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On the n-th Derivative of the Exponential Integral Functions

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Abstract In this paper, we present inequalities involving the n-th derivative of the exponential integral functions.

1. Introduction

For any $n \in \mathbb{N}_0$, the exponential integral function E_n (see [1]) is defined by

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt$$

for all $x > 0$.

For any $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, the derivative of E_n, denoted by $E_n^{(k)}$, is given by

$$E_n^{(k)}(x) = (-1)^k \int_1^\infty t^{k-n} e^{-xt} dt$$

for all $x > 0$.

In 2012, Sulaiman [2] presented inequalities involving the n-th derivative of the exponential integral functions as follows.

For any $x, y > 0$, $p > 1 = \frac{1}{p} + \frac{1}{q}$, $m+n, pm, qn \in \mathbb{N}_0$, and k is an even integer such that $k > m+n$,

$$E_{m+n}^{(k)}\left(\frac{x}{p} + \frac{y}{q}\right) \le \left(E_{pm}^{(k)}(x)\right)^{1/p} \left(E_{qn}^{(k)}(y)\right)^{1/q}. \tag{1.1}$$

For any $x > 0$, $0 < y \le 1$, $n \in \mathbb{N}_0$, $p > 1$, $0 < r < 1$ and $\frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{r} + \frac{1}{s}$, and k is an even integer such that $k > n$,

$$E_n^{(k)}(xy) \ge \left(E_n^{(k)}\left(\frac{rx^p}{p}\right)\right)^{1/r} \left(E_n^{(k)}\left(\frac{sy^q}{q}\right)\right)^{1/s}. \tag{1.2}$$

In this paper, we present the generalizations for inequalities (1.1) and (1.2).

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2. Results

Theorem 2.1. *Assume that* $n-1 \in \mathbb{N}$*,* $x_1, x_2, \ldots, x_n > 0$*,* $p_1, p_2, \ldots, p_n > 1$*,* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$*,* $m = \sum_{i=1}^{n} m_i$*,* $p_1 m_1, p_2 m_2, \ldots, p_n m_n$*,* $m \in \mathbb{N}_0$*, and* k *is an even integer such that* $k > m$*. Then*

$$E_m^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right) \le \prod_{i=1}^{n} \left(E_{p_i m_i}^{(k)}(x_i)\right)^{1/p_i}.$$

Proof. We obtain that

$$\begin{aligned}
E_m^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right) &= (-1)^k \int_1^\infty t^{k-m} e^{-t\sum_{i=1}^{n} \frac{x_i}{p_i}} \, dt \\
&= \int_1^\infty t^{k\sum_{i=1}^{n}\frac{1}{p_i} - \sum_{i=1}^{n} m_i} e^{-t\sum_{i=1}^{n} \frac{x_i}{p_i}} \, dt \\
&= \int_1^\infty t^{\sum_{i=1}^{n}\left(\frac{k}{p_i} - m_i\right)} e^{-t\sum_{i=1}^{n} \frac{x_i}{p_i}} \, dt \\
&= \int_1^\infty \prod_{i=1}^{n} t^{\frac{k}{p_i} - m_i} e^{-t\frac{x_i}{p_i}} \, dt.
\end{aligned}$$

By the generalized Hölder inequality,

$$\begin{aligned}
E_m^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right) &\le \prod_{i=1}^{n} \left(\int_1^\infty t^{k-p_i m_i} e^{-x_i t} \, dt\right)^{1/p_i} \\
&= \prod_{i=1}^{n} \left(E_{p_i m_i}^{(k)}(x_i)\right)^{1/p_i}.
\end{aligned}$$

□

Corollary 2.2. *Assume that* $n-1 \in \mathbb{N}$*,* $x_1, x_2, \ldots, x_n > 0$*,* $p_1, p_2, \ldots, p_n > 1$*,* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$*,* $m \in \mathbb{N}_0$*, and* k *is an even integer such that* $k > m$*. Then*

$$E_m^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{p_i}\right) \le \prod_{i=1}^{n} \left(E_m^{(k)}(x_i)\right)^{1/p_i}.$$

Proof. This follows from Theorem 2.1 in case $m_1 = \frac{m}{p_1}, m_2 = \frac{m}{p_2}, \ldots, m_n = \frac{m}{p_n}$. □

Corollary 2.3. *Assume that* $n-1 \in \mathbb{N}$*,* $x_1, x_2, \ldots, x_n > 0$*,* $m \in \mathbb{N}_0$*, and* k *is an even integer such that* $k > m$*. Then*

$$\left(E_m^{(k)}\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)\right)^n \le \prod_{i=1}^{n} \left(E_m^{(k)}(x_i)\right).$$

Proof. This follows from Corollary 2.2 in case $p_1 = p_2 = \ldots = p_n = n$. □

We note on Theorem 2.1 that if $n = 2$ then we obtain the inequality (1.1).

