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Research Article

On Generalized Fibonacci Hybrinomials

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Abstract. In this paper, we define generalized Fibonacci hybrinomials, which are generalization of both Fibonacci type hybrinomials and Lucas type hybrinomials. We introduce these hybrinomials in two types as generalized Fibonacci type hybrinomials and generalized Lucas type hybrinomials. We obtain matrix representations, Binet formulas, generating functions and some properties of the generalized Fibonacci hybrinomials. Moreover, we give relationship between the generalized Fibonacci type hybrinomials and generalized Lucas type hybrinomials.

Keywords. Fibonacci polynomial, Lucas polynomial, Fibonacci hybrinomial, Lucas hybrinomial

Mathematics Subject Classification (2020). 11B37, 11B39, 11B83

1. Introduction

Fibonacci sequence, which is a sequence of integers, is the most popular second order sequence with fascinating application and theory in various areas of modern science. The Fibonacci sequence, $\{F_n\}$ is defined by recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial values $F_0 = 0$, $F_1 = 1$ for $n \geq 2$ and there are many studies on this sequence and its generalizations [1–27]. Using the recurrence relation of Fibonacci sequence and different initial conditions, new number sequences can be created as Fibonacci type sequences. For instance, Lucas sequence, $\{L_n\}$ is defined by recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial values $L_0 = 2$, $L_1 = 1$ for $n \geq 2$. There is no unique generalization of this sequence. Falcon and Plaza introduced the k-Fibonacci sequences and k-Lucas sequences by recurrence relations $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ with $F_{k,0} = 0$,

$F_{k,1}=1$ and $L_{k,n}=kL_{k,n-1}+L_{k,n-2}$ with $L_{k,0}=2$, $L_{k,1}=k$ for $n\geq 2$, $k\geq 1$, respectively [2,3].

Large classes of polynomials are emerged as the well-known generalizations of Fibonacci sequences. Such polynomials, called the Fibonacci polynomials, are defined by

$$f_0(x)=0 \text{ and } f_1(x)=1,\ f_n(x)=xf_{n-1}(x)+f_{n-2}(x),\quad n\geq 2.$$

Using the recurrence relation of Fibonacci polynomials and different initial conditions, Lucas polynomials, are defined by

$$l_0(x)=2 \text{ and } l_1(x)=x,\ l_n(x)=xl_{n-1}(x)+l_{n-2}(x),\quad n\geq 2.$$

The Fibonacci polynomials have been studied by many authors and are called generalized Fibonacci polynomials. For instance, bivariate Fibonacci polynomials and bivariate Lucas polynomials are defined as

$$F_n(x,y)=xF_{n-1}(x,y)+yF_{n-2}(x,y),\ F_n(x,y)=0,\ F_1(x,y)=1,\quad n\geq 2$$

and

$$L_n(x,y)=xL_{n-1}(x,y)+yL_{n-2}(x,y),\ L_0(x,y)=2,\ L_1(x,y)=x,\quad n\geq 2,$$

$x,y\neq 0$ and $x^2+4y\neq 0$, respectively. Generalized identities of these polynomials are obtained [15]. In [12], the authors defined $h(x)$-Fibonacci polynomials as a generalization of Fibonacci polynomials and $h(x)$-Lucas polynomials as a generalization of Lucas polynomials by recurrence relations

$$F_{h,n}(x)=h(x)F_{h,n-1}(x)+F_{h,n-2}(x),\ F_{h,n}(x)=0,\ F_{h,1}(x)=1,\quad n\geq 2$$

and

$$L_{h,n}(x)=h(x)L_{h,n-1}(x)+L_{h,n-2}(x),\ L_{h,0}(x)=2,\ L_{h,1}(x)=h(x),\quad n\geq 2,$$

where $h(x)$ is a polynomial with real coefficients, respectively. Several authors presented generating functions, exponential generating functions, Binet-like formulas, sums formulas, matrix representations, periods according to the m modulo of Fibonacci polynomial sequences and obtained many generalizations of these sequences [4–7, 14]. Koshy introduced one of the most comprehensive sources contains the applications, generalizations and recurrence relations of Fibonacci and Lucas sequences [9]. As can be seen from the studies in the literature, F_n and L_n are very closely related and hence generalized Fibonacci numbers and polynomials are studied as generalized Fibonacci type and generalized Lucas type numbers and polynomials, respectively.

Motivated by of the above-cited studies, it is introduced a new generalization of the Fibonacci type and Lucas type numbers and polynomials called generalized Fibonacci polynomials. The generalized Fibonacci polynomial sequence, $\{G_n(x)\}_{n\geq 0}$ is defined by the recurrence relation

$$G_0(x)=p_0(x), G_1(x)=p_1(x),\ G_n(x)=d(x)G_{n-1}(x)+g(x)G_{n-2}(x),\quad n\geq 2, \tag{1.1}$$

where $p_0(x)$ is a constant and $p_1(x)$, $d(x)$ and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ with $\gcd(d(x),g(x))=1$ [4,5]. If eq. (1.1) satisfies the following recurrence relation,

$$F_0(x)=0, F_1(x)=1,\ \ F_n(x)=d(x)F_{n-1}(x)+g(x)F_{n-2}(x),\quad n\geq 2, \tag{1.2}$$

where $d(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$, it is called Fibonacci type polynomials, denoted by $G_n(x) = F_n(x)$. Obviously, for $d(x) = x$ and $g(x) = 1$ we obtain classical Fibonacci polynomial and $F_n(1) = F_n$ where F_n is the nth classical Fibonacci number. If eq. (1.1) satisfies the following recurrence relation,

$$L_0(x) = p_0,\ L_1(x) = p_1(x),\ L_n(x) = d(x)L_{n-1}(x) + g(x)L_{n-2}(x), \quad n \geq 2, \tag{1.3}$$

where $|p_0| = 1$ or 2 and $p_1(x), d(x) = \alpha p_1(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ with α an integer of the form $2/p_0$, it is called Lucas type polynomials, denoted by $G_n(x) = L_n(x)$. Also, for $p_0 = 2, p_1(x) = x, d(x) = x$ and $g(x) = 1$ we obtain classical Lucas polynomial and $L_n(1) = L_n$ where L_n is the nth classical Lucas number.

