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Research Article

On Secure Total Domination Cover Pebbling Number

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Abstract. In this paper, we introduce a new graph invariant called the secure total domination cover pebbling number, a combination of two graph invariants, namely, 'secure total domination' and 'cover pebbling number'. The secure total domination cover pebbling number of a graph G, denoted by $f_{stdp}(G)$, is the minimum number of pebbles that are required to place on $V(G)$, such that after a sequence of pebbling moves, the set of vertices with pebbles forms a total secure dominating set under any configuration of pebbles to the vertices of graph G. The secure total domination cover pebbling number for join of two graphs $G(p,q)$ and $G'(p',q')$ is determined. Also, a generalization of secure total domination cover pebbling number for some families of graphs such as complete graph K_n, complete bipartite graph $K_{p,q}$, complete y-partite graph $K_{p_1,p_2,\ldots,p_y}$ and path P_n is found.

Keywords. Graph pebbling, Secure total domination, Cover pebbling number, Secure total domination cover pebbling number

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1. Introduction

Let $G(V,E)$ be a connected, simple graph. The pebbling number of a graph G, $f(G)$, is the least n such that however n pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves [3]. The objective of this network optimization model is for the transportation of resources that are consumed in the transit. For a survey of additional results refer [8]. Secure total domination cover pebbling number is combination of two graph invariants, namely, cover pebbling and secure total domination.

2. Preliminaries

Definition 2.1 ([9])**.** A set of vertices S in G is a *dominating set* in G, if every vertex in G is either in S or adjacent to some element in S. The minimum number of vertices in the set S is called domination number and is denoted by $\gamma(G)$. A subset D of vertices of G is called a *total dominating set* of G if for every $u \in V$, there exists a vertex $v \in D$ such that $uv \in E(G)$.

Definition 2.2 ([4])**.** A dominating set S in G is called a *secure dominating set* in G denoted by $\gamma_s(G)$, if for every $v \in V(G)\backslash S$, there exists $u \in S \cap N(v)$ such that $(S\backslash\{u\}) \cup \{v\}$ is a dominating set where $N(v) = \{t \in V(G) : vt \in E(G)\}$. The minimum cardinality of a secure dominating set is called the secure domination number of G.

Definition 2.3 ([9])**.** A total dominating set D of a graph G is called a *secure total dominating set* of G if for every $u \in V\backslash D$, there exists a vertex $v \in D$ such that $uv \in E(G)$ and $(D\backslash\{v\}) \cup \{u\}$ is a total dominating set of G. The secure total domination number of G, denoted by $\gamma_{st}(G)$, is the minimum cardinality of a secure total dominating set of G.

Definition 2.4 ([5])**.** *The cover pebbling number* of a graph G denoted by $\lambda(G)$, is the minimum number of pebbles required to place a pebble on every vertex simultaneously under any initial configuration.

Definition 2.5 ([6])**.** *The domination cover pebbling number*, $\psi(G)$, of a graph G is the minimum number of pebbles required such that after a sequence of pebbling moves, the set of vertices with pebbles forms a domination set of G, regardless of the initial configuration of pebbles.

Definition 2.6 ([10])**.** *Secure domination cover pebbling number*, $f_{sdp}(G)$, of a graph G is the minimum number of pebbles that must be placed on $V(G)$, such that after a sequence of pebbling moves the set of vertices with pebbles forms a secure dominating set regardless of the initial configuration.

Definition 2.7 ([2])**.** Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be a connected simple graph. Then $G_1 \cup G_2$ is the graph $G(V,E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ and $G_1 + G_2$ is $G_1 \cup G_2$ together with the edges joining elements of V_1 to elements of V_2.

3. Motivation

The motivation behind this topic is as follows: For instance, consider a graph formed with vertices denoting the states of the country and the edges are drawn if two vertices(states) share a common boundary. Consider the problem of finding the minimum number of guards required to protect the country during an attack in such a way that each unguarded state(vertex) should be adjacent to a guarded state(vertex) as well as each guarded state(vertex) should also be adjacent to a guarded state(vertex). This problem is similar to the secure total domination problem. Consequently, our country will be more safer during an attack.

