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Research Article

Some Results on Core Inverses of Block Matrices Over Skew Fields

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Abstract. In this paper, necessary and sufficient conditions are given for the existence of the core inverse of the block matrix $\begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$ over any skew field, where A, B are both square and $rk(B) \geq rk(A)$. The representation of this core inverse and some relative additive results are also given.

Keywords. Skew fields, Block matrix, Core inverse

Mathematics Subject Classification (2020). 15A09, 15A24

1. Introduction

The core inverse for a complex matrix introduced by Baksalary and Trenkler [2] in 2010. Let A be a $n \times n$ complex matrix and $P_{R(A)}$ be the orthogonal projector onto $R(A)$. An $n \times n$ complex matrix $A^{\text{#}}$ satisfying $AA^{\text{#}} = P_{R(A)}$ and $R(A^{\text{#}}) \subseteq R(A)$ is the core inverse of A. A complex matrix has core inverse if and only if it is core invertible, and the core inverse is unique when it exists.

In 2015, Mieliniczuk [7] investigated C-inverse of a core matrix. Weighted core-EP inverse of an operator between Hilbert spaces established by Mosaić [8] in 2017. In 2018, Xu *et al*. [12] developed the concept of new characterization of the CMP inverse of matrices. Three limit representation of the core – EP inverse studied by Zhou *et al*. [13] in 2018. In 2019, Zho and Wang [14] investigated Weighted pseudo core inverses in rings. Core invertibility of triangular matrices over a ring developed by Xu [11] in 2019. In 2019, Ke *et al*. [6] extended the core inverse of a product and 2×2 matrices. Group inverse for a class 2×2 block matrices over skew fields studied by Bu *et al*. [4] in 2008.

2. Preliminaries

Definition 2.1 ([1])**.** A matrix A is Hermitian if $A^* = A$, and A is called an idempotent if $A^2 = A$. A Hermitian idempotent is said to be a projection.

Definition 2.2 ([2])**.** Let $A \in M_{n\times n}(\mathbb{C})$. A matrix $A^{\#} \in M_{n\times n}(\mathbb{C})$ satisfying:

(i) $AA^{\#} = P_A$, and

(ii) $R(A^{\#}) \subseteq R(A)$ is called core inverse of A.

Definition 2.3 ([2])**.** The core inverse of $A \in M_{n\times n}(\mathbb{C})$ is the matrix $X \in M_{n\times n}(\mathbb{C})$ which satisfies

$$(1)\ AXA = A \qquad (2)\ XAX = X \qquad (3)\ (AX)^* = AX \qquad (6)\ XA^2 = A \qquad (7)\ AX^2 = X$$

The matrix X is unique if it exist and is denoted by $A^{\#}$.

Definition 2.4 ([2])**.** A matrix A is said to be core-EP $AA^{\#} = A^{\#}A$.

Definition 2.5 ([1])**.** A matrix A is said to be invertible if $AB = BA = I$.

Lemma 2.6 ([5])**.** *Let $A \in M_{n\times n}(\mathbb{C})$. Then*

$$A = \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix},$$

where $KK^* + LL^* = I_r$, $\Sigma = \mathrm{diag}(\sigma_1 I_{r1}, \dots, \sigma_t I_{rt})$, $r_1 + \dots + r_t = r = rk(A)$ and $\sigma_1 > \sigma_2 > \dots > \sigma_t > 0$.

3. Some Results

Lemma 3.1. *Let $A, B \in M_{n\times n}(\mathbb{C})$. If $rk(A) = r$, $rk(B) = rk(AB) = rk(BA)$, then there are invertible matrices $P, Q \in M_{n\times n}(\mathbb{C})$ such that*

$$A = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q, \quad B = Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}P^{-1},$$

where $B_1 \in M_{r\times r}(\mathbb{C})$, $X \in M_{r\times(n-r)}(\mathbb{C})$ and $Y \in M_{(n-r)\times r}(\mathbb{C})$.

Proof. Since $rk(A) = r$, there are invertible matrices $P, Q \in M_{n\times n}(\mathbb{C})$ such that

$$A = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q, \quad B = Q^{-1}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}P^{-1},$$

where $B_1 \in M_{r\times r}(\mathbb{C})$, $B_2 \in M_{r\times(n-r)}(\mathbb{C})$, $B_3 \in M_{(n-r)\times r}(\mathbb{C})$, $B_4 \in M_{(n-r)\times(n-r)}(\mathbb{C})$.