Now, we present the Binet formulas for the generalized Fibonacci polynomials. For $d^2(x) + 4g(x) > 0$, the explicit formula for the recurrence relation in (1.1) is given by

$$G_n(x) = A\sigma^n(x) + B\rho^n(x), \quad n \geq 0, \tag{1.4}$$

where $\sigma(x)$ and $\rho(x)$ are the roots of the quadratic equation $t^2 - d(x)t - g(x) = 0$ of eq. (1.1) with

$$A = \frac{p_1(x) - p_0(x)\rho(x)}{\sigma(x) - \rho(x)} \quad \text{and} \quad B = \frac{-p_1(x) + p_0(x)\sigma(x)}{\sigma(x) - \rho(x)}.$$

So if A and B are used in eq. (1.4), we obtain the Binet formulas of the generalized Fibonacci polynomials by

$$G_n(x) = \frac{p_1(x)(\sigma^n(x) - \rho^n(x)) + p_0(x)g(x)(\sigma^{n-1}(x) - \rho^{n-1}(x))}{\sigma(x) - \rho(x)}. \tag{1.5}$$

If eq. (1.5) is used for recurrence relations (1.2) and (1.3) with $\alpha = 2/p_0$, the Binet formulas for the Fibonacci type and Lucas type polynomials are given by

$$F_n(x) = \frac{\sigma^n(x) - \rho^n(x)}{\sigma(x) - \rho(x)} \tag{1.6}$$

and

$$L_n(x) = \frac{\sigma^n(x) + \rho^n(x)}{\alpha}, \tag{1.7}$$

respectively. Note that roots $\sigma(x)$ and $\rho(x)$ hold

$$\sigma(x) + \rho(x) = d(x),$$

$$\sigma(x)\rho(x) = -g(x),$$

$$\sigma(x) - \rho(x) = \sqrt{d^2(x) + 4g(x)}.$$

where $d(x)$ and $g(x)$ are the polynomials defined for recurrence relations (1.2) and (1.3). The readers can find more detailed information about the generalized Fibonacci polynomials in [4, 5].

Özdemir [13] introduced the set of hybrid numbers denoted by $\mathbb{K}$ which a new generalization complex, dual and hyperbolic numbers. Hybrid numbers, which have applications in different fields of mathematics, are a new number system that is not commutative. The set of hybrid numbers is defined as

$$\mathbb{K} = \{a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\}.$$

For $z_1 = a+bi+c\varepsilon+dh$ and $z_2 = x+yi+z\varepsilon+th$ are defined as

Equality:	$z_1 = z_2$ only if $a = x$, $b = y$, $c = z$, $d = t$
Addition and subtraction:	$z_1 \pm z_2 = (a\pm x)+(b\pm y)i+(c\pm z)\varepsilon+(d\pm t)h$
Multiplication:	$z_1z_2 = ax-by+dt+bz+cy+i(ay+bx+bt-dy)$ $+(az+cx+bt-dy+dz-ct)\varepsilon+(at+dx+cy-bz)h$

The multiplication of a hybrid number $z = a+bi+c\varepsilon+dh$ by the real scalar k is defined as

$$kz = ka+kbi+kc\varepsilon+kdh.$$

Multiplication of hybrid numbers is associative and not commutative. Addition in the hybrid numbers is both associative and commutative. Zero $0 = 0+0i+0\varepsilon+0h$ is the null element. The additive inverse of z, hybrid number is $-z = -a-bi-c\varepsilon-dh$. This implies that, $(\mathbb{K};+)$ is an Abelian group. The readers can find more detailed information about the hybrid numbers in [13]. The multiplication table of the basis of hybrid numbers are as follows:

Table 1. The multiplication of hybrid units of $\mathbb{K}$

$\cdot$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon+i$
ε	ε	$1+h$	0	$-\varepsilon$
h	h	$-\varepsilon-i$	ε	1

Many studies on new type of hybrid numbers by using special integer sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Tribonacci and Tribonacci-Lucas numbers, etc have been introduced, called hybrid sequences in recent years. For instance, in [21–24], the authors defined the Fibonacci hybrid numbers, the Pell and Pell-Lucas hybrid numbers and the Jacosthal and Jacosthal-Lucas hybrid numbers, Horadam hybrid numbers, generalized their results and obtained various properties of these numbers. Polatlı obtained the divisibility properties of the Fibonacci and Lucas hybrid numbers [17]. Tasyurdu [25] and Yağmur [27] introduced the Tribonacci and Tribonacci-Lucas hybrid numbers and expressed many various properties of these hybrid numbers. Bilgici introduced unrestricted Gibonacci hybrid numbers by using Gibonacci sequence is a generalization of Fibonacci sequence [1].

On the other hand, new generalizations of the hybrid numbers by using special polynomials sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Tribonacci and Narayana polynomials, etc. have been introduced, called hybrinomial sequences. In [11, 19, 20], the authors introduced the Pell hybrinomials, the Fibonacci and Lucas hybrinomials and generalized Fibonacci-Pell hybrinomials, respectively. In particular, Kızılateş generalized their results and introduced the Horadam hybrid polynomials called Horadam hybrinomials and some special cases of these hybrinomials [8]. Taşyurdu and Polat defined Tribonacci and Tribonacci-Lucas hybrinomials and derived these hybrinomials by the matrices [26]. In [10, 16], the authors introduced the Narayana hybrinomials, the Jacobsthal representation hybrinomials, respectively.

The aim of this study is to introduce a new generalization of the Fibonacci type and Lucas type hybrid numbers and polynomials called generalized Fibonacci hybrinomials. It is also to give special cases of generalized Fibonacci hybrinomials by generalizing all the results for Fibonacci type hybrid polynomials and Lucas type hybrid polynomials.

2. The Generalized Fibonacci and Lucas Hybrinomials

In this section, we define the generalized Fibonacci hybrinomials, also called generalized Fibonacci type hybrinomials and generalized Lucas type hybrinomials, and derive these hybrinomials by the matrices. Moreover, we obtain Binet formulas and generating functions of these hybrinomials.

Definition 1. The nth generalized Fibonacci hybrinomial, $G_nH(x)$ is defined by

$$G_nH(x) = G_n(x) + iG_{n+1}(x) + \varepsilon G_{n+2}(x) + hG_{n+3}(x), \tag{2.1}$$

where $G_n(x)$ is the nth generalized Fibonacci polynomial and hybrid units i,ε,h satisfy the equations $i^2 = -1$, $\varepsilon^2 = 0$, $h^2 = 1$, $ih = -hi = \varepsilon + i$.

Definition 1 is the general model of the various generalizations of the hybrid numbers introduced by using special polynomials sequences. In the following table we state special cases of generalized Fibonacci hybrinomials according to initial conditions $G_0(x) = p_0(x)$, $G_1(x) = p_1(x)$ and the related parameters $d(x)$, $g(x)$ in given eq. (1.1).