In graph pebbling, we are finding the minimum number of guards required in order to place a guard in the root vertex. Thus, by appplying the concept of graph pebbling in total secure domination, we can find the minimum number of guards needed to safeguard the country more securely.

Thus, with this motivation, we define *secure total domination cover pebbling number*, $f_{stdp}(G)$, of a graph G as the minimum number of pebbles that are required to place on $V(G)$, such that after a sequence of pebbling moves, the set of vertices with pebbles forms a secure total dominating set regardless of the initial configuration. In this paper, secure total domination number, $f_{stdp}(G)$, of join of two graphs $G(p,q)$ and $G'(p',q')$ is found. Also, a generalization of secure total domination cover pebbling number for complete graph K_n, the complete bipartite graph $K_{p,q}$, complete y-partite graph $K_{p_1,p_2,\ldots,p_y}$ and path P_n is determined.

Result 3.1 ([3])**.** *$f(G) \geq n(G)$, where $n(G)$ is the number of vertices of G.*

Result 3.2 ([3])**.** $f(P_{t+1}) = 2^t$.

4. Main Results

Result 4.1. $\psi(G) \leq f_{sdp}(G) \leq f_{stdp}(G)$.

Proof. The result follows from the relation $\psi(G) \leq \gamma_s(G) \leq \gamma_{st}(G)$. □

Result 4.2. *For a simple connected non-trivial graph, $f_{stdp}(G) \geq 3$.*

Proof. The result follows from the relation $\gamma_{st}(G) \geq 2$. □

Notation. $p(v)$ denotes the number of pebbles placed at the vertex v.

Note 4.1. Let u be the target vertex of G and assume that $uv \in E(G)$. If $p(u) \geq 1$, then there is nothing to prove. Also, if $p(v) \geq 2$, then we can easily move a pebble to the target vertex u by a pebbling move. So, without loss of generality, we assume that $p(u) = 0$ and $p(v) \leq 1$.

Theorem 4.1. *Let G and G' be simple connected graphs which are not complete of order n and n', respectively. Suppose that $\gamma_{st}(G) = 4$, then $f_{stdp}(G + G') = n + n' + 7$.*

Proof. Let a, b and c, d be two arbitrary vertices of $V(G)$ and $V(G')$, respectively. Then $\{a, b, c, d\}$ forms the total secure dominating set [7]. We can place a pebble on each a and b by a pebbling move in any one of the following cases:

(i) Existence of minimum of two vertices in G with atleast 4 pebbles each.
(ii) Existence of a vertex in G with 8 pebbles.
(iii) Existence of a minimum of two vertices in G' with atleast 2 pebbles each.
(iv) Existence of a vertex in G' with atleast 4 pebbles

Similarly, we can place a pebble on both c and d by a sequence of pebbling moves in any one of the following cases:

(i) Existence of a minimum of two vertices in G with atleast 2 pebbles each.
(ii) Existence of a vertex in G with 4 pebbles.
(iii) Existence of a minimum of two vertices in G' with atleast 4 pebbles each.
(iv) Existence of a vertex in G' with atleast 8 pebbles.

So, consider the case where all vertices in $G + G'$ has a single pebble on it. Therefore, by placing 12 pebbles on the exceptional vertex, the non-pebbled vertices in the total secure dominating set are forced to have a pebble by a pebbling move and the result follows. □

Corollary 4.1. *The total secure domination cover pebbling number of a graph G, $f_{stdp}(G) = n+1$ if there exists $a, b \in V(G)$ such that $ab \in E(G)$ and $N(a) = V(G)\backslash\{a\}$ and $N(b) = V(G)\backslash\{b\}$.*

Proof. Let $a, b \in V(G)$ such that $ab \in E(G)$ and $N(a) = V(G)\backslash\{a\}$ and $N(b) = V(G)\backslash\{b\}$. Then $\{a, b\}$ forms the total secure dominating set [7] and the result follows from Theorem 4.1. □

Corollary 4.2. *Let G be a simple connected graph of order m and K_n be the complete graph of order $n \geq 2$. Then $f_{stdp}(G + K_n) = m + n + 3$.*

Corollary 4.3. *Let G and G' be simple connected graphs which are not complete of order n and n', respectively. Then $f_{stdp}(G + G') = n + n' + 3$ if*

(i) $\gamma_{st}(G) = 2$ *(or)*

(ii) $\gamma_{st}(G') = 2$ *(or)*

(iii) $\Delta(G) = n - 1$ *and* $\Delta(G') = n - 1$.

