volume 4 (2013), Number 2, pp. 145–148

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Inequalities for the Incomplete Exponential Integral Functions

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Abstract In this paper, new inequalities involving the incomplete exponential integral function were presented.

1. Introduction

For any $x > 0$ and $n \in \mathbb{N}_0$, we denote

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt\,.$$

The function E_n is called the exponential integral function (see [1, 2]).

For any $1 < a < b$ and $n \in \mathbb{N}_0$, we define the incomplete exponential integral function ${}_a^bE_n$ by

$${}_a^bE_n(x) = \int_a^b t^{-n} e^{-xt} dt$$

for all $x > 0$.

In this paper, we will show that the function ${}_a^bE_n$ is non-increasing. Moreover, we will also prove the inequalities as follows.

For any $1 < a < b$, $x, y > 0$, $p > 1 = \frac{1}{p} + \frac{1}{q}$, and $m + n, pm, qn \in \mathbb{N}_0$,

$${}_a^bE_{m+n}\left(\frac{x}{p} + \frac{y}{q}\right) \le \left({}_a^bE_{pm}(x)\right)^{1/p} \left({}_a^bE_{qn}(y)\right)^{1/q}.$$

For any $1 < a < b$, $x, y > 1$, $n \in \mathbb{N}_0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x + y \le xy$,

$${}_a^bE_n(xy) \le \left({}_a^bE_n(px)\right)^{1/p} \left({}_a^bE_n(qy)\right)^{1/q}.$$

For any $1 < a < b$, $x > 0$, $0 < y < 1$, $n \in \mathbb{N}_0$, $0 < p < 1 = \frac{1}{p} + \frac{1}{q}$ and $x + y \ge xy$,

$${}_a^bE_n(xy) \ge \left({}_a^bE_n(px)\right)^{1/p} \left({}_a^bE_n(qy)\right)^{1/q}.$$

2010 *Mathematics Subject Classification*. 26D15.
Key words and phrases. Exponential integral function; Inequality .

For any $1<a<b$, $x,y>1$, $n\in\mathbb{N}_0$, $p>1$, $0<r<1$ and $\frac{1}{p}+\frac{1}{q}=1=\frac{1}{r}+\frac{1}{s}$,

$$ {}_a^bE_n(xy)\geq\left({}_a^bE_n\left(\frac{rx^p}{p}\right)\right)^{1/r}\left({}_a^bE_n\left(\frac{sy^q}{q}\right)\right)^{1/s}. $$

2. Main Results

Lemma 2.1. *Assume that $1<a<b$ and $n\in\mathbb{N}_0$. Then ${}_a^bE_n$ is non-increasing.*

Proof. For any $x>0$,

$$ {}_a^bE_n'(x)=-\int_a^b t^{1-n}e^{-xt}dt\leq 0. \qquad\square $$

Theorem 2.2. *Assume that $1<a<b$, $x,y>0$, $p>1=\frac{1}{p}+\frac{1}{q}$, and $m+n, pm$, $qn\in\mathbb{N}_0$. Then*

$$ {}_a^bE_{m+n}\left(\frac{x}{p}+\frac{y}{q}\right)\leq\left({}_a^bE_{pm}(x)\right)^{1/p}\left({}_a^bE_{qn}(y)\right)^{1/q}. $$

Proof. By the Hölder inequality,

$$ \begin{aligned} {}_a^bE_{m+n}\left(\frac{x}{p}+\frac{y}{q}\right)&=\int_a^b t^{-m-n}e^{-t\left(\frac{x}{p}+\frac{y}{q}\right)}dt\\ &=\int_a^b t^{-m}e^{-t\frac{x}{p}}t^{-n}e^{-t\frac{y}{q}}dt\\ &\leq\left(\int_a^b t^{-pm}e^{-xt}dt\right)^{1/p}\left(\int_a^b t^{-qn}e^{-yt}dt\right)^{1/q}\\ &=\left({}_a^bE_{pm}(x)\right)^{1/p}\left({}_a^bE_{qn}(y)\right)^{1/q}. \qquad\square \end{aligned} $$

