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Research Article

Bayesian Analysis for Two Parameter Lomax Distribution Under Different Loss Functions

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Abstract. Statistical analysis via Bayesian approach is a common practice applied to draw inference about unknown parameter(s) and reliability characteristics of the probability distribution. The article include Bayesian inference of unknown shape parameter of two-parameter Lomax distribution. An attempt has been made to obtain Bayes estimators by using extension of Jeffrey's prior and Gamma prior under Entropy loss function and Precautionary loss function. Comparison has been made by using mean square error through simulation study with varying sample sizes.

Keywords. Lomax distribution, Informative priors, Entropy loss function, Precautionary loss function, R software

Mathematics Subject Classification (2020). 62C10, 62F15, 62N02

1. Introduction

The Lomax distribution also called "Pareto type II" distribution by Lomax [12], the Lomax model belongs to the family of decreasing failure rate as indicate by Chahkandi and Ganjali [7]. In recent years, assumed opposition of important in the field of life testing because of its uses to fit business failure data. This distribution has found wide applications such as the analysis of the business failure life time data, income and wealth inequality, size of cities, actuarial science, medical and biological sciences, engineering, lifetime and reliability modeling. Furthermore, it has been applied to model data obtained from income and wealth by Harris [11], and Atkinson *et al*. [5] and Hirsch-related statistics by Glänzel [10]. It may describe the lifetime of a decreasing failure rate component as a heavy tailed alternative to the exponential distribution. Lomax distribution was introduced by Lomax. Abd-Elfattah and Alharbey [1] estimated the parameters of Lomax distribution based on generalized probability weighted moment. Nasiri

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and Hosseini [13] performs comparisons of *Maximum Likelihood Estimation* (MLE) based on records and a proper prior distribution to attain a Bayes estimation (both informative and non-informative) based on records under quadratic loss and squared error loss functions. Afaq *et al*. [2] estimates the parameters of Lomax distribution using Jeffery's and extension of Jeffery's prior under Asymmetric loss functions. The graphs of pdf, cdf and survival function are shown in Figures 1, 2 and 3.

1.1 The Model

The cumulative distribution function of Lomax distribution is given by

$$F(x:\theta,\lambda) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\theta}. \tag{1.1}$$

Therefore, the corresponding probability density function is given by

$$f(x:\theta,\lambda) = \frac{\theta}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\theta+1)}, \tag{1.2}$$

where θ and α are shape and scale parameters, respectively.

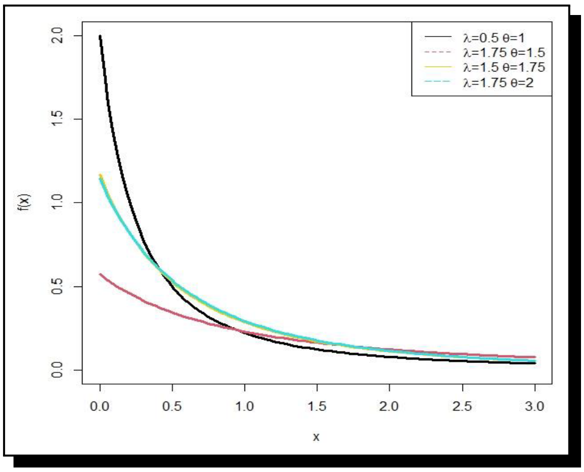

Figure 1

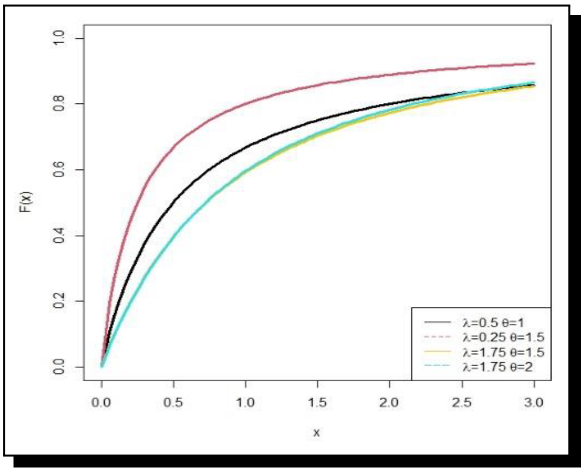

Figure 2

The survival function is given by:

$$R(x:\theta,\lambda) = \left(1 + \frac{x}{\lambda}\right)^{-\theta}. \tag{1.3}$$