Corollary 4.4. *Let G and G' be simple connected graphs which are not complete of order n and n', respectively. Suppose that $\gamma_{st}(G) = 3$. Then $f_{stdp}(G + G') = n + n' + 8$ if either $\gamma(G) = 2$ or $\gamma(G') = 2$ or $\Delta(G) = n - 1$ or $\Delta(G') = n - 1$.*

Note 4.2. Let G and G' be simple connected graphs which are not complete of order n and n', respectively. Then $n + n' + 5 \leq f_{stdp}(G + G') \leq n + n' + 7$.

Theorem 4.2. *For a complete graph K_n, $f_{stdp}(K_n) = n + 1$, $n \geq 2$.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of K_n, $n \geq 2$. Then $\{u_1, u_2\}$ forms the total secure dominating set [1]. The result is obvious in the following two cases:

(i) If any one of the vertices in K_n has atleast 4 pebbles on it.

(ii) If any two of the vertices in K_n has atleast 2 pebbles each.

Therefore, consider the case where all the vertices except one with a single pebble on it. Then, if we place 4 pebbles on the exceptional vertex, the non-pebbled vertices in the total secure dominating set are forced to have a pebble on it by a sequence of pebbling moves. Thus, $f_{stdp}(K_n) = n + 1$, $n \geq 2$. □

Theorem 4.3. *For a complete bipartite graph $K_{p,q}$,*

$$f_{stdp}(K_{p,q}) = \begin{cases} 4q - 1, & p = 1, \\ q + 6, & p = 2, \\ p + q + 8, & p \geq 3. \end{cases}$$

Proof. Let $P = \{u_1, u_2, \ldots, u_p\}$ and $Q = \{v_1, v_2, \ldots, v_q\}$ be the partition of the complete bipartite graph $K_{p,q}$.

Case 1: $p = 1$

For $p = 1$, $\{u_1, v_1, v_2, \ldots, v_q\}$ forms the total secure dominating set [1]. If we place $4q - 2$ pebbles on any of the vertices in the partition Q of $K_{p,q}$, then it is not possible to place atleast one pebble on each of the vertices in the total secure dominating set by a sequence of pebbling moves. Therefore, $f_{stdp}(K_{p,q}) \geq 4q - 1$, $p = 1$.

The result is obvious if we distribute all the $4q-1$ pebbles on the partition P of $K_{p,q}$, since we need only a maximum of $2q+1$ pebbles.

Imagine the case where x vertices in the partition Q have a single pebble on it. Then, we are left out with $4q-1-x$ pebbles and it is sufficient enough to place a pebble on all the vertices in the target vertex set by a pebbling move.

If we place all the $4q-1$ pebbles on any of the vertices in the partition Q of $K_{p,q}$, we are able to place a pebble on each of the vertices of the total secure dominating set by a pebbling move and the result follows.

Case 2: $p=2$

For $p=2$, $\{u_1,u_2,v_1\}$ forms the total secure dominating set [1]. Placing $q+5$ pebbles on any of the vertices of $K_{p,q}$ does not produce a total secure domination cover solution since we are not able to place atleast one pebble on each of the vertices in the total secure dominating set by a pebbling move. Therefore, $f_{stdp}(K_{p,q}) \geq q+6$, $p=2$.

If we place all the pebbles on the vertices of partition P of $K_{p,q}$, then there is nothing to prove.

Consider the case where all the pebbles are distributed to the vertices of the partition Q of $K_{p,q}$. We can place atleast one pebble on all the vertices in the secure total dominating set by a pebbling move in any one of the following cases:

(i) Existence of a vertex in the partition Q with a minimum of 8 pebbles on it.