Theorem 2.4. *Assume that $n \in \mathbb{N}$, $x_1, \dots, x_n > 0$, $0 < y \le 1$, $m \in \mathbb{N}_0$, $p_1, \dots, p_n > 1$, $0 < r_1, \dots, r_n < 1$, $\left(\sum_{i=1}^{n} \frac{1}{p_i}\right) + \frac{1}{q} = 1 = \left(\sum_{i=1}^{n} \frac{1}{r_i}\right) + \frac{1}{s}$, and k is an even integer such that $k > m$. Then*

$$E_m^{(k)}\left(y \prod_{i=1}^{n} x_i\right) \ge \left(E_m^{(k)}\left(\frac{sy^q}{q}\right)\right)^{1/s} \left(\prod_{i=1}^{n} E_m^{(k)}\left(\frac{r_i x_i^{p_i}}{p_i}\right)\right)^{1/r_i}.$$

Proof. For any $x > 0$,

$$E_m^{(k+1)}(x) = (-1)^{k+1} \int_1^\infty t^{k+1-m} e^{-xt}\, dt = -\int_1^\infty t^{k+1-m} e^{-xt}\, dt \le 0.$$

Thus, $E_m^{(k)}$ is non-increasing.

We note that

$$y \prod_{i=1}^{n} x_i \le \frac{y^q}{q} + \sum_{i=1}^{n} \frac{x_i^{p_i}}{p_i}.$$

It follows that

$$\begin{aligned}
E_m^{(k)}\left(y \prod_{i=1}^{n} x_i\right) &\ge E_m^{(k)}\left(\frac{y^q}{q} + \sum_{i=1}^{n} \frac{x_i^{p_i}}{p_i}\right) \\
&= (-1)^k \int_1^\infty t^{k-m} e^{-t\left(\frac{y^q}{q} + \sum_{i=1}^{n} \frac{x_i^{p_i}}{p_i}\right)}\, dt \\
&= \int_1^\infty t^{k-m} e^{-\frac{y^q}{q}t + \sum_{i=1}^{n}\left(-\frac{x_i^{p_i}}{p_i}t\right)}\, dt \\
&= \int_1^\infty t^{\frac{k-m}{s}} e^{-\frac{y^q}{q}t} \prod_{i=1}^{n} t^{\frac{k-m}{r_i}} e^{-\frac{x_i^{p_i}}{p_i}t}\, dt.
\end{aligned}$$

By the generalized reverse Hölder inequality,

$$\begin{aligned}
E_m\left(y \prod_{i=1}^{n} x_i\right) &\ge \left(\int_1^\infty t^{k-m} e^{\frac{-sy^q t}{q}}\, dt\right)^{\frac{1}{s}} \prod_{i=1}^{n} \left(\int_1^\infty t^{k-m} e^{\frac{-r_i x_i^{p_i} t}{p_i}}\, dt\right)^{\frac{1}{r_i}} \\
&= \left(E_m^{(k)}\left(\frac{sy^q}{q}\right)\right)^{1/s} \left(\prod_{i=1}^{n} E_m^{(k)}\left(\frac{r_i x_i^{p_i}}{p_i}\right)\right)^{1/r_i}.
\end{aligned}$$

□

Corollary 2.5. *Assume that $n \in \mathbb{N}$, $x_1, \dots, x_n > 0$, $m \in \mathbb{N}_0$, and k is an even integer such that $k > m$. Then*

$$E_m^{(k)}\left(\prod_{i=1}^{n} x_i\right) \ge \left[E_m^{(k)}\left(\frac{1}{(n+1)^2}\right) \prod_{i=1}^{n} E_m^{(k)}\left(\frac{x_i^{n+1}}{(n+1)^2}\right)\right]^{n+1}.$$

Proof. This follows from Theorem 2.4 in case $p_1 = p_2 = \dots = p_n = n+1$, $r_1 = r_2 = \dots = r_n = \frac{1}{n+1}$ and $y = 1$. □

We note on Theorem 2.4 that if $n = 1$ then we obtain the inequality (1.2).

References

[1] M. Aabromowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas*, Dover Publications, New York, 1965.

[2] W.T. Sulaiman, Turan inequalities for the exponential integral functions, *Commun. Optim. Theory* **1** (2012), 35–41.

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