Theorem 2.3. *Assume that $1<a<b$, $x,y>1$, $n\in\mathbb{N}_0$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $x+y\leq xy$. Then*

$$ {}_a^bE_n(xy)\leq\left({}_a^bE_n(px)\right)^{1/p}\left({}_a^bE_n(qy)\right)^{1/q}. $$

Proof. By Lemma 2.1, we obtain that ${}_a^bE_n$ is non-increasing. It follows that

$$ \begin{aligned} {}_a^bE_n(xy)&\leq{}_a^b E_n(x+y)\\ &=\int_a^b t^{-n}e^{-t(x+y)}dt\\ &=\int_a^b t^{-\frac{n}{p}}e^{-xt}t^{-\frac{n}{q}}e^{-yt}dt. \end{aligned} $$

By the Hölder inequality,

$$ {}_a^bE_n(xy) \leq \left(\int_a^b t^{-n} e^{-pxt} dt \right)^{1/p} \left(\int_a^b t^{-n} e^{-qyt} dt \right)^{1/q} $$

$$ = \left({}_a^bE_n(px) \right)^{1/p} \left({}_a^bE_n(qy) \right)^{1/q} . \qquad \square $$

Theorem 2.4. *Assume that* $1 < a < b$, $x > 0$, $0 < y < 1$, $n \in \mathbb{N}_0$, $0 < p < 1 = \frac{1}{p} + \frac{1}{q}$ *and* $x + y \geq xy$. *Then*

$$ {}_a^bE_n(xy) \geq \left({}_a^bE_n(px) \right)^{1/p} \left({}_a^bE_n(qy) \right)^{1/q} . $$

Proof. By Lemma 2.1, we obtain that ${}_a^bE_n$ is non-increasing. It follows that

$$ {}_a^bE_n(xy) \geq_a^b E_n(x + y) $$

$$ = \int_a^b t^{-n} e^{-t(x+y)} dt $$

$$ = \int_a^b t^{-\frac{n}{p}} e^{-xt} t^{-\frac{n}{q}} e^{-yt} dt . $$

By the reverse Hölder inequality,

$$ {}_a^bE_n(xy) \geq \left(b_a t^{-n} e^{-pxt} dt \right)^{1/p} \left(b_a t^{-n} e^{-qyt} dt \right)^{1/q} $$

$$ = \left({}_a^bE_n(px) \right)^{1/p} \left({}_a^bE_n(qy) \right)^{1/q} . \qquad \square $$

Theorem 2.5. *Assume that* $1 < a < b$, $x, y > 1$, $n \in \mathbb{N}_0$, $p > 1$, $0 < r < 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{r} + \frac{1}{s}$. *Then*

$$ {}_a^bE_n(xy) \geq \left({}_a^bE_n \left(\frac{rx^p}{p} \right) \right)^{1/r} \left({}_a^bE_n \left(\frac{sy^q}{q} \right) \right)^{1/s} . $$

Proof. We note that

$$ xy \leq \frac{x^p}{p} + \frac{y^q}{q}. $$

By Lemma 2.1, we obtain that ${}_a^bE_n$ is non-increasing. It follows that

$$ {}_a^bE_n(xy) \geq_a^b E_n \left(\frac{x^p}{p} + \frac{y^q}{q} \right) $$

$$ = \int_a^b t^{-n} e^{-t\left(\frac{x^p}{p} + \frac{y^q}{q} \right)} dt $$

$$ = \int_a^b t^{-\frac{n}{r}} e^{-\frac{x^p}{p} t} t^{-\frac{n}{s}} e^{-\frac{y^q}{q} t} dt . $$

By the reverse Hölder inequality,

$$ {}_a^bE_m(xy) \geq \left(\int_a^b t^{-m} e^{-\frac{rx^p t}{p}}\,dt\right)^{1/r} \left(\int_a^b t^{-m} e^{-\frac{sy^q t}{q}}\,dt\right)^{1/s} $$

$$ = \left({}_a^bE_n\left(\frac{rx^p}{p}\right)\right)^{1/r} \left({}_a^bE_n\left(\frac{sy^q}{q}\right)\right)^{1/s}. \qquad \square $$

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Received December 4, 2012
Accepted December 23, 2012