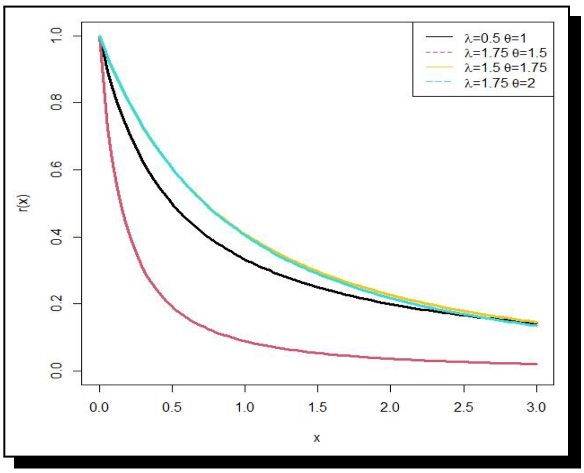

Figure 3

The hazard function is given by

$$h(x:\theta,\lambda)=\frac{\theta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(2\theta+1)}. \tag{1.4}$$

2. Maximum Likelihood Estimation

Let us consider a random sample $\underline{x}=(x_1,x_2,\dots,x_n)$ of size n from Lomax distribution. Then the log-likelihood function for the given sample observation is

$$\log L(\theta,\lambda)=n\log\theta-n\log\lambda-(\theta+1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right). \tag{2.1}$$

As the parameter λ is assumed to be known, the ML estimator ofθ is obtained by solving the equation

$$\frac{\partial\log L(\theta,\lambda)}{\partial\theta}=0,$$

$$\Rightarrow\quad \frac{n}{\theta}-\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)=0,$$

$$\widehat{\theta}_{ML}=\frac{n}{\sum\limits_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)}.$$

3. Material and Method

3.1 Prior and Loss Functions

The Bayesian inference requires appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one's subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use non-informative prior(s). In this paper, we consider both types of priors: the extended Jeffrey's prior and the natural conjugate prior.

The extended Jeffrey's prior proposed by Al-Kutubi [3] is given as

$$g_1(\theta)\propto[I(\theta)]^C,\quad C\in R^+,$$

where $I[(\theta)]=-nE\left[\frac{\partial^2\log f(x:\theta\lambda)}{\partial\theta^2}\right]$ is the Fisher's information matrix. For the model (1.2)

$$g_1(\theta)=\frac{1}{\theta^{2c}}. \tag{3.1}$$

The gamma distribution is used as a conjugate prior for θ with hyper parameters a and b which is also a conjugate prior for the class of distribution, so the prior distribution is

$$g_2(\theta)=\frac{a^b}{\Gamma b}e^{-a\theta}\theta^{b-1},\quad a,b,\theta>0. \tag{3.2}$$

3.2 Loss Function

(a) Entropy Loss Function

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $(\widehat{\theta}/\theta)$. In this case, Calabria and Pulcini [6] point out that a useful asymmetric loss function is the entropy loss function:

$$L(\delta^p)\propto[\delta^p-p\log(\delta)-1].$$

(b) Precautionary Loss Function

The precautionary loss function given by

$$l(\widehat{\theta},\theta)=\frac{(\widehat{\theta}-\theta)^2}{\widehat{\theta}}$$

which is an asymmetric loss function. This loss function is interesting in the sense that a slight modification of squared error loss introduces asymmetry.

4. Bayesian Estimation of Lomax Distribution Under the Extension of Jeffrey's Prior by Using Different Loss Function

Combining the prior distribution in (2.1) and the likelihood function, the posterior density of $\theta$is derived as follows:

$$\pi_1(\theta\mid \underline{x})\propto\left(\frac{\theta}{\lambda}\right)^n\prod_{i=1}^{n}\left(1+\frac{x_i}{\lambda}\right)^2\frac{1}{\theta^{2c}}$$

$$\Rightarrow\quad \pi_1(\theta\mid \underline{x})\Rightarrow\frac{\theta^{n-2c}}{\lambda^n}\exp\left(\left(-(\theta+1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)\right)\right)$$

$$\Rightarrow\quad \pi_1(\theta\mid \underline{x})=k\theta^{n-2c}\exp\left(-\theta\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)\right)$$

where k is independent of θ.