Table 2. Special cases of the sequences $\{G_nH(x)\}_{n\geq 0}$

Generalized Fibonacci hybrinomials	$G_nH(x)$	$G_nH(x)(x:p_0(x),p_1(x);d(x),g(x))$
Fibonacci hybrinomials [20]	$F_nH(x)$	$F_nH(x)(x:0,1;x,1)$
Fibonacci hybrinomials with two variables	$\mathfrak{f}_nH(x)$	$\mathfrak{f}_nH(x)(x:0,1;x,y)$
$h(x)$-Fibonacci hybrinomials	$F_{h,n}H(x)$	$F_{h,n}H(x)(x:0,1;h(x),1)$
Lucas hybrinomials [20]	$L_nH(x)$	$L_nH(x)(x:2,x;x,1)$
Lucas hybrinomials with two variables [18]	$l_nH(x)$	$l_nH(x)(x:2,x;x,y)$
$h(x)$-Lucas hybrinomials	$L_{h,n}H(x)$	$L_{h,n}H(x)(x:2,h(x);h(x),1)$
Pell hybrinomials [11]	$P_nH(x)$	$P_nH(x)(x:0,1;2x,1)$
Pell-Lucas hybrinomials [11]	$Q_nH(x)$	$Q_nH(x)(x:2,2x;2x,1)$
Pell-Lucas-prime hybrinomials	$Q'_nH(x)$	$Q'_nH(x)\ (x:1,x;2x,1)$
Fermat hybrinomials	$\Phi_nH(x)$	$\Phi_nH(x)(x:0,1;3x,-2)$
Fermat-Lucas hybrinomials	$\vartheta_nH(x)$	$\vartheta_nH(x)(x:2,3x;3x,-2)$
Chebyshev first kind hybrinomials	$T_nH(x)$	$T_nH(x)(x:1,x;2x,-1)$
Chebyshev second kind hybrinomials	$U_nH(x)$	$U_nH(x)(x:0,1;2x,-1)$
Jacobsthal hybrinomials [10]	$J_nH(x)$	$J_nH(x)(x:0,1;1,2x)$
Jacobsthal-Lucas hybrinomials [10]	$j_nH(x)$	$j_nH(x)(x:2,1;1,2x)$
Morgan-Voyce first kind hybrinomials	$B_nH(x)$	$B_nH(x)(x:0,1;x+2,-1)$
Morgan-Voyce second kind hybrinomials	$C_nH(x)$	$C_nH(x)(x:2,x+2;x+2,-1)$
Vieta hybrinomials	$V_nH(x)$	$V_nH(x)(x:0,1;x,-1)$
Vieta-Lucas hybrinomials	$v_nH(x)$	$v_nH(x)\ (x:2,x;x,-1)$
Horadam hybrinomials [8]	$H_nH(x)$	$H_nH(x)(x:a,bx;px,q)$

We present the generalized Fibonacci hybrinomials in two types as the generalized Fibonacci type hybrinomials $\{GF_nH(x)\}_{n\geq 0}$ and the generalized Lucas type hybrinomials $\{GL_nH(x)\}_{n\geq 0}$. Fibonacci type hybrinomials are Fibonacci hybrinomials $F_nH(x)$, Fibonacci hybrinomials with two variables $f_nH(x)$, $h(x)$-Fibonacci hybrinomials $F_{h,n}H(x)$, Pell hybrinomials $P_nH(x)$, Fermat hybrinomials $\Phi_nH(x)$, Chebyshev second kind hybrinomials $U_nH(x)$, Jacobsthal hybrinomials $J_nH(x)$, and Morgan-Voyce first kind hybrinomials $B_nH(x)$, Vieta hybrinomials $V_nH(x)$. Lucas type hybrinomials are Lucas hybrinomials $L_nH(x)$, Lucas hybrinomials with two variables $l_nH(x)$, $h(x)$-Lucas hybrinomials $L_{h,n}H(x)$, Pell-Lucas hybrinomials $Q_nH(x)$, Pell-Lucas-prime hybrinomials $Q'_nH(x)$, Fermat-Lucas hybrinomials $\vartheta_nH(x)$, Chebyshev first kind hybrinomials $T_nH(x)$, Jacobsthal-Lucas hybrinomials $j_nH(x)$ and Morgan-Voyce second type hybrinomials $C_nH(x)$, Vieta-Lucas hybrinomials $v_nH(x)$. Since the all results given throughout the study are provided for all generalized Fibonacci type hybrinomial sequences and generalized Lucas type hybrinomial sequences, the values given in Table 2 can be used in the theorem or corollary of any hybrinomial sequences which are the generalized Fibonacci type hybrinomial sequences or generalized Lucas type hybrinomial sequences.

Note that if we take $p_0(x)$, $p_1(x)$, $d(x)$ and $g(x)$ as the values in Table 2, Definition 1 gives the recurrence relations for the generalized Fibonacci type hybrinomial sequences and the generalized Lucas type hybrinomial sequences.

Definition 2. The nth generalized Fibonacci type hybrinomial, $GF_nH(x)$ and generalized Lucas type hybrinomial, $GL_nH(x)$ are defined by

$$GF_nH(x) = F_n(x) + iF_{n+1}(x) + \varepsilon F_{n+2}(x) + hF_{n+3}(x), \tag{2.2}$$

$$GL_nH(x) = L_n(x) + iL_{n+1}(x) + \varepsilon L_{n+2}(x) + hL_{n+3}(x), \tag{2.3}$$

where $F_n(x)$ and $L_n(x)$ are the nth generalized Fibonacci type polynomial and the nth generalized Lucas type polynomial, respectively. Here hybrid units i, ε, h satisfy the equations $i^2 = -1$, $\varepsilon^2 = 0$, $h^2 = 1$, $ih = -hi = \varepsilon + i$.

Using eq. (2.1) we can give the following recurrence relations of the generalized Fibonacci hybrinomial sequences, $\{G_nH(x)\}_{n\geq 0}$ which are the generalized Fibonacci type hybrinomial sequences, $\{GF_nH(x)\}_{n\geq 0}$ and generalized Lucas type hybrinomial sequences $\{GL_nH(x)\}_{n\geq 0}$.