(ii) Existence of minimum of four vertices in the partition Q with 2 pebbles each.

(iii) Existence of minimum of two vertices in the partition Q with atleast 4 pebbles each.

(iv) Existence of two vertices in the partition Q such that one vertex with 2 pebbles and other one with 6 pebbles.

So, consider the case where all vertices except one has a single pebble on it. Then by placing all the remaining 8 pebbles on the exceptional vertex we are able to place atleast one pebble on all the vertices in the secure dominating set by a pebbling move.

Consider the case where any one of the vertices in the partition Q has 2 pebbles on it. Then, we are able to place a pebble on either u_1 or u_2 in the total secure dominating set by a pebbling move. Without loss of generality, let the pebbled vertex in the total secure dominating set be u_1. Therefore, now we have the newly obtained target vertex set as $\{u_2,v_1\}$. We can place a pebble on all the vertices of the newly obtained target vertex set by a pebbling move in any one of the following cases:

(i) Existence of a vertex in the partition Q with atleast 6 pebbles on it.

(ii) Existence of minimum of three vertices in the partition Q with atleast 2 pebbles each.

(iii) Existence of two vertices in the partition Q such that one vertex has 2 pebbles and the other one has 4 pebbles.

Therefore, consider the case where all the vertices except one has a single pebble on it. Then, we are left with 7 pebbles and it is sufficient to place atleast one pebble on each of the vertices in the newly obtained target vertex set by a pebbling move.

Consider the case where any two of the vertices in the partition Q have 2 pebbles each. Henceforth, we can place a pebble on both u_1 and u_2 in the total secure dominating set by a pebbling move. Further, the new target vertex set reduces to $\{v_1\}$. We can place a pebble on v_1 by a pebbling move in any one of the following cases:

(i) Existence of a vertex in the partition Q with atleast 4 pebbles on it.

(ii) Existence of minimum of two vertices in the partition Q with atleast 2 pebbles each.

So, consider the case where all the vertices except one has a single pebble on it. Then, we are left with 6 pebbles and these left out pebbles are sufficient enough to place a pebble on v_1 by a pebbling move.

Consider the case where any three of the vertices in the partition Q has 2 pebbles each. Thus, we can place a pebble on each of u_1 and u_2 by a pebbling move. Consequently, we have $q-5$ non-pebbled vertices with $q-1$ pebbles on it. Then, there must exist a vertex in the partition Q of $K_{p,q}$ with atleast 2 pebbles on it. Otherwise, the total number of pebbles distributed in the partition Q is atmost $q-1$ which is a contradiction. Finally, we have a minimum of two vertices in the partition Q with 2 pebbles each. Eventually we can place a pebble on the root vertex v_1 by a sequence of pebbling moves.

Now, consider the case where the vertex v_1 in the total secure dominating set has a single pebble on it. Thus, the new target vertex set becomes $\{u_1, u_2\}$. Now, we are left with $q+5$ pebbles and it is sufficient to place a pebble on each of the non-pebbled vertices in the newly obtained target vertex set. The detailed proof is mentioned in [10].

Let us consider the case where any one of the u_i in the total secure dominating set has one pebble on it. Without loss of generality, assume that the pebbled vertex in the total secure dominating set is u_2. Consequently, the newly obtained target vertex set becomes $\{u_1, v_1\}$. We can place a pebble on any of the vertices in the newly obtained target vertex set by a pebbling move in any one of the following cases:

(i) Existence of a vertex in the partition Q with atleast 6 pebbles.

(ii) Existence of atleast 2 vertices in Q in which one of the vertices has 2 pebbles and other one has 4 pebbles on it.

Hence, consider the case where all the vertices except one has a single pebble on it. Subsequently, we are left with 8 pebbles and it is sufficient to place a pebble on all the non-pebbled vertices in the newly obtained target vertex set.