From $rk(B) = rk(AB)$, we have

$$B_3 = YB_1, \quad B_4 = YB_2, \quad Y \in M_{(n-r)\times r}(\mathbb{C}).$$

Since $rk(B) = rk(BA)$, we obtain

$$B_2 = B_1X,\ B_4 = B_3X,\ X \in M_{r\times(n-r)}(\mathbb{C}).$$

$$B = Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix} P^{-1}.$$ □

Lemma 3.2. *Let $A \in M_{r\times r}(\mathbb{C})$, $B \in M_{(n-r)\times r}(\mathbb{C})$, $M = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \in M_{n\times n}(\mathbb{C})$. Then the core inverse of M exists if and only if the core inverse of A exists and $rk(A) = \begin{pmatrix} A \\ B \end{pmatrix}$. If the core inverse of M exists, then $M^{\#} = \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix}$.*

Proof. Since $M = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$. Suppose core inverse of $A^{\#}$ exists. $rk(A) = \begin{pmatrix} A \\ B \end{pmatrix}$.
Now $rk(M) = rk\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = rk\begin{pmatrix} A & B \end{pmatrix} = rk(A)$.
But $rk(A) = rk(A)^2$ as $(A)^{\#}$ exists.
This implies $rk(M) = rk(M^2)$. Therefore, $M^{\#}$ exists.
Let $M^{\#} = X = \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix}$. Then

(1) $$MXM = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} AA^{\#} & 0 \\ BA^{\#} & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} AA^{\#}A & 0 \\ BA^{\#}A & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = M,$$

(2) $$XMX = \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix} = \begin{pmatrix} A^{\#}A & 0 \\ B(A^{\#})^2A & 0 \end{pmatrix}\begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix} = \begin{pmatrix} A^{\#}AA^{\#} & 0 \\ B(A^{\#})^2AA^{\#} & 0 \end{pmatrix} = \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix} = X,$$

(3) $(MX)^* = MX$,

$$M = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}, \quad X = \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}, \quad \text{where } KK^* + LL^* = I_r,$$

$$(AA^{\#})(AA^{\#})^* + (BA^{\#})(BA^{\#})^* = I_r,$$
$$BA^{\#} = 0, \tag{3.1}$$
$$MX = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix} = \begin{pmatrix} AA^{\#} & 0 \\ BA^{\#} & 0 \end{pmatrix},$$
$$(MX)^* = \begin{pmatrix} AA^{\#} & 0 \\ BA^{\#} & 0 \end{pmatrix}^* = \begin{pmatrix} (AA^{\#})^* & BA^{\#} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} AA^{\#} & BA^{\#} \\ 0 & 0 \end{pmatrix} = MX,$$

(6) $$XM^2 = \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} A^{\#}A & 0 \\ B(A^{\#})^2A & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} A^{\#}AA & 0 \\ B(A^{\#})^2AA & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = M,$$

(7) $$MX^2 = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix} \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix} = \begin{pmatrix} AA^{\#} & 0 \\ BA^{\#} & 0 \end{pmatrix} \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix} = \begin{pmatrix} AA^{\#}A^{\#} & 0 \\ BA^{\#}A^{\#} & 0 \end{pmatrix} = \begin{pmatrix} A^{\#} & 0 \\ B(A^{\#})^2 & 0 \end{pmatrix} = X.$$

Conversely, suppose that the core inverse of M exists,

$$rk(M) = rk(M^2)$$
$$\Rightarrow \quad rk\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = rk\begin{pmatrix} A^2 & 0 \\ BA & 0 \end{pmatrix}$$
$$\Rightarrow \quad rk\begin{pmatrix} A \\ B \end{pmatrix} = rk\begin{pmatrix} A^2 \\ BA \end{pmatrix}$$
$$\Rightarrow \quad rk(A) = rk(A^2).$$

Therefore the core inverse of A exists.

Also $rk(M) = rk\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = rk\begin{pmatrix} A \\ B \end{pmatrix} = rk(A)$. □

Lemma 3.3. *Let $A \in M_{r\times r}(\mathbb{C})$, $B \in M_{r\times(n-r)}(\mathbb{C})$, $M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \in M_{n\times n}(\mathbb{C})$. Then the core inverse of M exists if and only if the core inverse of A exists and $rk(A) = \begin{pmatrix} A & B \end{pmatrix}$. If the core inverse of M exists, then $M^{\circledast} = \begin{pmatrix} A^{\circledast} & (A^{\circledast})^2B \\ 0 & 0 \end{pmatrix}$.*

Proof. Since $M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$. Suppose core inverse of $A^{\circledast}$ exists. $rk(A) = \begin{pmatrix} A \\ B \end{pmatrix}$.