Theorem 1. *For $n \geq 2$, the following recurrence relation for the generalized Fibonacci hybrinomial sequences, $\{G_nH(x)\}_{n\geq 0}$ holds*

$$G_nH(x) = d(x)G_{n-1}H(x) + g(x)G_{n-2}H(x) \tag{2.4}$$

with

$$G_0H(x) = p_0(x) + ip_1(x) + \varepsilon(d(x)p_1(x) + g(x)p_0(x)) + h(d^2(x)p_1(x) + d(x)g(x)p_0(x) + g(x)p_1(x)),$$

$$\begin{aligned}G_1H(x) = {} & p_1(x) + i(d(x)p_1(x) + g(x)p_0(x)) + \varepsilon(d^2(x)p_1(x) + d(x)g(x)p_0(x) + g(x)p_1(x)) \\ & + h(d^3(x)p_1(x) + d^2(x)g(x)p_0(x) + 2d(x)g(x)p_1(x) + g^2(x)p_0(x)).\end{aligned}$$

Proof. Using the eqs. (1.1) and (2.1), we obtain

$$d(x)G_{n-1}H(x) + g(x)G_{n-2}(x)H(x)$$

$$= d(x)(G_{n-1}(x)+iG_n(x)+\varepsilon G_{n+1}(x)+hG_{n+2}(x))+g(x)(G_{n-2}(x)+iG_{n-1}(x)+\varepsilon G_n(x)+hG_{n+1}(x))$$
$$= d(x)G_{n-1}(x)+g(x)G_{n-2}(x)+i(d(x)G_n(x)+g(x)G_{n-1}(x))$$
$$= \varepsilon(d(x)G_{n+1}(x)+g(x)G_n(x))+h(d(x)G_{n+2}(x)+g(x)G_{n+1}(x))$$
$$= G_n(x)+iG_{n+1}(x)+\varepsilon G_{n+2}(x)+hG_{n+3}(x)$$
$$= G_nH(x)$$

which ends the proof. □

Note that if we take $p_0(x)$, $p_1(x)$, $d(x)$ and $g(x)$ as the values in Table 2, Theorem 1 gives the recurrence relations for the generalized Fibonacci type hybrinomial sequences and the generalized Lucas type hybrinomial sequences.

Corollary 1. *For $n \geq 2$, the following recurrence relations for the generalized Fibonacci type hybrinomial sequences and generalized Lucas type hybrinomial sequences hold*

$$GF_nH(x) = d(x)GF_{n-1}H(x)+g(x)GF_{n-2}(x)H(x),$$
$$GL_nH(x) = d(x)GL_{n-1}H(x)+g(x)GL_{n-2}(x)H(x)$$

with

$$GF_0(x) = i+\varepsilon d(x)+h(d^2(x)+g(x)),$$
$$GF_1(x) = 1+id(x)+\varepsilon(d^2(x)+g(x))+h(d^3(x)+2d(x)g(x)),$$

where $d(x)$ and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ and

$$GL_0(x) = p_0+ip_1(x)+\varepsilon(d(x)p_1(x)+g(x)p_0)+h(d^2(x)p_1(x)+d(x)g(x)p_0+g(x)p_1(x)),$$
$$GL_1H(x) = p_1(x)+i(d(x)p_1(x)+g(x)p_0)+\varepsilon(d^2(x)p_1(x)+d(x)g(x)p_0+g(x)p_1(x)) + h(d^3(x)p_1(x)+d^2(x)g(x)p_0+2d(x)g(x)p_1(x)+g^2(x)p_0),$$

where $|p_0| = 1$ or 2 and $p_1(x), d(x) = \alpha p_1(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ with α an integer of the $2/p_0$, respectively.

Now we derive the matrix representations of the generalized Fibonacci hybrinomials with next theorem

Theorem 2. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} G_{n+2}H(x) & G_{n+1}H(x) \\ G_{n+1}H(x) & G_nH(x) \end{bmatrix} = \begin{bmatrix} G_2H(x) & G_1H(x) \\ G_1H(x) & G_0H(x) \end{bmatrix} \begin{bmatrix} d(x) & 1 \\ g(x) & 0 \end{bmatrix}^n$$

where $G_nH(x)$ is the nth generalized Fibonacci hybrinomial.

Proof. By using induction on n, if $n = 0$, then the result is obvious. Now assume that for any $n \geq 0$ holds

$$\begin{bmatrix} G_{n+2}H(x) & G_{n+1}H(x) \\ G_{n+1}H(x) & G_nH(x) \end{bmatrix} = \begin{bmatrix} G_2H(x) & G_1H(x) \\ G_1H(x) & G_0H(x) \end{bmatrix} \begin{bmatrix} d(x) & 1 \\ g(x) & 0 \end{bmatrix}^n.$$

Then, we need to show that above equality holds for $n+1$. That is,

$$\begin{bmatrix} G_{n+3}H(x) & G_{n+2}H(x) \\ G_{n+2}H(x) & G_{n+1}H(x) \end{bmatrix} = \begin{bmatrix} G_2H(x) & G_1H(x) \\ G_1H(x) & G_0H(x) \end{bmatrix} \begin{bmatrix} d(x) & 1 \\ g(x) & 0 \end{bmatrix}^{n+1}.$$

Then by induction hypothesis we obtain

$$\begin{aligned}
\begin{bmatrix} G_2H(x) & G_1H(x) \\ G_1H(x) & G_0H(x) \end{bmatrix} \begin{bmatrix} d(x) & 1 \\ g(x) & 0 \end{bmatrix}^{n+1} &= \begin{bmatrix} G_2H(x) & G_1H(x) \\ G_1H(x) & G_0H(x) \end{bmatrix} \begin{bmatrix} d(x) & 1 \\ g(x) & 0 \end{bmatrix}^{n} \begin{bmatrix} d(x) & 1 \\ g(x) & 0 \end{bmatrix} \\
&= \begin{bmatrix} G_{n+2}H(x) & G_{n+1}H(x) \\ G_{n+1}H(x) & G_nH(x) \end{bmatrix} \begin{bmatrix} d(x) & 1 \\ g(x) & 0 \end{bmatrix} \\
&= \begin{bmatrix} d(x)G_{n+2}H(x) + g(x)G_{n+1}H(x) & G_{n+2}H(x) \\ d(x)G_{n+1}H(x) + g(x)G_nH(x) & G_{n+1}H(x) \end{bmatrix} \\
&= \begin{bmatrix} G_{n+3}H(x) & G_{n+2}H(x) \\ G_{n+2}H(x) & G_{n+1}H(x) \end{bmatrix},
\end{aligned}$$

which ends the proof. □

Note that if we take $p_0(x)$, $p_1(x)$,$d(x)$ and $g(x)$ as the values in Table 2, Theorem 2 gives the matrix representations for the generalized Fibonacci type hybrinomial sequences and the generalized Lucas type hybrinomial sequences.

Corollary 2. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} GF_{n+2}H(x) & GF_{n+1}H(x) \\ GF_{n+1}H(x) & GF_nH(x) \end{bmatrix} = \begin{bmatrix} GF_2H(x) & GF_1H(x) \\ GF_1H(x) & GF_0H(x) \end{bmatrix} \begin{bmatrix} d(x) & 1 \\ g(x) & 0 \end{bmatrix}^n,$$

$$\begin{bmatrix} GL_{n+2}H(x) & GL_{n+1}H(x) \\ GL_{n+1}H(x) & GL_nH(x) \end{bmatrix} = \begin{bmatrix} GL_2H(x) & GL_1H(x) \\ GL_1H(x) & GL_0H(x) \end{bmatrix} \begin{bmatrix} d(x) & 1 \\ g(x) & 0 \end{bmatrix}^n,$$

where $GF_nH(x)$ and $GL_nH(x)$ are the nth generalized Fibonacci type hybrinomial and the nth generalized Lucas type hybrinomial, respectively.