Hence posterior distribution of θ is given by

$$\begin{aligned}\pi_1(\theta\mid \underline{x})&=\left[\frac{\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)}{\Gamma(n-2c+1)}\right]\theta^{n-2c}\exp\left(-\left(\sum_{i=0}^{\infty}1+\frac{x_i}{\lambda}\right)\theta\right)\\&=\frac{t^{n-2c+1}}{\Gamma(n-2c+1)}\theta^{n-2c}e^{-t\theta}\end{aligned}\tag{4.1}$$

where $t=\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)$.

4.1 Estimation Under Entropy Loss Function

By using entropy loss function $L(\delta)\propto[\delta-\log(\delta)-1]$ for some constant b the risk function is given by

$$\begin{aligned}R(\widehat{\theta},\theta)&=\int_0^\infty b(\delta-\log(\delta)-1)\pi_1(\theta\mid \underline{x})d\theta\cdot\int_0^\infty\left(b\frac{\widehat{\theta}}{\theta}-\log\left(\frac{\widehat{\theta}}{\theta}\right)-1\right)\theta^{n-2c}e^{-\theta t}d\theta\\&=b\frac{t^{n-2c+1}}{\Gamma(n-2c+1)}\left[\widehat{\theta}\frac{\Gamma(n-2c)}{t^{n-2c}}-\log(\widehat{\theta})\frac{\Gamma(n-2c+1)}{t^{n-2c+1}}\pm\frac{\Gamma(n-2c+1)}{t^{n-2c+1}}\right]\\&=b\left[\frac{\widehat{\theta}t}{n-2c}-\log(\widehat{\theta})-\frac{\Gamma'(n-2c+1)}{\Gamma(n-2c+1)}-1\right].\end{aligned}$$

Now solving $\frac{\partial R(\widehat{\theta},\theta)}{\partial\widehat{\theta}}=0$, we obtain the Bayes estimator as

$$\widehat{\theta}=\frac{n-2c}{t}.\tag{4.2}$$

We get the Jeffrey's non-informative prior for $c=1.5$ and the Hartigan's non-informative prior for $c=1.75$.

4.2 Estimation Under Precautionary Loss Function

By using precautionary loss function $l(\widehat{\theta},\theta)=\frac{(\widehat{\theta},\theta)^2}{\widehat{\theta}}$ the risk function is given by

$$R(\widehat{\theta})\int_0^\infty \frac{(\widehat{\theta},\theta)^2}{\widehat{\theta}}\frac{t^{n-2c+1}}{\Gamma(n-2c+1)}\theta^{n-2c}\exp(-\theta t)d\theta$$
$$=\frac{t^{n-2c+1}}{\widehat{\theta}\Gamma(n-2c+1)}\int_0^\infty(\widehat{\theta}-\theta)^2\theta^{n-2c}\exp(-\theta t)d\theta$$

Now solving $\frac{\partial R(\widehat{\theta},\theta)}{\partial\widehat{\theta}}=0$, we obtain the Bayes estimator as

$$\widehat{\theta}_p=\frac{[(n-2c+2)(n-2c+1)]^{1/2}}{t}. \tag{4.3}$$

We get the Jeffrey's non-informative prior for $c=1.5$ and the Hartigan's non-informative prior for $c=1.75$.