Now $rk(M) = rk\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = rk\begin{pmatrix} A & B \end{pmatrix} = rk(A)$.

But $rk(A) = rk(A)^2$ as $(A)^{\circledast}$ exists.

This implies $rk(M) = rk(M^2)$. Therefore, $M^{\circledast}$ exists.

Let $M^{\circledast} = X = \begin{pmatrix} A^{\circledast} & (A^{\circledast})^2B \\ 0 & 0 \end{pmatrix}$. Then,

$$
\begin{aligned}
(1)\quad MXM &= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{\circledast} & (A^{\circledast})^2B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} AA^{\circledast} & A(A^{\circledast})^2B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} AA^{\circledast}A & AA^{\circledast}B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\
&= M,
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad XMX &= \begin{pmatrix} A^{\circledast} & (A^{\circledast})^2B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{\circledast} & (A^{\circledast})^2B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} A^{\circledast}A & A^{\circledast}B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{\circledast} & (A^{\circledast})^2B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} A^{\circledast}AA^{\circledast} & A^{\circledast}A(A^{\circledast})^2B \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} A^{\circledast} & (A^{\circledast})^2B \\ 0 & 0 \end{pmatrix} \\
&= X,
\end{aligned}
$$

$$(3)\quad (MX)^* = MX.$$

By using Lemma 2.6, we get

$$M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} A^{\circledast} & (A^{\circledast})^2B \\ 0 & 0 \end{pmatrix},$$

$$MX = \begin{pmatrix} AA^{\#} & A(A^{\#})^2B \\ 0 & 0 \end{pmatrix},$$

$$\begin{aligned}(MX)^* &= \begin{pmatrix} AA^{\#} & 0 \\ A(A^{\#})^2B & 0 \end{pmatrix}^* \quad (\text{since } A(A^{\#})^2B = 0)\\ &= \begin{pmatrix} (AA^{\#})^* & A(A^{\#})^2B \\ 0 & 0 \end{pmatrix}\\ &= \begin{pmatrix} AA^{\#} & A(A^{\#})^2B \\ 0 & 0 \end{pmatrix}\\ &= MX,\end{aligned}$$

$$\begin{aligned}(6)\quad XM^2 &= \begin{pmatrix} A^{\#} & (A^{\#})^2B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\\ &= \begin{pmatrix} A^{\#}A & A^{\#}B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\\ &= \begin{pmatrix} A^{\#}AA & A^{\#}AB \\ 0 & 0 \end{pmatrix}\\ &= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\\ &= M,\end{aligned}$$

$$\begin{aligned}(7)\quad MX^2 &= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{\#} & (A^{\#})^2B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{\#} & (A^{\#})^2B \\ 0 & 0 \end{pmatrix}\\ &= \begin{pmatrix} AA^{\#} & A(A^{\#})^2B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{\#} & (A^{\#})^2B \\ 0 & 0 \end{pmatrix}\\ &= \begin{pmatrix} AA^{\#}A^{\#} & AA^{\#}(A^{\#})^2B \\ 0 & 0 \end{pmatrix}\\ &= X.\end{aligned}$$

Conversely, suppose the core inverse of M exists then

$$\begin{aligned} & rk(M) = rk(M^2)\\ \Rightarrow\quad & rk\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = rk\begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix}\\ \Rightarrow\quad & rk\begin{pmatrix} A & B \end{pmatrix} = rk\begin{pmatrix} A^2 & AB \end{pmatrix}\\ \Rightarrow\quad & rk(A) = rk(A^2).\end{aligned}$$

Therefore the core inverse of A exists.

Also $rk(M) = rk\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = rk\begin{pmatrix} A & B \end{pmatrix} = rk(A)$. □

Lemma 3.4. *Let $A, B \in M_{n\times n}(\mathbb{C})$. If $rk(A) = rk(B) = rk(AB) = rk(BA)$, then the following conclusions hold:*

(i) $AB(AB)^{\#}A = A$,

(ii) $A(BA)^{\#}BA = A$,

(iii) $BA(BA)^{\circledast}B = B$,

(iv) $B(AB)^{\circledast}A = BA(BA)^{\circledast}$,

(v) $A(BA)^{\circledast} = (AB)^{\circledast}A$.