Binet formulas for the generalized Fibonacci hybrinomial sequences are given in the next theorem.

Theorem 3. *For $n \geq 0$, the Binet formulas for the generalized Fibonacci hybrinomial sequences are given by*

$$G_nH(x) = \frac{p_1(x)(\hat{\sigma}(x)\sigma^n(x) - \hat{\rho}(x)\rho^n(x)) + p_0(x)g(x)(\hat{\sigma}(x)\sigma^{n-1}(x) - \hat{\rho}(x)\rho^{n-1}(x))}{\sigma(x) - \rho(x)},$$

respectively, where $\hat{\sigma}(x) = 1 + i\sigma(x) + \varepsilon\sigma^2(x) + h\sigma^3(x)$, $\hat{\rho}(x) = 1 + i\rho(x) + \varepsilon\rho^2(x) + h\rho^3$ and $\sigma(x)$ and $\rho(x)$ are the roots of the quadratic equation $z^2 - d(x)z - g(x) = 0$.

Proof. By considering the Binet formula for the nth generalized Fibonacci polynomial given in eq. (1.5) and eq. (2.1), we have

$$\begin{aligned}
G_nH(x) &= G_n(x) + iG_{n+1}(x) + \varepsilon G_{n+2}(x) + hG_{n+3}(x) \\
&= \left(\frac{p_1(x)(\sigma^n(x) - \rho^n(x)) + p_0(x)g(x)(\sigma^{n-1}(x) - \rho^{n-1}(x))}{\sigma(x) - \rho(x)} \right) \\
&\quad + i \left(\frac{p_1(x)(\sigma^{n+1}(x) - \rho^{n+1}(x)) + p_0(x)g(x)(\sigma^n(x) - \rho^n(x))}{\sigma(x) - \rho(x)} \right)
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\left(\frac{p_1(x)(\sigma^{n+2}(x)-\rho^{n+2}(x))+p_0(x)g(x)(\sigma^{n+1}(x)-\rho^{n+1}(x))}{\sigma(x)-\rho(x)}\right)\\
&+h\left(\frac{p_1(x)(\sigma^{n+3}(x)-\rho^{n+3}(x))+p_0(x)g(x)(\sigma^{n+2}(x)-\rho^{n+2}(x))}{\sigma(x)-\rho(x)}\right)\\
=&\frac{p_1(x)(1+i\sigma(x)+\varepsilon\sigma^2(x)+h\sigma^3(x))\sigma^n(x)}{\sigma(x)-\rho(x)}-\frac{p_1(x)(1+i\rho(x)+\varepsilon\rho^2(x)+h\rho^3(x))\rho^n(x)}{\sigma(x)-\rho(x)}\\
&+\frac{p_0(x)g(x)(1+i\sigma(x)+\varepsilon\sigma^2(x)+h\sigma^3(x))\sigma^{n-1}(x)}{\sigma(x)-\rho(x)}\\
&-\frac{p_0(x)g(x)(1+i\rho(x)+\varepsilon\rho^2(x)+h\rho^3(x))\rho^{n-1}(x)}{\sigma(x)-\rho(x)}\\
=&\frac{p_1(x)(\hat{\sigma}(x)\sigma^n(x)-\hat{\rho}(x)\rho^n(x))+p_0(x)g(x)(\hat{\sigma}(x)\sigma^{n-1}(x)-\hat{\rho}(x)\rho^{n-1}(x))}{\sigma(x)-\rho(x)},
\end{aligned}
$$

where

$$
\begin{aligned}
&\hat{\sigma}(x)=1+i\sigma(x)+\varepsilon\sigma^2(x)+h\sigma^3(x),\\
&\hat{\rho}(x)=1+i\rho(x)+\varepsilon\rho^2(x)+h\rho^3(x),
\end{aligned}
$$

which ends the proof. □

Note that if we take $p_0(x)$, $p_1(x)$, $d(x)$ and $g(x)$ as the values in Table 2, Theorem 3 gives the Binet formulas for the generalized Fibonacci type hybrinomials and the generalized Lucas type hybrinomials.

Corollary 3. *For $n \geq 0$, the Binet formulas for the generalized Fibonacci type hybrinomials and generalized Lucas type hybrinomials are given by*

$$GF_nH(x)=\frac{\hat{\sigma}(x)\sigma^n(x)-\hat{\rho}(x)\rho^n(x)}{\sigma(x)-\rho(x)}, \tag{2.5}$$

$$GL_nH(x)=\frac{\hat{\sigma}(x)\sigma^n(x)+\hat{\rho}(x)\rho^n(x)}{\alpha}, \tag{2.6}$$

respectively, where $\hat{\sigma}(x)=1+i\sigma(x)+\varepsilon\sigma^2(x)+h\sigma^3(x)$, $\hat{\rho}(x)=1+i\rho(x)+\varepsilon\rho^2(x)+h\rho^3$ and $\sigma(x)$ and $\rho(x)$ are the roots of the quadratic equation $z^2-d(x)z-g(x)=0$.

Now, we give the generating functions for the generalized Fibonacci hybrinomial sequences the with following theorem.

Theorem 4. *The generating functions for the generalized Fibonacci hybrinomial sequences are*

$$g(t)=\sum_{n=0}^{\infty}G_nH(x)t^n=\frac{G_0H(x)+G_1H(x)t-d(x)G_0H(x)t}{1-d(x)t-g(x)t^2}.$$

Proof. Let $g(t)=\sum_{n=0}^{\infty}G_nH(x)t^n$ be the generating functions for the generalized Fibonacci hybrinomial sequences. Then

$$g(t)=\sum_{n=0}^{\infty}G_nH(x)t^n$$

$$
\begin{aligned}
&= G_0H(x) + G_1H(x)t + \sum_{n=2}^{\infty} G_nH(x)t^n \\
&= G_0H(x) + G_1H(x)t + \sum_{n=2}^{\infty} (d(x)G_{n-1}H(x) + g(x)G_{n-2}H(x))t^n \\
&= G_0H(x) + G_1H(x)t + d(x)t\sum_{n=0}^{\infty} G_nH(x)t^n - d(x)G_0H(x)t + g(x)t^2\sum_{n=0}^{\infty} G_n(x)Ht^n \\
&= G_0H(x) + G_1H(x)t + d(x)tg(t) - d(x)G_0H(x)t + g(x)t^2g(t)
\end{aligned}
$$

and we obtain that

$$(1 - d(x)t - g(x)t^2)g(t) = G_0H(x) + G_1H(x)t - d(x)G_0H(x)t.$$

So the generating functions for the generalized Fibonacci hybrinomial sequences are

$$g(t) = \frac{G_0H(x) + G_1H(x)t - d(x)G_0H(x)t}{1 - d(x)t - g(x)t^2}. \qquad \square$$

Note that if we take $p_0(x)$, $p_1(x)$, $d(x)$ and $g(x)$ as the values in Table 2, Theorem 4 gives the generating functions for the generalized Fibonacci type hybrinomials and the generalized Lucas type hybrinomials.

