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Eigenvalues of Geometric Operators Under the List's Flow

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Abstract We consider the evolution equation of eigenvalues of the operator $-\Delta + cS$ under the (normalized) List's flow. As an application, we derive monotonicity formulas for eigenvalues of $-\Delta + \frac{1}{2}S$.

1. Introduction

Let $(M^n, g(t))$ be a compact Riemannian manifold, $g(t)$ be a solution to the following List's flow which was introduced by B. List [4]:

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\mathrm{Ric} + 2\alpha d\varphi \otimes d\varphi\,, \\ \varphi_t = \Delta\varphi\,, \end{cases} \tag{1.1}$$

where $\alpha > 0$ is a constant, $\varphi = \varphi(t)$ is a smooth function on M^n and Δ denotes the Laplacian given by $g(t)$. The motivation to study the system (1.1) stems from its connection to general relativity. Denote by $S_{ij} = R_{ij} - \alpha\varphi_i\varphi_j$ a symmetric two-tensor. Then (1.1) becomes

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} = -2S_{ij}\,, \\ \varphi_t = \Delta\varphi\,. \end{cases} \tag{1.2}$$

Throughout this paper, we will use the Einstein summation convention freely. Let $S = g^{ij}S_{ij} = R - \alpha|\nabla\varphi|^2$ be the trace of the two-tensor S_{ij}. In [5], List proved that the functional

$$E = \int_{M^n} (S + |\nabla f|^2)e^{-f}\, dV_g$$

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is nondecreasing along (1.2). If we define

$$\mu(g)=\inf E(g,f),$$

where the infimum is taken over all smooth function f satisfying

$$\int_{M^n} e^{-f}dV_g=1,$$

then $\mu(g)$ is the lowest eigenvalue of the operator $-\Delta+\frac{1}{4}S$ and the nondecreasing of the functional E implies the nondecreasing of $\mu(g)$. Therefore, studying the eigenvalues of the geometric operator $-\Delta+cS$ is a very powerful tool for the understanding of Riemannian manifolds. For the research of eigenvalues of such geometric operator under the Ricci flow (for example, see [2, 1, 8, 7, 3]).

The rest of this paper is organized as follows: In Section 2, we first derive the evolution of a geometric operator under the general geometric flow. As applications, we obtain the monotonicity formula of eigenvalues of $-\Delta+\frac{1}{2}S$ along the List's flow; in Section 3, we study the evolution of a geometric operator under the normalized geometric flow. We also obtain the monotonicity formula of eigenvalues of $-\Delta+\frac{1}{2}S$ along the normalized List's flow.

2. Evolution Under the List's Flow

We consider the metric evolves by

$$\frac{\partial}{\partial t}g_{ij}=v_{ij}. \tag{2.1}$$

Then $\frac{\partial}{\partial t}g^{ij}=-v^{ij}$ and $\frac{\partial}{\partial t}dV_g=\frac{v}{2}dV_g$, where $v=g^{ij}v_{ij}$ denotes the trace of v_{ij}. Let λ be a eigenvalue of the operator

$$\Big(-\Delta-\frac{c}{2}v\Big)f=\lambda f \tag{2.2}$$

with

$$\int_{M^n} f^2dV_g=1, \tag{2.3}$$

where c is a constant.

Lemma 2.1. *If λ is an eigenvalue of the operator $-\Delta-\frac{c}{2}v$, f is the eigenvalue corresponding to λ, that is,*

$$\Big(-\Delta-\frac{c}{2}v\Big)f=\lambda f$$

and the metric evolves by (2.1), *then we have*

$$\frac{d}{dt}\lambda=\int_{M^n}\Big\{\Big[v^{ij}f_{ij}-\frac{c}{2}\frac{\partial v}{\partial t}f\Big]f+\Big[v^{ij}{}_{,j}f_i-\frac{1}{2}\nabla f\nabla v\Big]f\Big\}dV_g. \tag{2.4}$$