Proof. Suppose $rk(A) = r$. By Lemma 3.1, we have

$$A = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q, \quad B = Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}P^{-1},$$

where $B_1 \in M_{r\times r}(\mathbb{C})$, $X \in M_{r\times(n-r)}(\mathbb{C})$, $Y \in M_{(n-r)\times r}$. Then

$$AB = P\begin{pmatrix} B_1 & B_1X \\ 0 & 0 \end{pmatrix}P^{-1}, \quad BA = Q^{-1}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix}Q.$$

Since $rk(A) = rk(B)$, we have that B_1 is invertible. By using Lemma 3.2 and Lemma 3.3, we get

$$(AB)^{\circledast} = P\begin{pmatrix} B_1^{-1} & B_1^{-1}X \\ 0 & 0 \end{pmatrix}P^{-1}, \quad (BA)^{\circledast} = Q^{-1}\begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix}Q.$$

Then

$$\begin{aligned}
\text{(i)}\quad AB(AB)^{\circledast}A &= P\begin{pmatrix} B_1 & B_1X \\ 0 & 0 \end{pmatrix}P^{-1}P\begin{pmatrix} B_1^{-1} & B_1^{-1}X \\ 0 & 0 \end{pmatrix}P^{-1}P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q \\
&= P\begin{pmatrix} B_1 & B_1X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1^{-1} & B_1^{-1}X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q \\
&= P\begin{pmatrix} B_1B_1^{-1} & B_1B_1^{-1}X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q \\
&= P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q \\
&= A,
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad A(BA)^{\circledast}BA &= P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}QQ^{-1}\begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix}QQ^{-1}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix}Q \\
&= P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix}Q \\
&= P\begin{pmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix}Q \\
&= P\begin{pmatrix} B_1^{-1}B_1 & 0 \\ 0 & 0 \end{pmatrix}Q \\
&= P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q \\
&= A,
\end{aligned}$$

$$\begin{aligned}
\text{(iii)}\quad BA(BA)^{\circledast}B &= Q^{-1}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix}QQ^{-1}\begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix}QQ^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}P^{-1} \\
&= Q^{-1}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix}\begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}P^{-1} \\
&= Q^{-1}\begin{pmatrix} B_1B_1^{-1} & 0 \\ YB_1B_1^{-1} & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}P^{-1}
\end{aligned}$$

$$= Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix} P^{-1}$$
$$= B,$$

(iv) $$B(AB)^{\#}A = Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix} P^{-1}P\begin{pmatrix} B_1^{-1} & B_1^{-1}X \\ 0 & 0 \end{pmatrix} P^{-1}P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q$$
$$= Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}\begin{pmatrix} B_1^{-1} & B_1^{-1}X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q$$
$$= Q^{-1}\begin{pmatrix} B_1B_1^{-1} & B_1B_1^{-1}X \\ YB_1B_1^{-1} & YB_1B_1^{-1}X \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q$$
$$= Q^{-1}\begin{pmatrix} I_r & 0 \\ Y & 0 \end{pmatrix} Q,$$
$$BA(BA)^{\#} = Q^{-1}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix} QQ^{-1}\begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix} Q$$
$$= Q^{-1}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix}\begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix} Q$$
$$= Q^{-1}\begin{pmatrix} I_r & 0 \\ Y & 0 \end{pmatrix} Q,$$
$$B(AB)^{\#}A = BA(BA)^{\#},$$

(v) $$A(BA)^{\#} = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} QQ^{-1}\begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix} Q$$
$$= P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix} Q$$
$$= P\begin{pmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q,$$
$$(AB)^{\#}A = P\begin{pmatrix} B_1^{-1} & B_1^{-1}X \\ 0 & 0 \end{pmatrix} P^{-1}P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q$$
$$= P\begin{pmatrix} B_1^{-1} & B_1^{-1}X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q$$
$$= P\begin{pmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q,$$
$$A(BA)^{\#} = (AB)^{\#}A. \qquad \square$$

4. Main Results

Theorem 4.1. Let $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$, where $A, B \in M_{n\times n}(\mathbb{C})$, $rk(B) \geq rk(A) = r$. Then

(i) the core inverse of M exists if and only if $rk(A) = rk(B) = rk(AB) = rk(BA)$.