Corollary 4. *The generating functions for the generalized Fibonacci type hybrinomials and generalized Lucas type hybrinomials are given by*

$$f(t) = \sum_{n=0}^{\infty} GF_nH(x)t^n = \frac{GF_0H(x) + GF_1H(x)t - d(x)GF_0H(x)t}{1 - d(x)t - g(x)t^2},$$

$$l(t) = \sum_{n=0}^{\infty} GL_nH(x)t^n = \frac{GL_0H(x) + GL_1H(x)t - d(x)GL_0H(x)t}{1 - d(x)t - g(x)t^2},$$

respectively.

3. Identities of the Generalized Fibonacci and Lucas Hybrinomials

In this section, we present interesting properties such as Catalan's identity, Cassini's identity, d'Ocagne's identity, Honsberger's identity for the generalized Fibonacci type hybrinomials and generalized Lucas type hybrinomials. Also, some relations between the generalized Fibonacci hybrinomials are given.

Now, we give the Catalan's identity for the generalized Fibonacci type hybrinomials and generalized Lucas type hybrinomials. Note that the Cassini's identity is a special case of the Catalan's identity and we only prove the Catalan's identity.

Theorem 5 (Catalan's Identity)**.** *For $0 \le r \le n$, we have*

(i) $GF_{n+r}H(x)GF_{n-r}H(x) - (GF_nH(x))^2 = \frac{(-g(x))^n[\hat{\sigma}(x)\hat{\rho}(x)(1-\sigma^r(x)\rho^{-r}(x)) + \hat{\rho}(x)\hat{\sigma}(x)(1-\rho^r(x)\sigma^{-r}(x))]}{(\sigma(x)-\rho(x))^2}$,

(ii) $GL_{n+r}H(x)GL_{n-r}H(x) - (GL_nH(x))^2 = \frac{(-g(x))^n[\hat{\sigma}(x)\hat{\rho}(x)(\sigma^r(x)\rho^{-r}(x)-1) + \hat{\rho}(x)\hat{\sigma}(x)(\rho^r(x)\sigma^{-r}(x)-1)]}{\alpha^2}$.

Proof. (i) By using the Binet formulas of the generalized Fibonacci type hybrinomials given in eq. (2.5), we have

$$\begin{aligned}
&GF_{n+r}H(x)GF_{n-r}H(x)-(GF_nH(x))^2\\
&\quad=\frac{(\hat{\sigma}(x)\sigma^{n+r}(x)-\hat{\rho}(x)\rho^{n+r}(x))(\hat{\sigma}(x)\sigma^{n-r}(x)-\hat{\rho}(x)\rho^{n-r}(x))}{(\sigma(x)-\rho(x))^2}\\
&\qquad-\frac{(\hat{\sigma}(x)\sigma^{n}(x)-\hat{\rho}(x)\rho^{n}(x))(\hat{\sigma}(x)\sigma^{n}(x)-\hat{\rho}(x)\rho^{n}(x))}{(\sigma(x)-\rho(x))^2}\\
&\quad=\frac{\hat{\sigma}(x)\hat{\rho}(x)\sigma^n(x)\rho^n(x)(1-\sigma^r(x)\rho^{-r}(x))+\hat{\rho}(x)\hat{\sigma}(x)\rho^n(x)\sigma^n(x)(1-\rho^r(x)\sigma^{-r}(x))}{(\sigma(x)-\rho(x))^2}\\
&\quad=\frac{(-g(x))^n[\hat{\sigma}(x)\hat{\rho}(x)(1-\sigma^r(x)\rho^{-r}(x))+\hat{\rho}(x)\hat{\sigma}(x)(1-\rho^r(x)\sigma^{-r}(x))]}{(\sigma(x)-\rho(x))^2}.
\end{aligned}$$

(ii) As the way used in (i), we get Catalan's identity for the generalized Lucas type hybrinomials by using the Binet formulas of the generalized Lucas type hybrinomials given in eq. (2.6). □

Note that if we take $r=1$ in Theorem 5, we obtain the Cassini's identity for the generalized Fibonacci type hybrinomials and generalized Lucas type hybrinomials. So we can write following corollary.

Corollary 5 (Cassini's Identity)**.** *For* $1\le n$*, we have*

(i) $GF_{n+1}H(x)GF_{n-1}H(x)-(GF_nH(x))^2=\frac{(-g(x))^n[\hat{\sigma}(x)\hat{\rho}(x)(1-\sigma(x)\rho^{-1}(x))+\hat{\rho}(x)\hat{\sigma}(x)(1-\rho(x)\sigma^{-1}(x))]}{(\sigma(x)-\rho(x))^2}$,

(ii) $GL_{n+1}H(x)GL_{n-1}H(x)-(GL_nH(x))^2=\frac{(-g(x))^n[\hat{\sigma}(x)\hat{\rho}(x)(\sigma(x)\rho^{-1}(x)-1)+\hat{\rho}(x)\hat{\rho}(x)(\rho(x)\sigma^{-1}(x)-1)]}{\alpha^2}$.

Theorem 6 (Honsberger's Identity)**.** *For* $n,m\ge 0$*, we have*

(i) $GF_nH(x)GF_mH(x)+GF_{n+1}H(x)GF_{m+1}H(x)$

$$=\frac{\hat{\sigma}^2(x)\sigma^{n+m}(x)(1+\sigma^2(x))+(g(x)-1)(\hat{\sigma}(x)\hat{\rho}(x)\sigma^n(x)\rho^m(x)+\hat{\rho}(x)\hat{\sigma}(x)\rho^n(x)\sigma^m(x))+(\hat{\rho}^2(x)\rho^{n+m}(1+\rho^2(x))}{(\sigma(x)-\rho(x))^2},$$

(ii) $GL_nH(x)GL_mH(x)+GL_{n+1}H(x)GL_{m+1}H(x)$

$$=\frac{\hat{\sigma}^2(x)\sigma^{n+m}(x)(1+\sigma^2(x))+(1-g(x))(\hat{\sigma}(x)\hat{\rho}(x)\sigma^n(x)\rho^m(x)+\hat{\rho}(x)\hat{\sigma}(x)\rho^n(x)\sigma^m(x))+(\hat{\rho}^2(x)\rho^{n+m}(1+\rho^2(x))}{\alpha^2}.$$