Proof. Since we can define

$$\lambda = \int_{M^n} f\left[-\Delta - \frac{c}{2}v\right] f\, dV_g = \int_{M^n}\left[|\nabla f|^2 - \frac{c}{2}vf^2\right] dV_g\,, \tag{2.5}$$

where f satisfies (2.3). Hence,

$$\lambda' = \int_{M^n}\left[(g^{ij})' f_i f_j + 2\nabla f \nabla f' - \frac{c}{2}v' f^2 - \frac{c}{2}v(f^2)'\right] dV_g + \int_{M^n}\left[|\nabla f|^2 - \frac{c}{2}vf^2\right]\frac{v}{2} dV_g\,. \tag{2.6}$$

Applying

$$\begin{aligned}\int_{M^n} 2\nabla f \nabla f' dV_g &= -\int_{M^n} 2f'\Delta f\, dV_g \\ &= \int_{M^n} 2f'\left(\lambda f + \frac{c}{2}vf\right) \\ &= \lambda\int_{M^n}(f^2)' dV_g + \frac{c}{2}\int_{M^n} v(f^2)' dV_g\end{aligned}$$

and

$$\begin{aligned}\int_{M^n}|\nabla f|^2\frac{v}{2}dV_g &= -\int_{M^n} f\left(\frac{v}{2}\Delta f + \nabla f\nabla\frac{v}{2}\right) dV_g \\ &= \int_{M^n} f\left(\lambda f + \frac{c}{2}vf\right)\frac{v}{2}dV_g - \frac{1}{2}\int_{M^n} f\nabla f\nabla v dV_g \\ &= \frac{\lambda}{2}\int_{M^n} f^2 v dV_g + \frac{c}{4}\int_{M^n} f^2 v^2 dV_g - \frac{1}{2}\int_{M^n} f\nabla f\nabla v dV_g\end{aligned}$$

into (2.6) yields

$$\begin{aligned}\lambda' &= \int_{M^n}\left[-v^{ij}f_i f_j + \lambda(f^2)' - \frac{c}{2}v' f^2 + \frac{\lambda}{2}f^2 v - \frac{1}{2}f\nabla f\nabla v\right] dV_g \\ &= \int_{M^n}\left[-v^{ij}f_i f_j - \frac{c}{2}v' f^2 - \frac{1}{2}f\nabla f\nabla v\right] dV_g \\ &\quad + \lambda\int_{M^n}\left[(f^2)' + \frac{1}{2}f^2 v\right] dV_g \\ &= \int_{M^n}\left[-v^{ij}f_i f_j - \frac{c}{2}v' f^2 - \frac{1}{2}f\nabla f\nabla v\right] dV_g\,,\end{aligned}$$

where the last equality used

$$\int_{M^n}\left[(f^2)' + \frac{1}{2}f^2 v\right] dV_g = 0$$

from (2.3). Using integrating by parts again, we complete the proof of the lemma. □

Lemma 2.2. *As in Lemma 2.1, let $v_{ij}=-2S_{ij}$, then the evolution of the eigenvalue of the operator $-\Delta+cS$ under the List's flow satisfies*:

$$\begin{aligned}\frac{d}{dt}\lambda=&(4c-2)\int_{M^n}S^{ij}f_{ij}f\,dV_g+4c\int_{M^n}S^{ij}f_if_jdV_g\\&+2c\int_{M^n}|S_{ij}|^2f^2dV_g+2c\alpha\int_{M^n}|\Delta\varphi|^2f^2dV_g\\&-(4c-2)\alpha\int_{M^n}(\Delta\varphi)f\nabla\varphi\nabla f\,dV_g.\end{aligned}\tag{2.7}$$

Proof. By the definition of S_{ij} and the contracted Bianchi indentity, we have

$$\begin{aligned}S_{ij,j}&=R_{ij,j}-\alpha(\varphi_i\varphi_j)_{,j}\\&=\frac{1}{2}(S+\alpha|\nabla\varphi|^2)_{,i}-\alpha(\varphi_i\varphi_j)_{,j}\\&=\frac{1}{2}S_{,i}-\alpha(\Delta\varphi)\varphi_i\end{aligned}$$

which shows that

$$\begin{aligned}\frac{1}{2}\Delta S&=S_{ij,ji}+\alpha[(\Delta\varphi)\varphi_i]_{,i}\\&=S_{ij,ji}+\alpha|\Delta\varphi|^2+\alpha\nabla\varphi\nabla\Delta\varphi.\end{aligned}\tag{2.8}$$