(ii) if the core inverse of M exists, then
$$M^{\#} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where

$$M_{11} = (AB)^{\#}A - (AB)^{\#}A^2(BA)^{\#}B,$$
$$M_{12} = (AB)^{\#}A,$$
$$M_{21} = (BA)^{\#}B - B(AB)^{\#}A^2(BA)^{\#} + B(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}B,$$
$$M_{22} = -B(AB)^{\#}A^2(BA)^{\#}.$$

Proof. (i) Given $rk(B) \geq rk(A) = r$.
Suppose $rk(A) = rk(B)$ then, $rk(A)^2 = rk(AB)$.
Since, $rk(AB) = rk(A)$ so, $rk(A)^2 = rk(A)$.
Now, the core inverse of M exists if $rk(M) = rk(M)^2$. Therefore,

$$rk(M) = rk\begin{pmatrix} A & A \\ B & 0 \end{pmatrix} = rk\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = rk(A) + rk(B).$$

Since, $rk(A) = rk(B)$. Therefore, $rk(M) = 2rk(A)$.
Also, using elementary transformation, we have

$$rk(M^2) = rk\begin{pmatrix} A^2 + AB & A^2 \\ BA & BA \end{pmatrix} = rk\begin{pmatrix} AB & A^2 \\ 0 & BA \end{pmatrix}.$$

We have

$$rk(A)^2 = rk(A) \text{ and } rk(A) = rk(AB) = rk(BA).$$

Then,

$$\begin{aligned} rk(M^2) &= rk\begin{pmatrix} AB & AB \\ 0 & BA \end{pmatrix} \\ &= rk\begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \\ &= rk(AB) + rk(BA) \\ &= rk(A) + rk(A) \\ &= 2rk(A). \end{aligned}$$

Thus,

$$rk(M) = rk(M^2) = 2rk(A).$$

Hence, the core inverse of M exists.
Now, we will show that the condition is necessary

$$rk(M) = rk\begin{pmatrix} A & A \\ B & 0 \end{pmatrix} = rk\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = rk(A) + rk(B),$$
$$rk(M^2) = rk\begin{pmatrix} A^2 + AB & A^2 \\ BA & BA \end{pmatrix} = rk\begin{pmatrix} AB & A^2 \\ 0 & BA \end{pmatrix}.$$

Since the core inverse of M exists if and only if $rk(M) = rk(M^2)$, we have

$$rk(A) + rk(B) = rk(M^2)$$

$$\leq rk(AB) + rk\begin{pmatrix} A^2 \\ BA \end{pmatrix}$$
$$\leq rk(AB) + rk\left(\begin{pmatrix} A \\ B \end{pmatrix} A\right)$$
$$\leq rk(AB) + rk(A).$$

Also,

$$\begin{aligned} rk(A) + rk(B) &= rk(M^2) \\ &\leq rk\begin{pmatrix} AB & A^2 \end{pmatrix} + rk(BA) \\ &\leq rk\left(A\begin{pmatrix} B & A \end{pmatrix}\right) + rk(BA) \\ &\leq rk(A) + rk(BA). \end{aligned}$$

Then, $rk(B) \leq rk(AB) \leq rk(B)$ and $rk(B) \leq rk(BA)$. Therefore,

$$rk(B) = rk(AB) = rk(BA).$$

From $rk(B) = rk(AB) \leq rk(A)$ and $rk(A) = rk(AB) \leq rk(B)$, we have

$$rk(A) = rk(B).$$

Since $rk(A) + rk(B) \leq rk\begin{pmatrix} AB & A^2 \end{pmatrix} + rk(BA)$ and $rk\begin{pmatrix} AB & A^2 \end{pmatrix} \leq rk(A) \leq rk\begin{pmatrix} AB & A^2 \end{pmatrix}$, we get

$$rk\begin{pmatrix} AB & A^2 \end{pmatrix} = rk(A).$$

Thus,

$$rk\begin{pmatrix} AB & A^2 \end{pmatrix} = rk(AB)$$

then there exists a matrix $U \in M_n(\mathbb{C})$ such that $ABU = A^2$. Then,

$$rk(M^2) = rk\begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} = rk(AB) + rk(BA).$$

So, we get $rk(A) = rk(B) = rk(AB) = rk(BA)$.