Proof. (i) By using the Binet formulas of the generalized Fibonacci type hybrinomials given in eq. (2.5), we have

$$\begin{aligned}
&GF_nH(x)GF_mH(x)+GF_{n+1}H(x)GF_{m+1}H(x)\\
&\quad=\frac{(\hat{\sigma}(x)\sigma^{n}(x)-\hat{\rho}(x)\rho^{n}(x))(\hat{\sigma}(x)\sigma^{m}(x)-\hat{\rho}(x)\rho^{m}(x))}{(\sigma(x)-\rho(x))^2}\\
&\qquad+\frac{(\hat{\sigma}(x)\sigma^{n+1}(x)-\hat{\rho}(x)\rho^{n+1}(x))(\hat{\sigma}(x)\sigma^{m+1}(x)-\hat{\rho}(x)\rho^{m+1}(x))}{(\sigma(x)-\rho(x))^2}\\
&\quad=\frac{\hat{\sigma}^2(x)\sigma^{n+m}(x)(1+\sigma^2(x))-\hat{\sigma}(x)\hat{\rho}(x)\sigma^n(x)\rho^m(x)(1+\sigma(x)\rho(x))}{(\sigma(x)-\rho(x))^2}
\end{aligned}$$

$$-\frac{\hat{\rho}(x)\hat{\sigma}(x)\rho^n(x)\sigma^m(x)(1+\rho(x)\sigma(x))-\rho^2(x)\rho^{n+m}(1+\rho^2(x))}{(\sigma(x)-\rho(x))^2}$$
$$=\frac{\hat{\sigma}^2(x)\sigma^{n+m}(x)(1+\sigma^2(x))+(g(x)-1)(\hat{\sigma}(x)\hat{\rho}(x)\sigma^n(x)\rho^m(x)+\hat{\rho}(x)\hat{\sigma}(x)\rho^n(x)\sigma^m(x))}{(\sigma(x)-\rho(x))^2}$$
$$+\frac{(\hat{\rho}^2(x)\rho^{n+m}(1+\rho^2(x))}{(\sigma(x)-\rho(x))^2}.$$

(ii) As the way used in (i), we get Honsberger's identity for the generalized Lucas type hybrinomials by using the Binet formulas of the generalized Lucas type hybrinomials given in eq. (2.6). □

Theorem 7 (d'Ocagne's Identity). *For $n \le m$, we have*

(i) $GF_mH(x)GF_{n+1}H(x)-GF_{m+1}H(x)GF_nH(x)=(-g(x))^n\left[\frac{\hat{\sigma}(x)\hat{\rho}(x)\sigma^{m-n}(x)-\hat{\rho}(x)\hat{\alpha}(x)\rho^{m-n}(x)}{\sigma(x)-\rho(x)}\right]$,

(ii) $GL_mH(x)GL_{n+1}H(x)-GL_{m+1}H(x)GL_nH(x)$
$=(-g(x))^n(\sigma(x)-\rho(x))\left[\frac{-\hat{\sigma}(x)\hat{\rho}(x)\sigma^{m-n}(x)+\hat{\rho}(x)\hat{\sigma}(x)\rho^{m-n}(x)}{\alpha^2}\right]$.

Proof. (i) By using the Binet formulas of the generalized Fibonacci type hybrinomials given in eq. (2.5), we have

$$GF_mH(x)GF_{n+1}H(x)-GF_{m+1}H(x)GF_nH(x)$$
$$=\frac{(\hat{\sigma}(x)\sigma^m(x)-\hat{\rho}(x)\rho^m(x))(\hat{\sigma}(x)\sigma^{n+1}(x)-\hat{\rho}(x)\rho^{n+1}(x))}{(\sigma(x)-\rho(x))^2}$$
$$-\frac{(\hat{\sigma}(x)\sigma^{m+1}(x)-\hat{\rho}(x)\rho^{m+1}(x))(\hat{\sigma}(x)\sigma^n(x)-\hat{\rho}(x)\rho^n(x))}{(\sigma(x)-\rho(x))^2}$$
$$=\frac{\hat{\sigma}(x)\hat{\rho}(x)\sigma^m(x)\rho^n(x)(-\rho(x)+\sigma(x))-\hat{\rho}(x)\hat{\sigma}(x)\rho^m(x)\sigma^n(x)(\sigma(x)-\rho(x))}{(\sigma(x)-\rho(x))^2}$$
$$=\frac{\hat{\sigma}(x)\hat{\rho}(x)\sigma^m(x)\rho^n(x)-\hat{\rho}(x)\hat{\sigma}(x)\rho^m(x)\sigma^n(x)}{\sigma(x)-b(x)}$$
$$=(-g(x))^n\left[\frac{\hat{\sigma}(x)\hat{\rho}(x)\sigma^{m-n}(x)-\hat{\rho}(x)\hat{\sigma}(x)\rho^{m-n}(x)}{\sigma(x)-\rho(x)}\right].$$

(ii) As the way used in (i), we get d'Ocagne's identity for the generalized Lucas type hybrinomials by using the Binet formulas of the generalized Lucas type hybrinomials given in eq. (2.6). □

Other relations for the sequences $\{GF_nH(x)\}_{n\ge0}$ and $\{GL_nH(x)\}_{n\ge0}$ are given by the following theorem.

Theorem 8. *Let $\{GF_nH(x)\}_{n\ge0}$ and $\{GL_nH(x)\}_{n\ge0}$ be equivalent generalized Fibonacci hybrinomial sequences. If m and n are positive integers, then*

(i) $g(x)GF_{n-1}H(x)+GF_{n+1}H(x)=\alpha GL_nH(x)$,

(ii) $(\sigma(x)-\rho(x))^2GF_n^2H(x)+\sigma^2GL_n^2H(x)=2(\hat{\sigma}^2(x)\sigma^{2n}(x)+\hat{\rho}^2(x)\rho^{2n}(x))$,

(iii) $(\sigma(x)-\rho(x))^2GF_n^2H(x)-\sigma^2(x)GL_n^2H(x)=2(-g(x))^n(-\hat{\sigma}(x)\hat{\rho}(x)-\hat{\rho}(x)\hat{\sigma}(x))$,

(iv) $GF_{n+2}H(x)+g^2GF_{n-2}H(x)=(\sigma^2(x)+\rho^2(x))GF_nH(x)$,

(v) $\alpha(GL_mH(x)GF_nH(x)+GF_mH(x)GL_nH(x))=\frac{2\hat{\sigma}(x)^2\sigma^{n+m}(x)-2\hat{\rho}^2(x)\rho^{n+m}(x)}{\sigma(x)-\rho(x)}$.