On the other hand, under the List's flow (1.2), we have (see [4]) $S_t=\Delta S+2|S_{ij}|^2+2\alpha|\Delta\varphi|^2$. Putting $v=-2S$ into (2.4) gives

$$\begin{aligned}\frac{d}{dt}\lambda=&\int_{M^n}[-2S^{ij}f_{ij}+cf(\Delta S+2|S_{ij}|^2+2\alpha|\Delta\varphi|^2)]f\,dV_g\\&+\int_{M^n}[-2S_{ij,j}f_i+S_if_i]f\,dV_g\\=&\int_{M^n}[-2S^{ij}f_{ij}f+c(\Delta S)f^2+2c|S_{ij}|^2f^2+2c\alpha|\Delta\varphi|^2f^2]dV_g\\&+2\alpha\int_{M^n}(\Delta\varphi)\varphi_if_if\,dV_g.\end{aligned}\tag{2.9}$$

By virtue of (2.8),

$$\begin{aligned}c\int_{M^n}(\Delta S)f^2dV_g&=2c\int_{M^n}[S_{ij,ji}+\alpha(\Delta\varphi)^2+\alpha\nabla\varphi\nabla\Delta\varphi]f^2dV_g\\&=4c\int_{M^n}S_{ij}(f_{ij}f+f_if_j)dV_g-4c\alpha\int_{M^n}(\Delta\varphi)\varphi_if_if\,dV_g.\end{aligned}\tag{2.10}$$

Thus, inserting (2.10) into (2.9) completes the proof of Lemma 2.2. □

From the Lemma 2.2, we obtain the following result for the special geometric flow:

Lemma 2.3. *Let $(M^n, g(t))$, $t \in [0, T)$ be a solution to the List's flow (1.2) with $S_{ij} = \frac{S}{n} g_{ij}$. Then the evolution of the eigenvalue of the operator $-\Delta + cS$ satisfies:*

$$\frac{d}{dt}\lambda = \int_{M^n} \left[\frac{4c^2}{n} S^2 f^2 + \frac{2-4c}{n} \lambda S f^2 + \frac{4c}{n} S |\nabla f|^2 \right] dV_g + \int_{M^n} [2c\alpha |\Delta\varphi|^2 f^2 + \alpha(2-4c)(\Delta\varphi)\nabla f \nabla \varphi f] dV_g. \tag{2.11}$$

Proof. Using (2.8), we obtain from (2.9)

$$\begin{aligned}
\frac{d}{dt}\lambda &= \int_{M^n} [-2S^{ij} f_{ij} f + c(\Delta S) f^2 + 2c|S_{ij}|^2 f^2 + 2c\alpha|\Delta\varphi|^2 f^2] dV_g \\
&\quad + 2\alpha \int_{M^n} (\Delta\varphi)\varphi_i f_i f \, dV_g \\
&= \int_{M^n} \{-2S^{ij} f_{ij} f + 2c[S_{ij,ji} + \alpha|\Delta\varphi|^2 + \alpha \nabla\varphi \nabla\Delta\varphi] f^2 \\
&\quad + 2c|S_{ij}|^2 f^2 + 2c\alpha|\Delta\varphi|^2 f^2 + 2\alpha(\Delta\varphi)\varphi_i f_i f\} dV_g \\
&= \int_{M^n} \Big\{ -2S^{ij} f_{ij} f + 4cS_{ij}(f_{ij} f + f_i f_j) + 2c[\alpha|\Delta\varphi|^2 + \alpha\nabla\varphi\nabla\Delta\varphi] f^2 \\
&\quad + \frac{2c}{n} S^2 f^2 + 2c\alpha|\Delta\varphi|^2 f^2 + 2\alpha(\Delta\varphi)\varphi_i f_i f \Big\} dV_g \\
&= \int_{M^n} \Big\{ \frac{4c-2}{n} S(\Delta f) f + \frac{4c}{n} S|\nabla f|^2 + 2c[\alpha|\Delta\varphi|^2 + \alpha\nabla\varphi\nabla\Delta\varphi] f^2 \\
&\quad + \frac{2c}{n} S^2 f^2 + 2c\alpha|\Delta\varphi|^2 f^2 + 2\alpha(\Delta\varphi)\varphi_i f_i f \Big\} dV_g \\
&= \int_{M^n} \Big\{ \frac{4c-2}{n} S(cSf - \lambda f) f + \frac{4c}{n} S|\nabla f|^2 + 2c[\alpha|\Delta\varphi|^2 + \alpha\nabla\varphi\nabla\Delta\varphi] f^2 \\
&\quad + \frac{2c}{n} S^2 f^2 + 2c\alpha|\Delta\varphi|^2 f^2 + 2\alpha(\Delta\varphi)\varphi_i f_i f \Big\} dV_g \\
&= \int_{M^n} \left[\frac{4c^2}{n} S^2 f^2 + \frac{2-4c}{n} \lambda S f^2 + \frac{4c}{n} S|\nabla f|^2 \right] dV_g \\
&\quad + \int_{M^n} \{2c\alpha\nabla\varphi\nabla\Delta\varphi f^2 + 4c\alpha|\Delta\varphi|^2 f^2 + 2\alpha(\Delta\varphi)\varphi_i f_i f\} dV_g.
\end{aligned} \tag{2.12}$$