Consider the case where any one of the u_i, $i = 1, 2$, say, u_1 and v_1 in the total secure dominating set has a single pebble on it. Thus, we have $q-1$ vertices with $q+3$ pebbles. Then by the Pigeonhole principle, there exists a vertex v_i, $i \neq 1$ in the partition Q of $K_{p,q}$ with atleast 2 pebbles and the result follows.

Consider the case where both u_1 and u_2 in the total secure dominating set has a single pebble on it. Consequently, we have $q-1$ vertices with $q+3$ pebbles on it. If there exists a vertex in the partition Q with atleast 4 pebbles on it, then there is nothing to prove. In this, there exists minimum of two vertices with atleast 2 pebbles on it. Otherwise, the total number of pebbles distributed is atmost $q+4$ which is a contradiction.

Case 3: $p \geq 3$

For $p \geq 3$, $\{u_1, u_2, v_1, v_2\}$ forms the secure domination set [1]. We need a minimum of $p+q+8$ pebbles to place a pebble on each of the vertices of the target vertex set by a pebbling move. The detailed proof is discussed in [10]. □

Theorem 4.4. *Let $x_1, x_2, \dots, x_y$ be the number of vertices in the vertex classes $p_1, p_2, \dots p_y$ of the complete y-partite graph $K_{p_1,p_2,\dots,p_y}$. Then,*

$$f_{stdp}(K_{p_1,p_2,\dots,p_y}) = \begin{cases} x_1 + x_2 + \dots + x_y + 1, & p_i = 1 \ \forall\ i, \\ x_3 + x_4 + \dots + x_y + 3, & p_1 = p_2 = 1, \\ x_1 + x_2 + \dots + x_y + 4, & otherwise. \end{cases}$$

Proof. Let $p_1 = \{u_{11}, u_{12}, \dots, u_{1x_1}\}$, $p_2 = \{u_{21}, u_{22}, \dots, u_{2x_2}\}, \dots$, $p_y = \{u_{y1}, u_{y2}, \dots, u_{yx_y}\}$ be the vertices of the complete y-partite graph $K_{p_1,p_2,\dots,p_y}$.

Case 1: $p_i = 1 \ \ \forall \ \ i$

If $p_i = 1\ \forall\ i$, then we have a complete graph with $x_1 + x_2 + \dots + x_y$ vertices. Thus, by Theorem 4.2, we have $f_{stdp}(K_{p_1,p_2,\dots,p_y}) = x_1 + x_2 + \dots + x_y + 1$.

Case 2: $p_1 = p_2 = 1$

For $p_1 = p_2 = 1$, $\{u_{11}, u_{21}\}$ forms the secure total dominating set [1] for $K_{p_1,p_2,\dots,p_y}$. Consider the case where all the vertices except one say, u_{ij}, are occupied by a single pebble. Then, placing 3 pebbles on the exceptional vertex u_{ij} does not produce a secure total domination cover solution. Therefore, $f_{stdp}(K_{p_1,p_2,\dots,p_y}) \geq x_3 + x_4 + \dots + x_y + 3$. We can place a pebble on each of the vertices of the secure total dominating set by a pebbling move in any one of the following cases:

(i) Existence of a vertex with atleast 4 pebbles.

(ii) Existence of minimum of two vertices with 2 pebbles each.

Consider the case where all the vertices except one has a pebble on it. Thus, we are left with 4 pebbles and these remaining 4 pebbles are sufficient to place a pebble on both target vertices u_{11} and u_{21} by a sequence of pebbling moves.

Consider the case where any one of the vertices of $K_{p_1,p_2,\dots,p_y}$ has 2 pebbles on it. Consequently, we can place a pebble on any one of the vertices of the total secure dominating set by a pebbling move. Eventually, we have $x_3 + x_4 + \dots + x_y - 1$ vertices with $x_3 + x_4 + \dots + x_y + 1$ pebbles. Thus, by the Pigeon hole principle there must exist a vertex in $K_{p_1,p_2,\dots,p_y}$ with atleast 2 pebbles and the result follows.