Proof. By using the Binet formulas for the Fibonacci type hybrinomials and Lucas type hybrinomials given in eqs. (2.5) and (2.6), we get

(i) $g(x)GF_{n-1}H(x)+GF_{n+1}H(x)$

$$\begin{aligned}
&= g(x)\left(\frac{\hat{a}(x)\sigma^{n-1}(x)-\hat{\rho}\ (x)\rho^{n-1}(x)}{\sigma(x)-\rho(x)}\right)+\left(\frac{\hat{\sigma}(x)\sigma^{n+1}(x)-\hat{\rho}(x)\ \rho^{n+1}(x)}{\sigma(x)-\rho(x)}\right)\\
&= \frac{\hat{\sigma}(x)\sigma^{n}(x)(g(x)\sigma^{-1}(x)+\sigma(x))+\hat{\rho}(x)\rho^{n}(x)(-g(x)\rho^{-1}(x)-\rho(x))}{\sigma(x)-\rho(x)}\\
&= \hat{\sigma}(x)\sigma^{n}(x)+\hat{\rho}(x)\rho^{n}(x)\\
&= \alpha GL_nH(x),
\end{aligned}$$

(ii) $(\sigma(x)-\rho(x))^2GF_n^2H(x)+\sigma^2GL_n^2H(x)$

$$\begin{aligned}
&= (\sigma(x)-\rho(x))^2\left(\frac{\hat{\sigma}(x)\sigma^{n}(x)-\hat{\rho}\ (x)\rho^{n}(x)}{\sigma(x)-\rho(x)}\right)^2+\alpha^2\left(\frac{\hat{\sigma}(x)\sigma^{n}(x)+\hat{\rho}(x)\rho^{n}(x)}{\alpha}\right)^2\\
&= (\sigma(x)-\rho(x))^2\left(\frac{\hat{\sigma}^2(x)\sigma^{2n}(x)-\hat{\sigma}(x)\hat{\rho}(x)\sigma^{n}(x)\rho^{n}(x)-\hat{\rho}(x)\hat{\sigma}(x)\rho^{n}(x)\sigma^{n}(x)+\hat{\rho}^2(x)\rho^{2n}(x)}{(\sigma(x)-\rho(x))^2}\right)\\
&\quad+\sigma^2\left(\frac{\hat{\sigma}^2(x)\sigma^{2n}(x)+\hat{\sigma}(x)\hat{\rho}(x)\sigma^{n}(x)\rho^{n}(x)+\hat{\rho}(x)\hat{\sigma}(x)\rho^{n}(x)\sigma^{n}(x)+\hat{\rho}^2(x)\rho^{2n}(x)}{\alpha^2}\right)\\
&= 2(\hat{\sigma}^2(x)\sigma^{2n}(x)+\hat{\rho}^2(x)\rho^{2n}(x)),
\end{aligned}$$

(iii) $(\sigma(x)-\rho(x))^2GF_n^2H(x)-\sigma^2GL_n^2H(x)$

$$\begin{aligned}
&= (\sigma(x)-\rho(x))^2\left(\frac{\hat{\sigma}(x)\sigma^{n}(x)-\hat{\rho}(x)\rho^{n}(x)}{\sigma(x)-\rho(x)}\right)^2-\sigma^2\left(\frac{\hat{\sigma}(x)\sigma^{n}(x)+\hat{\rho}(x)\rho^{n}(x)}{\alpha}\right)^2\\
&= (\sigma(x)-\rho(x))^2\left(\frac{\hat{\sigma}^2(x)\sigma^{2n}(x)-\hat{\sigma}(x)\hat{\rho}(x)\sigma^{n}(x)\rho^{n}(x)-\hat{\rho}(x)\hat{\sigma}(x)\rho^{n}(x)\sigma^{n}(x)+\hat{\rho}^2(x)\rho^{2n}(x)}{\left(\sigma(x)-\rho(x)\right)^2}\right)\\
&\quad-\sigma^2\left(\frac{\hat{\sigma}^2(x)\sigma^{2n}(x)+\hat{\sigma}(x)\hat{\rho}(x)\sigma^{n}(x)\rho^{n}(x)+\hat{\rho}(x)\hat{\sigma}(x)\rho^{n}(x)\sigma^{n}(x)+\hat{\rho}^2(x)\rho^{2n}(x)}{\alpha^2}\right)\\
&= 2(-g(x))^n(-\hat{\sigma}(x)\hat{\rho}(x)-\hat{\rho}(x)\hat{\sigma}(x)),
\end{aligned}$$

(iv) $GF_{n+2}H(x)+g^2GF_{n-2}H(x)$

$$\begin{aligned}
&= \frac{(\hat{\sigma}(x)\sigma^{n+2}(x)-\hat{\rho}(x)\rho^{n+2}(x))+g^2(x)(\hat{\sigma}\ (x)\sigma^{n-2}(x)-\hat{\rho}(x)\rho^{n-2}(x))}{\sigma(x)-\rho(x)}\\
&= \frac{\hat{\sigma}(x)\sigma^{n}(x)(\sigma^2(x)+g^2\sigma^{-2}(x))-\hat{\rho}(x)\rho^{n}(x)(\rho^2(x)+g^2\rho^{-2}(x))}{\sigma(x)-\rho(x)}\\
&= \frac{(\hat{\sigma}(x)\sigma^{n}(x)-\hat{\rho}(x)\rho^{n}(x))(\sigma^2(x)+\rho^2(x))}{\sigma(x)-\rho(x)}\\
&= (\sigma^2(x)+\rho^2(x))GF_nH(x),
\end{aligned}$$

(v) $\alpha(GL_mH(x)GF_nH(x)+GF_mH(x)GL_nH(x))$

$$= \alpha\left[\left(\frac{\hat{\sigma}(x)\sigma^m(x)+\hat{\rho}(x)\rho^m(x)}{\alpha}\right)\left(\frac{\hat{\sigma}(x)\sigma^n(x)-\hat{\rho}(x)\rho^n(x)}{\sigma(x)-\rho(x)}\right)\right.$$

$$\left.+\left(\frac{\hat{\sigma}(x)\sigma^m(x)-\hat{\rho}(x)\rho^m(x)}{\sigma(x)-\rho(x)}\right)\left(\frac{\hat{\sigma}(x)\ \sigma^n(x)+\hat{\rho}(x)\rho^n(x)}{\alpha}\right)\right]$$

$$= \frac{2\hat{\sigma}^2(x)\sigma^{n+m}(x)-2\hat{\rho}^2(x)\rho^{n+m}(x)}{\sigma(x)-\rho(x)},$$

where $\sigma(x)\rho(x) = -g(x)$. So proof is completed. □

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Competing Interests

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Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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