By integrating by parts, we obtian

$$\int_{M^n} 2c\alpha\nabla\varphi\nabla\Delta\varphi f^2 dV_g = \int_{M^n} [-2c\alpha|\Delta\varphi|^2 f^2 - 4c\alpha(\Delta\varphi)\varphi_i f_i f]dV_g.$$

Therefore, we obtain (2.11) from (2.12). We complete the proof of Lemma 2.3. □

From the Lemma 2.2, we obtain the following result by letting $c = \frac{1}{2}$ directly:

Theorem 2.1. *Let $(M^n, g(t))$, $t \in [0, T)$ be a solution to the List's flow (1.2). Then the evolution of the eigenvalue of the operator $-\Delta + \frac{1}{2}S$ satisfies:*

$$\frac{d}{dt}\lambda = \int_{M^n} [2S^{ij} f_i f_j + |S_{ij}|^2 f^2 + \alpha|\Delta\varphi|^2 f^2]dV_g. \tag{2.13}$$

Remark 2.1. In particular, taking $\alpha = 2$, Theorem 2.1 becomes the Theorem 1.10 in [6].

On the other hand, the nonnegativity of the operator S_{ij} is preserved by the List's flow, hence, we have the following result from Theorem 2.1:

Corollary 2.2. *If $S_{ij}(g(0)) \geq 0$, then the eigenvalue of the operator $-\Delta + \frac{1}{2}S$ are nondecreasing under the List's flow.*

Let $c = \frac{1}{2}$ in (2.11). We obtain

Theorem 2.3. *Let $(M^n, g(t))$, $t \in [0, T)$ be a solution to the List's flow (1.2) with $S_{ij} = \frac{S}{n}g_{ij}$. Then the evolution of the eigenvalue of the operator $-\Delta + \frac{1}{2}S$ satisfies:*

$$\frac{d}{dt}\lambda = \int_{M^n} [\frac{1}{n}S^2 f^2 + \frac{2}{n}S|\nabla f|^2 + \alpha|\Delta\varphi|^2 f^2]dV_g. \tag{2.14}$$

Hence, the following result is clear:

Corollary 2.4. *Let $(M^n, g(t))$, $t \in [0, T)$ be a solution to the following List's flow (1.2) with $S_{ij} = \frac{S}{n}g_{ij}$ and $S \geq 0$. Then the eigenvalue of the operator $-\Delta + \frac{1}{2}S$ are nondecreasing.*

3. Evolution under the Normalized List's Flow

In this part, we consider the metric evolves by

$$\frac{\partial}{\partial t}g_{ij} = v_{ij} - \frac{r}{n}g_{ij}, \tag{3.1}$$

where $r = \frac{\int_{M^n} v dV_g}{\int_{M^n} dV_g}$ is the average of $v = g^{ij}v_{ij}$. Then $\frac{\partial}{\partial t}g^{ij} = -(v^{ij} - \frac{r}{n}g^{ij})$ and $\frac{\partial}{\partial t}dV_g = \frac{1}{2}(v - r)dV_g$. Clearly, under the normalized geometric flow, the volume of $(M^n, g(t))$ is a constant for all t. Let λ be a eigenvalue of the operator

$$\left(-\Delta - \frac{c}{2}v\right)f = \lambda f \tag{3.2}$$

with $\int_{M^n} f^2 dV_g = 1$. By taking derivative of (3.2), we derive easily that