Consider the case where any of the vertices u_{11} or u_{21} has a pebble on it. Without loss of generality, assume the vertex to be u_{11}. Subsequently, we have $x_3 + x_4 + \dots + x_y$ non-pebbled vertices with $x_3 + x_4 + \dots + x_y + 2$ pebbles. Then, there should exist a vertex in $K_{p_1,p_2,\dots,p_y}$ with atleast 2 pebbles on it. Otherwise, the total number of pebbles distributed to the complete y-graph $K_{p_1,p_2,\dots,p_y}$ is atmost $x_3 + x_4 + \dots + x_y + 3$ which is a contradiction. Henceforth we can place a pebble on the target vertex u_{21} by a pebbling move.

Case 3: Otherwise

For all other cases, we see that $\{u_{11}, u_{21}, u_{31}\}$ forms the secure total dominating set for the complete y-partite graph $K_{p_1,p_2,\dots,p_y}$ [1]. We need atleast $x_1 + x_2 + \dots + x_y + 4$ pebbles to place a pebble on each vertices of the secure total dominating set of $K_{p_1,p_2,\dots,p_y}$ by a sequence of pebbling moves. The detailed proof is mentioned in [10]. □

Theorem 4.5. *The secure total domination cover pebbling number of a path P_n, $n \ge 9$ is*

$$f_{stdp}(P_{7t+s}) = \begin{cases} 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+27(2^{7t-5}), & s=0, \\ 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+55(2^{7t-5}), & s=1, \\ 3+440\left(\frac{2^{7t}-1}{2^7-1}\right), & s=2, \\ 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+\sum_{s=3}^{5} 2^{7t+s-1}, & s=3,4,5, \\ 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+11(2^{7t+2}), & s=6, \end{cases}$$

where $t \ge 1$.

Proof. Denote the vertices of the path P_n by $v_1, v_2, \dots, v_n$. Let the path P_{n-2} with vertices $\{v_3, v_4, \dots, v_n\}$ be divided into $\lfloor \frac{n-2}{7} \rfloor$ subpaths P_7^i of order 7 and one(possibly empty) subpath P_x with vertex set $\{y_1, y_2, \dots, y_x\}$ of order $x = n-2(mod 7)$. Denote the vertices of the subpaths P_7^i by $\{w_1^i, w_2^i, \dots w_7^i\}, i = 1, 2, \dots \lfloor \frac{n-2}{7} \rfloor$. Let $X = \{v_1, v_2\}$, $Y = \cup_{i=1}^{\lfloor \frac{n-2}{7} \rfloor} \{w_2^i, w_3^i, w_4^i, w_6^i, w_7^i\}$ and $Z = V(P_x) - \{y_{x-2} : x \ge 4\}$. Then $X \cup Y \cup Z$ forms the secure total domination set [1].

We have $f_{stdp}(P_s) = \lambda(P_s)$ for $s = 1, 2, \dots, 5$, where $\lambda(P_s)$ denotes the cover pebbling number of P_s. Whereas, for $s = 6, 7, 8$, the secure total domination cover pebbling number, $f_{stdp}(P_s)$, is as follows.

$$f_{stdp}(P_s) = \begin{cases} 47, & s=6, \\ 111, & s=7, \\ 223, & s=8 \end{cases}$$

and the result is obvious.

Case 1: $n = 7t$, $t \ge 1$

Consider the configuration of distributing all the pebbles on v_1. Then, a minimum of $1+2+55(2^3+2^{10}+2^{17}+\dots)+2^{7(t-1)+2}+2^{7(t--1)+3}+2^{7(t-1)+5}+2^{7(t-1)+6}$ pebbles are required to place atleast one pebble on all the vertices of the total secure dominating set of P_{7t} by a sequence of pebbling moves. Consequently, under this configuration, we have $f_{stdp}(P_{7t}) \ge 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+27(2^{7t-5})$, $t \ge 1$.