(ii) Let $X = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, we will prove that the matrix X satisfies the conditions of the core inverse. Firstly, we will compute

$$MX = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} AM_{11} + AM_{21} & AM_{12} + AM_{22} \\ BM_{11} & BM_{12} \end{pmatrix},$$

$$XM = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} A & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} M_{11}A + M_{12}B & M_{11}A \\ M_{21} + M_{22}B & M_{21}A \end{pmatrix}.$$

Applying Lemma 3.4(i), (iii) and (v), we have

$$\begin{aligned} AM_{11} + AM_{21} &= A(AB)^{\#}A - A(AB)^{\#}A^2(BA)^{\#}B + A(BA)^{\#}B - AB(AB)^{\#}A^2(BA)^{\#} \\ &\quad + AB(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}B \\ &= A(AB)^{\#}A - A(AB)^{\#}A^2(BA)^{\#}B + A(BA)^{\#}B - AB(AB)^{\#}AA(BA)^{\#} \\ &\quad + A(AB)^{\#}A^2(BA)^{\#}B \\ &= A(AB)^{\#}A + A(BA)^{\#}B - A(AB)^{\#}A \\ &= A(BA)^{\#}B, \end{aligned}$$

$$\begin{aligned} M_{11}A + M_{12}B &= (AB)^{\#}AA - (AB)^{\#}A^2(BA)^{\#}BA + (AB)^{\#}AB \\ &= (AB)^{\#}A^2 - (AB)^{\#}A^2 + A(AB)^{\#}B \\ &= A(BA)^{\#}B. \end{aligned}$$

From Lemma 3.4(ii), we obtain

$$\begin{aligned} AM_{11} + AM_{21} &= M_{11}A + M_{12}B, \\ AM_{12} + AM_{22} &= A(AB)^{\#}A - AB(AB)^{\#}A^2(BA)^{\#} \\ &= A(AB)^{\#}A - AA(BA)^{\#} \\ &= A(AB)^{\#}A - A(BA)^{\#}A \\ &= 0, \\ M_{11}A &= (AB)^{\#}A^2 - (AB)^{\#}A^2(BA)^{\#}BA \\ &= (AB)^{\#}A^2 - (AB)^{\#}A^2 \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} AM_{12} + AM_{22} &= M_{11}A, \\ BM_{11} &= B(AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B, \\ M_{21}A + M_{22}B &= (BA)^{\#}BA - B(AB)^{\#}A^2(BA)^{\#}A + B(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}BA \\ &\quad - B(AB)^{\#}A^2(BA)^{\#}B \\ &= B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}A + B(AB)^{\#}AA(BA)^{\#}A \\ &\quad - B(AB)^{\#}A^2(BA)^{\#}B \\ &= B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B. \end{aligned}$$

Thus,

$$\begin{aligned} BM_{11} &= M_{21}A + M_{22}B, \\ BM_{12} &= B(BA)^{\#}A, \\ M_{21}A &= B(AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}A + B(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}BA \\ &= B(AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}A + B(AB)^{\#}AA(BA)^{\#}A \\ &= B(AB)^{\#}A. \end{aligned}$$

Thus,

$$BM_{12} = M_{21}A.$$

Therefore,

$$MX = \begin{pmatrix} A(BA)^{\#}B & 0 \\ B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B & B(AB)^{\#}A \end{pmatrix},$$

$$XM = \begin{pmatrix} A(BA)^{\#}B & 0 \\ B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B & B(AB)^{\#}A \end{pmatrix}.$$

(1) $$MXM = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}\begin{pmatrix} A(BA)^{\#}B & 0 \\ B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B & B(AB)^{\#}A \end{pmatrix}$$
$$= \begin{pmatrix} A^2(BA)^{\#}B + AB(BA)^{\#}A - AB(AB)^{\#}A^2(BA)^{\#}B & AB(AB)^{\#}A \\ BA(BA)^{\#}B & 0 \end{pmatrix}$$
$$= \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & 0 \end{pmatrix}.$$

Applying Lemma 3.4(i) and (iii), we compute

$$X_{11} = A^2(BA)^{\#}B + AB(BA)^{\#}A - AB(AB)^{\#}A^2(BA)^{\#}B$$
$$= A^2(BA)^{\#}B + A - A^2(BA)^{\#}B$$
$$= A,$$
$$X_{12} = AB(AB)^{\#}A = A,$$
$$X_{21} = BA(BA)^{\#}B = B.$$

Thus, $MXM = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix} = M$.