Lemma 3.1. *If λ is an eigenvalue of the operator $-\Delta - \frac{c}{2}v$, f is the eigenvalue corresponding to λ, that is,*

$$\left(-\Delta - \frac{c}{2}v\right)f = \lambda f$$

and the metric evolves by (3.1), *then we have*

$$\frac{d}{dt}\lambda = \int_{M^n}\left\{\left[v^{ij}f_{ij} - \frac{c}{2}\frac{\partial v}{\partial t}f\right]f + \left[v^{ij}{}_{,j}f_i - \frac{1}{2}\nabla f\nabla v\right]f + \frac{r}{n}|\nabla f|^2\right\}dV_g. \quad (3.3)$$

Lemma 3.2. *As in Lemma 3.1, let $v_{ij} = -2S_{ij}$, then the evolution of the eigenvalue of the operator $-\Delta + cS$ under the normalized List's flow*

$$\begin{cases} \dfrac{\partial}{\partial t}g_{ij} = -2\left(S_{ij} - \dfrac{\tilde{r}}{n}g_{ij}\right), \\ \varphi_t = \Delta\varphi \end{cases} \quad (3.4)$$

satisfies:

$$\begin{aligned} \frac{d}{dt}\lambda = {} & (4c-2)\int_{M^n} S^{ij}f_{ij}f\,dV_g + 4c\int_{M^n} S^{ij}f_if_j dV_g \\ & + 2c\int_{M^n}|S_{ij}|^2 f^2 dV_g + 2c\alpha\int_{M^n}|\Delta\varphi|^2 f^2 dV_g \\ & - (4c-2)\alpha\int_{M^n}(\Delta\varphi)f\nabla\varphi\nabla f\,dV_g - \frac{2}{n}\tilde{r}\lambda, \end{aligned} \quad (3.5)$$

where $\tilde{r} = \frac{\int_{M^n} S dV_g}{\int_{M^n} dV_g}$.

Proof. Under the assumption of the lemma, (3.1) becomes

$$\frac{\partial}{\partial t}g_{ij} = -2\left(S_{ij} - \frac{\tilde{r}}{n}g_{ij}\right). \quad (3.6)$$

It has been shown that (see Lemma 1.4 in [4])

$$\begin{aligned} \frac{\partial}{\partial t}R &= 2\Delta(S-\tilde{r}) + g^{pq}g^{rs}\left[-2\left(S_{qs} - \frac{\tilde{r}}{n}g_{qs}\right)_{,rp} + 2R_{pr}\left(S_{qs} - \frac{\tilde{r}}{n}g_{qs}\right)\right] \\ &= 2\Delta S - 2S_{ij,ji} + 2R_{ij}S_{ij} - \frac{2}{n}\tilde{r}R \\ &= \Delta S + 2\alpha|\Delta\varphi|^2 + 2\alpha\nabla\varphi\nabla\Delta\varphi + 2R_{ij}S_{ij} - \frac{2}{n}\tilde{r}R, \end{aligned} \quad (3.7)$$

where the last equality used (2.8). Hence,

$$\frac{\partial}{\partial t}S = \frac{\partial}{\partial t}R - 2\alpha\left(S^{ij} - \frac{\tilde{r}}{n}g^{ij}\right)\varphi_i\varphi_j - 2\alpha\nabla\varphi\nabla\Delta\varphi$$

$$
\begin{aligned}
&= \frac{\partial}{\partial t}R - 2\alpha S^{ij}\varphi_i\varphi_j + 2\alpha\frac{\tilde{r}}{n}|\nabla\varphi|^2 - 2\alpha\nabla\varphi\nabla\Delta\varphi \\
&= \Delta S + 2|S_{ij}|^2 - \frac{2}{n}\tilde{r}S + 2\alpha|\Delta\varphi|^2. \qquad (3.8)
\end{aligned}
$$