5. Bayesian Estimation of Lomax Distribution Under Conjugate Prior by Using Different Loss Function

Combining the prior distribution in (3.2) and the likelihood function, the posterior density of θ is derived as follows:

$$\pi_2(\theta\mid\underline{x})=\left(\frac{\theta}{\lambda}\right)^n\prod_{i=1}^n\left(1+\frac{x}{\lambda}\right)^{-(\theta+1)}\frac{a^b}{\Gamma b}e^{-a\theta}\theta^{b-1}$$
$$\pi_2(\theta\mid\underline{x})\propto\frac{\theta^n}{\lambda^n}\exp\left((\theta+1)\sum_{i=1}^n\log\left(1+\frac{x}{n}\right)\right)\frac{a^b}{\Gamma b}e^{-a\theta}\theta^{b-1}$$
$$\Rightarrow\quad \pi_2(\theta\mid\underline{x})=k\theta^{n+b+1}\exp\left(-\left(a+\sum_{i=1}^n 1+\frac{x_i}{\lambda}\right)\right)$$
$$=\left[\frac{a+\sum\limits_{i=1}^n\log\left(1+\frac{x}{\lambda}\right)}{\Gamma(n+b)}\right]\theta^{n+b+1}\exp\left(-\left(a+\sum_{i=0}^\infty 1+\frac{x_i}{\lambda}\right)\theta\right)$$
$$=\frac{t^{n+b}}{\Gamma(n+b)}\theta^{n+b+1}e^{-t\theta} \tag{5.1}$$

which is the probability density function of gamma distribution with parameters $(t,n+b)$.

5.1 Estimation Under Entropy Loss Function

$$R(\widehat{\theta},\theta)=\int_0^\infty b(\delta-\log(\delta)-1)\pi_1(\theta\mid\underline{x})d\theta\int_0^\infty\left(b\frac{\widehat{\theta}}{\theta}-\log\left(\frac{\widehat{\theta}}{\theta}\right)-1\right)\frac{t^{n+b}}{\Gamma(a+b)}\theta^{n+b+1}e^{-\theta t}d\theta$$
$$=b\frac{t^{n+b}}{\Gamma(a+b)}\int_0^\infty\left(\frac{\widehat{\theta}}{\theta}-\log\left(\frac{\widehat{\theta}}{\theta}\right)-1\right)\theta^{n+b+1}e^{-\theta t}d\theta$$
$$=b\frac{t^{n+b}}{\Gamma(a+b)}\left[\widehat{\theta}\frac{\Gamma(n+b+1)}{t^{n+b+1}}-\log(\widehat{\theta})\frac{\Gamma(n+b)}{t^{n+b}}-\frac{\Gamma'(a+b)}{t^{n+b}}-\frac{\Gamma(n+b)}{t^{n+b}}\right]$$
$$=b\left[\frac{\widehat{\theta}t}{t^{n+b+1}}-\log(\widehat{\theta})-\frac{\Gamma'(n+b)}{\Gamma(n+b)}-1\right].$$

Now solving $\frac{\partial R(\widehat{\theta},\theta)}{\partial \widehat{\theta}}=0$, we obtain the Bayes estimator as

$$\widehat{\theta}=\frac{n-+b-1}{t}. \tag{5.2}$$

5.2 Estimation Under Precautionary Loss Function

By using precautionary loss function $l(\widehat{\theta},\theta)=\frac{(\widehat{\theta},\theta)^2}{\widehat{\theta}}$ the risk function is given by

$$\begin{aligned}
R(\widehat{\theta})&\int_0^\infty \frac{(\widehat{\theta},\theta)^2}{\widehat{\theta}}\frac{t^{n+b}}{\Gamma(n+b)}\theta^{n+b-1}\exp(-\theta t)d\theta\\
&=\frac{t^{n+b}}{\widehat{\theta}\Gamma(n+b)}\left[\widehat{\theta}^2\int_0^\infty \theta^{n+b+1}e^{-t\theta}d\theta-2\widehat{\theta}\int_0^\infty \theta^{n+b}e^{-t\theta}d\theta+\int_0^\infty \theta^{n+b+1}e^{-t\theta}d\theta\right]\\
&\cdot\frac{t^{n+b}}{\widehat{\theta}\Gamma(n+b)}\left[\widehat{\theta}^2\frac{\Gamma(n+b)}{t^{n+b}}+2\widehat{\theta}\frac{\Gamma(n+b+1)}{t^{n+b+1}}+\frac{\Gamma(n+b+2)}{t^{n+b+2}}\right]\\
&\cdot\widehat{\theta}-\frac{2(n+b)}{t}+\frac{(n+b)(n+b+1)}{\widehat{\theta}t^2}.
\end{aligned}$$