Now, we will prove that $f_{stdp}(P_{7t}) \le 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+27(2^{7t-5})$, $t \ge 1$ by induction on t. The result is obvious for $t = 1$. Let us assume that the result to be true for all P_{7i}, where $1 \le i \le t-1$. Consider the distribution of all $3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+27(2^{7t-5})$ pebbles to P_{7t}, $t \ge 1$. For $t \ge 1$, $\{v_{7(t-1)+1}, v_{7(t-1)+2}, v_{7(t-1)+3}, v_{7(t-1)+4}, v_{7(t-1)+6}, v_{7t}\}$ are the additional vertices of the total secure dominating set of P_{7t} when compared to $P_{7(t-1)}$. But by Result 3.2, we need only a maximum of $111(2^{7(t-1)})$ pebbles under any configuration to place a pebble on each of the vertices in the set $\{v_{7(t-1)+1}, v_{7(t-1)+2}, v_{7(t-1)+3}, v_{7(t-1)+4}, v_{7(t-1)+6}, v_{7t}\}$, $t \ge 1$ in a finite number of pebbling moves. Consequently, we are left with $3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+27(2^{7t-5})-111(2^{7(t-1)}) = 3+440\left(\frac{2^{7(t-1)}-1}{2^7-1}\right)+27(2^{7(t-1)-5}$ pebbles. By hypothesis, the remaining $3+440\left(\frac{2^{7(t-1)}-1}{2^7-1}\right)+27(2^{7(t-1)-5})$ pebbles are sufficient to place a pebble on each of the vertices of the total secure dominating set of $P_{7(t-1)}$ and the result follows by induction.

Case 2: $n = 7t+1$, $t \geq 1$

Consider the situation of distributing all the pebbles on the vertex v_1. Then, we need atleast $1+2+55(2^3+2^{10}+2^{17}+\ldots)+2^{7(t-1)+2}+2^{7(t-1)+3}+2^{7(t-1)+4}+2^{7(t-1)+6}+2^{7t}$ pebbles to place a pebble on each of the vertices of the total secure dominating set of P_{7t+1} by a sequence of pebbling moves. Thus, we have $f_{stdp}(P_{7t+1}) \geq 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+55(2^{7t-5}), t \geq 1$.

We will prove the result $f_{stdp}(P_{7t+1}) \leq 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+55(2^{7t-5})$, $t \geq 1$ by induction on t. The result is true for $t = 1$. Let us assume that the assertion is true for all P_{7i}, where $1 \leq i \leq t-1$. Let us distribute all the $3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+55(2^{7t-5})$ pebbles to P_{7t+1}, $t \geq 1$. Therefore, by Result 3.2, we need a maximum of $111(2^{7t-6})$ pebbles under any configuration to place a pebble on all the extra vertices $\{v_{7t-5}, v_{7t-4}, v_{7t-3}, v_{7t-2}, v_{7t}, v_{7t+1}\}$ of the total secure dominating set of P_{7t+1} when compared to $P_{7(t-1)+1}$ in a finite number of pebbling moves. Eventually, we are left with $3+440\left(\frac{2^{7t}-1}{2^7-1}\right)+55(2^{7t-5})-111(2^{7t-6}) = 3+440\left(\frac{2^{7(t-1)}-1}{2^7-1}\right)+55(2^{7(t-1)-5})$ pebbles. By induction hypothesis, the left out pebbles are sufficient to place atleast one pebble on each of the vertices of the total secure dominating set of $P_{7(t-1)}$ and the result follows by induction.

Case 3: $n = 7t+2$, $t \geq 1$

Consider the case of assigning all the pebbles to the first vertex v_1 of P_{7k+2}. Then, a minimum of $1+2+55(2^3+2^{10}+2^{17}+\ldots)$ pebbles are required to place atleast one pebble on all the vertices of the total secure dominating set by a sequence of pebbling moves. Thus, under this configuration, we have $f_{stdp}(P_{7t+2}) \geq 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)$, $t \geq 1$.