(2) $$XMX = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} A(BA)^{\#}B & 0 \\ B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B & B(AB)^{\#}A \end{pmatrix}$$
$$= \begin{pmatrix} M_{11}A(BA)^{\#}B + M_{12}B(AB)^{\#}A - M_{12}B(AB)^{\#}A^2(BA)^{\#}B & M_{12}B(AB)^{\#}A \\ M_{21}A(BA)^{\#}B + M_{22}B(AB)^{\#}A - M_{22}B(AB)^{\#}A^2(BA)^{\#}B & M_{22}B(AB)^{\#}A \end{pmatrix}$$
$$= \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

$$Y_{11} = (AB)^{\#}A^2(BA)^{\#}B - (AB)^{\#}A^2(BA)^{\#}BA(BA)^{\#}B + (AB)^{\#}AB(AB)^{\#}A$$
$$- (AB)^{\#}AB(AB)^{\#}A^2(BA)^{\#}B$$
$$= (AB)^{\#}A^2(BA)^{\#}B - (AB)^{\#}A^2(BA)^{\#}B + (AB)^{\#}A - (AB)^{\#}A^2(BA)^{\#}B$$
$$= (AB)^{\#}A - (AB)^{\#}A^2(BA)^{\#}B$$
$$= M_{11},$$
$$Y_{12} = (AB)^{\#}AB(AB)^{\#}A$$
$$= (AB)^{\#}A$$
$$= M_{12},$$
$$Y_{21} = (BA)^{\#}BA(BA)^{\#}B - B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B$$
$$+ B(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}BA(BA)^{\#}B$$
$$- B(AB)^{\#}A^2(BA)^{\#}B(AB)^{\#}A + B(AB)^{\#}A^2(BA)^{\#}B(AB)^{\#}A^2(BA)^{\#}B$$
$$= (BA)^{\#}B - B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B + B(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}B$$
$$- B(AB)^{\#}AA(BA)^{\#}ABA(BA)^{\#}A + B(AB)^{\#}A(AB)^{\#}AB(AB)^{\#}AA(BA)^{\#}B$$
$$= (BA)^{\#}B - B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B + B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B$$
$$- B(AB)^{\#}AA(BA)^{\#} + B(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}B$$

$$= (BA)^{\#}B - B(AB)^{\#}A^2(BA)^{\#} + B(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}B.$$

$$= M_{21},$$

$$Y_{22} = -B(AB)^{\#}A^2(BA)^{\#}B(AB)^{\#}A$$

$$= -B(AB)^{\#}A^2(BA)^{\#}$$

$$= M_{22}.$$

Therefore, $XMX = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = X$.

(3) $(MX)^* = MX$.

By using Lemma 2.6, we get

$$MX = \begin{pmatrix} A(BA)^{\#}B & 0 \\ B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B & B(AB)^{\#}A \end{pmatrix}.$$

Since $B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B = 0$

$$(MX)^* = \begin{pmatrix} A(BA)^{\#}B & 0 \\ B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B & B(AB)^{\#}A \end{pmatrix}^*$$

$$= \begin{pmatrix} (A(BA)^{\#}B)^* & B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B \\ 0 & (B(AB)^{\#}A)^* \end{pmatrix}$$

$$= \begin{pmatrix} A(BA)^{\#}B) & B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B \\ 0 & B(AB)^{\#}A \end{pmatrix}$$

$$= MX,$$

(6) $$XM^2 = \begin{pmatrix} A(BA)^{\#}B & 0 \\ B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B & B(AB)^{\#}A \end{pmatrix} \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A(BA)^{\#}BA & A(BA)^{\#}BA \\ \begin{pmatrix} AB(AB)^{\#}A - AB(AB)^{\#}A^2(BA)^{\#}B \\ +B(AB)^{\#}AB \end{pmatrix} & \begin{pmatrix} AB(AB)^{\#}A \\ -AB(AB)^{\#}A^2(BA)^{\#}B \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} A & A \\ A - A + B & A - A \end{pmatrix}$$

$$= \begin{pmatrix} A & A \\ B & A \end{pmatrix}$$

$$= M,$$

(7) $$MX^2 = \begin{pmatrix} A(BA)^{\#}B & 0 \\ B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B & B(AB)^{\#}A \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A(BA)^{\#}BM_{11} & A(BA)^{\#}BM_{12} \\ \begin{pmatrix} (B(AB)^{\#}A \\ -B(AB)^{\#}A^2(BA)^{\#}B)M_{11} \\ +M_{21}B(AB)^{\#}A \end{pmatrix} & \begin{pmatrix} (B(BA)^{\#}A \\ -B(AB)^{\#}A^2(BA)^{\#}B)M_{12} \\ +AM_{22} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