Now, inserting $v_{ij} = -2S_{ij}$, $v = -2S$ and $r = -2\tilde{r}$ into (3.3) gives

$$
\begin{aligned}
\frac{d}{dt}\lambda &= \int_{M^n}\Big\{-2S_{ij}f_{ij}f + cS'f^2 + [-2S^{ij}{}_{,j}f_i + S_if_i]f - \frac{2}{n}\tilde{r}|\nabla f|^2\Big\}dV_g \\
&= \int_{M^n}\Big\{-2S_{ij}f_{ij} + cf\Big(\Delta S + 2|S_{ij}|^2 - \frac{2}{n}\tilde{r}S + 2\alpha|\Delta\varphi|^2\Big)\Big\}f\,dV_g \\
&\quad + \int_{M^n}\Big\{\Big[-2S^{ij}{}_{,j}f_i + S_if_i\Big]f - \frac{2}{n}\tilde{r}|\nabla f|^2\Big\}dV_g. \qquad (3.9)
\end{aligned}
$$

Applying

$$
c\int_{M^n}(\Delta S)f^2dV_g = 4c\int_{M^n}S_{ij}(f_{ij}f + f_if_j)dV_g - 4c\alpha\int_{M^n}(\Delta\varphi)\varphi_if_if\,dV_g
$$

into (3.9) gives

$$
\begin{aligned}
\frac{d}{dt}\lambda &= (4c-2)\int_{M^n}S^{ij}f_{ij}f\,dV_g + 4c\int_{M^n}S^{ij}f_if_jdV_g + 2c\int_{M^n}|S_{ij}|^2f^2dV_g \\
&\quad + 2c\alpha\int_{M^n}|\Delta\varphi|^2f^2dV_g - (4c-2)\alpha\int_{M^n}(\Delta\varphi)f\nabla\varphi\nabla f\,dV_g \\
&\quad - \frac{2}{n}\tilde{r}\int_{M^n}(|\nabla f|^2 + cSf^2)dV_g \\
&= (4c-2)\int_{M^n}S^{ij}f_{ij}f\,dV_g + 4c\int_{M^n}S^{ij}f_if_jdV_g + 2c\int_{M^n}|S_{ij}|^2f^2dV_g \\
&\quad + 2c\alpha\int_{M^n}|\Delta\varphi|^2f^2dV_g - (4c-2)\alpha\int_{M^n}(\Delta\varphi)f\nabla\varphi\nabla f\,dV_g - \frac{2}{n}\tilde{r}\lambda. \qquad (3.10)
\end{aligned}
$$

□

From the Lemma 3.2, we obtain the following result by letting $c = \frac{1}{2}$ directly:

Theorem 3.1. *Let $(M^n, g(t))$, $t \in [0, T)$ be a solution to the normalized List's flow (3.4). Then the evolution of the eigenvalue of the operator $-\Delta + \frac{1}{2}S$ satisfies:*

$$
\frac{d}{dt}\lambda = \int_{M^n}[2S^{ij}f_if_j + |S_{ij}|^2f^2 + \alpha|\Delta\varphi|^2f^2]dV_g - \frac{2}{n}\tilde{r}\lambda. \qquad (3.11)
$$

Corollary 3.2. *Let $(M^n, g(t))$, $t \in [0, T)$ be a solution to the normalized List's flow (3.4). Then the evolution of the eigenvalue of the operator $-\Delta + \frac{1}{2}S$ satisfies:*

$$
\frac{d}{dt}(e^{\frac{2\tilde{r}t}{n}}\lambda) = e^{\frac{2\tilde{r}t}{n}}\int_{M^n}[2S^{ij}f_if_j + |S_{ij}|^2f^2 + \alpha|\Delta\varphi|^2f^2]dV_g. \qquad (3.12)
$$

In particular, if $S_{ij}(g(t)) \geq 0$, *then* $\frac{d}{dt}(e^{\frac{2\bar{r}t}{n}}\lambda) \geq 0$ *and hence* $e^{\frac{2\bar{r}t}{n}}\lambda$ *is nondecreasing.*

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