Now, we will prove that $f_{stdp}(P_{7t+2}) \leq 3+440\left(\frac{2^{7t}-1}{2^7-1}\right)$, $t \geq 1$ by induction on t. The result is obvious for $t = 1$. Let us assume that the assertion is true for all P_{7i}, where $1 \leq i \leq t-1$. Consider the distribution of all $3+440\left(\frac{2^{7t}-1}{2^7-1}\right)$ pebbles to P_{7t+2}, $t \geq 1$. By Result 3.2, we need a maximum of $55(2^{7t-4})$ pebbles under any configuration to place a pebble on $\{v_{7t+2}, v_{7t+1}, v_{7t-1}, v_{7t-2}, v_{7t-3}\}$ in a finite number of moves. Consequently, we are left with $3+440\left(\frac{2^{7t}-1}{2^7-1}\right)-55(2^{7t-4}) = 3+440\left(\frac{2^{7(t-1)}-1}{2^7-1}\right)$ pebbles. By hypothesis, the remaining pebbles $3+440\left(\frac{2^{7(t-1)}-1}{2^7-1}\right)$ pebbles are sufficient to place atleast one pebble on each of the vertices of the total secure dominating set of $P_{7(t-1)}$ and the result follows by induction.

Case 4: $n = 7t+3$

For $s = 3$, v_{7t+3} is the only additional vertex in the total secure dominating set of P_{7t+3} when compared to P_{7t+2}. Eventually, by Result 3.2, we need only a maximum of 2^{7t+2} pebbles to place a pebble on v_{7t+2} in a finite number of pebbling moves. Thus, adding 2^{7t+2} pebbles to $f_{stdp}(P_{7t+2})$, we get a secure total domination cover solution for P_{7t+3}, $t \geq 1$.

Case 5: $n = 7t+4$

For $s = 4$, when we compare to Case 3, v_{7t+4} is the only extra vertex in the total secure dominating set of P_{7t+4}. But by Result 3.2, we need only a maximum of 2^{7t+3} pebbles in order to place a pebble on v_{7t+3} by a sequence of pebbling moves. Thus, by adding 2^{7t+3} pebbles to $f_{stdp}(P_{7t+3})$, $t \geq 1$ the result follows.

Case 6: $n = 7t + 5$

For $s = 5$, when we compare to P_{7t+4}, the only additional vertex in the total secure dominating set of P_{7t+5} is v_{7t+5}, $t \geq 1$. Hence, by Result 3.2 we need only a maximum of 2^{7t+4} pebbles to place a pebble on v_{7t+5} by a pebbling move. Thus, by adding 2^{7t+4} pebbles to $f_{stdp}(P_{7t+4})$, we get a secure total domination cover solution for P_{7t+5}, $t \geq 1$.

Case 7: $n = 7t + 6$

For $s = 6$, $\{v_{7t+3}, v_{7t+5}, v_{7t+6}\}$ are the additional vertices in the total secure dominating set of P_{7t+6} when compared to P_{7t+2}. Thus, by Result 3.2 we need only a maximum of $11(2^{7t+2})$ pebbles to place a pebble on each vertices of the set $\{v_{7t+3}, v_{7t+5}, v_{7t+6}\}$ in a finite number of pebbling moves. Therefore, by adding $11(2^{7t+2})$ pebbles to $f_{stdp}(P_{7t+2})$, the result follows for P_{7t+6}, $t \geq 1$. □

5. Conclusion

Secure domination cover pebbling number is a rapidly developing area of research in graph theory. As a consequence, we introduced a new graph invariant namely, "Secure total domination cover pebbling number" which is a combination of two graph invariants, 'secure total domination' and 'cover pebbling number'. In this paper, the secure total domination cover pebbling number for join of two graphs $G(p,q)$ and $G'(p',q')$ are determined. Also, the total secure domination cover pebbling number for some special graphs such as complete graph K_n, complete bipartite graph $K_{m,n}$, complete y-partite graph $K_{p_1,p_2,\dots,p_y}$ and path P_n are found. Finding the secure total domination cover pebbling number for other class of graphs and networks is still open.

Given below are some interesting open problems for secure total domination cover pebbling number:

Problem 1. Find secure total domination cover pebbling number for other graph operations such as union, product etc.

Problem 2. Characterize the class of graphs for which $f_{stdp}(G) = n$.

Problem 3. Characterize the class of graphs for which $f_{sdp}(G) = f_{stdp}(G)$.

Problem 4. Is $f_{stdp}(G) \leq k$?

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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