Now solving $\frac{\partial R(\widehat{\theta})}{\partial \widehat{\theta}}=0$, we obtain the bayes estimator as

$$\widehat{\theta}_p=\frac{[(n+b)(n+b+1)]^{1/2}}{t}. \tag{5.3}$$

6. Elicitation of Hyperparameters

When significant amount of information is available pertaining to the model, then it first become essential to quantify such information in the form of a (prior) probability distribution and then properly use this prior into the subsequent Bayesian analysis. The process of quantifying the prior information accurately is known as elicitation. Garthwaite and Dickey [8], defined elicitation is the process of formulating a person's knowledge and attitude about one or more uncertain quantities into a probability distribution for those quantities. In the context of Bayesian statistical analysis, it arises most usually as a method for specifying the prior distribution for one or more unknown parameters of a statistical model. Aslam [4] proposed some new methods in his paper based on prior predictive distribution. The values of hyper parameters can also be taken directly by knowing the range of hyper-parameters in prior distribution.

7. Simulation Study

In our simulation study, we chose a sample size of $n=50$ and 100 to represent small, medium and large data set. The shape parameter is estimated for Lomax distribution with Maximum Likelihood and Bayesian using extension of Jeffrey's prior and gamma prior. For the shape parameter θ we have considered = 0.5, 0.75 and 1. The scale parameter λ has been fixed at 0.5, 0.75 and 1. The values of Jeffrey's extension were $c=0.5$ and 1.75. The value of loss parameter a and b are (0.5,0.1) and (1.0,0.5), respectively. This was iterated 1000 times and the shape parameter for each method was calculated. A simulation study was conducted R-software to examine and compare the performance of the estimates for different sample sizes with different values of loss functions. The results are presented in Tables 1 - 2 for different selections of the parameters.

Table 1. Mean square error for $(\widehat{\theta})$ under the extension of Jeffery's prior

			MLE		MSE		Bayes estimate	
n	θ	α	θ_{MLE}	α_{MLE}	ELF	PLF	ELF	PLF
50	0.5	0.75	0.537	0.8876	0.004	0.0050	0.490	0.5050
	0.75	0.5	0.839	0.6337	0.0702	0.0625	0.4851	0.5000
	0.75	1	0.857	1.3098	0.2601	0.0600	0.49	0.5049
	1	0.75	1.207	1.0372	0.2652	0.25003	0.4850	0.49997
100	0.5	0.75	0.511	0.80373	0.2401	0.25501	0.9900	1.00499
	0.75	0.5	0.816	0.51494	0.2352	0.24999	0.9850	0.9999
	0.75	1	0.788	1.1068	0.0576	0.06502	0.99	1.00499
	1	0.75	1.081	0.85485	0.0002	0.85485	0.985	0.99999

Table 2. Mean square error for $(\widehat{\theta})$ under the extension of conjugate prior

				MLE		Bayes estimate		MSE	
n	θ	α	b	θ_{MLE}	α_{MLE}	ELF	PLF	ELF	PLF
50	0.5	0.75	0.25	0.5326	0.8923	0.4925	0.5075	0.0040	0.0041
	0.75	0.5	0.75	0.8441	0.6389	0.6156	0.0181	0.6343	0.0134
	0.75	1	0.25	0.8438	1.2828	0.5472	0.0411	0.5639	0.0346
	1	0.75	0.75	1.1659	0.9813	0.55	0.2025	0.5666	0.1878
100	0.5	0.75	0.25	0.5148	0.808	1.1028	0.3633	1.1194	0.3837
	0.75	0.5	0.75	0.7855	0.5517	0.9925	0.0588	1.0075	0.0663
	0.75	1	0.25	0.7845	1.0979	1.1028	0.1245	1.1194	0.1365
	1	0.75	0.75	1.0711	0.8384	1.1083	0.0117	1.125	0.0156

8. Concluding Remarks

The study was conducted to find out an appropriate Bayes estimator for the parameter of Lomax distribution. Two informative and non-informative priors have been assumed under two loss functions for the posterior analysis. The performance of the different estimators has been evaluated under a detailed simulation study. The study proposed that in to order estimate the said parameter, the use of gamma prior under precautionary loss function can be preferred.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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