$$Y_{11} = A(BA)^{\#}B(AB)^{\#}A - A(BA)^{\#}B(AB)^{\#}A^2(BA)^{\#}B$$

$$= (AB)^{\#}A - (AB)^{\#}A^2(BA)^{\#}B$$
$$= M_{11},$$
$$Y_{12} = A(BA)^{\#}B(AB)^{\#}A$$
$$= (AB)^{\#}A$$
$$= M_{12},$$
$$Y_{21} = (B(BA)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B)M_{11} + M_{21}B(AB)^{\#}A$$
$$= B(AB)^{\#}A(AB)^{\#}A - (AB)^{\#}A^2(BA)^{\#}BB(AB)^{\#}A - (AB)^{\#}AB(AB)^{\#}A^2(BA)^{\#}B$$
$$+ (AB)^{\#}A^2(BA)^{\#}BB(AB)^{\#}A^2(BA)^{\#}B + B(AB)^{\#}A((BA)^{\#}B$$
$$- B(AB)^{\#}A^2(BA)^{\#} + B(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}B)$$
$$= (AB)^{\#}A - (AB)^{\#}A - (AB)^{\#}A + (AB)^{\#}A + (BA)^{\#}B - B(AB)^{\#}A^2(BA)^{\#}$$
$$+ B(AB)^{\#}A(AB)^{\#}A^2(BA)^{\#}B$$
$$= M_{21},$$
$$Y_{22} = B(BA)^{\#}A(AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B)(AB)^{\#}A$$
$$+ B(AB)^{\#}A(-B(AB)^{\#}A^2(BA)^{\#})$$
$$= (AB)^{\#}A - (AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}$$
$$= -B(AB)^{\#}A^2(BA)^{\#}$$
$$= M_{22},$$
$$MX^2 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

So we have $X = M^{\#}$.

Theorem 4.2. Let $M = \begin{pmatrix} A & B \\ A & 0 \end{pmatrix}$, where $A, B \in M_{n\times n}(\mathbb{C})$, $rk(B) \geq rk(A) = r$. Then

(i) the core inverse of M exists if and only if $rk(A) = rk(B) = rk(AB) = rk(BA)$.

(ii) if the core inverse of M exists, then $M^{\#} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$, where

$$Z_{11} = (AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#},$$
$$Z_{12} = B(AB)^{\#} - (AB)^{\#}A^2(BA)^{\#}B + B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B,$$
$$Z_{21} = (AB)^{\#}A,$$
$$Z_{22} = -(AB)^{\#}A^2(BA)^{\#}B.$$

Proof. (i) Given $rk(B) \geq rk(A) = r$.

Suppose $rk(A) = rk(B)$ then, $rk(A)^2 = rk(AB)$.

Since,

$rk(AB) = rk(A)$ so, $rk(A)^2 = rk(A)$.

Now, the core inverse of M exists if $rk(M) = rk(M)^2$.

Therefore,

$$rk(M) = rk\begin{pmatrix} A & B \\ A & 0 \end{pmatrix} = rk\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} = rk(A) + rk(B).$$

Since, $rk(A) = rk(B)$. Therefore, $rk(M) = 2rk(A)$.

Also, using elementary transformation, we have

$$rk(M^2) = rk\begin{pmatrix} A^2 + BA & AB \\ A^2 & AB \end{pmatrix} = rk\begin{pmatrix} BA & 0 \\ A^2 & AB \end{pmatrix}.$$

We have $rk(A)^2 = rk(A)$ and $rk(A) = rk(AB) = rk(BA)$. Then,

$$\begin{aligned} rk(M^2) &= rk\begin{pmatrix} BA & 0 \\ BA & AB \end{pmatrix} \\ &= rk\begin{pmatrix} BA & 0 \\ 0 & AB \end{pmatrix} \\ &= rk(BA) + rk(AB) \\ &= rk(A) + rk(A) \\ &= 2rk(A). \end{aligned}$$

Now, we will show that the condition is necessary.

$$rk(M) = rk\begin{pmatrix} A & A \\ B & 0 \end{pmatrix} = rk\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = rk(A) + rk(B),$$

$$rk(M^2) = rk\begin{pmatrix} A^2 + BA & AB \\ A^2 & AB \end{pmatrix} = rk\begin{pmatrix} BA & 0 \\ A^2 & AB \end{pmatrix}.$$

Since the core inverse of M exists if and only if $rk(M) = rk(M^2)$, we have

$$\begin{aligned} rk(A) + rk(B) &= rk(M^2) \\ &\le rk\begin{pmatrix} BA \\ A^2 \end{pmatrix} + rk(AB) \\ &\le rk\left(\begin{pmatrix} B \\ A \end{pmatrix} A\right) + rk(AB) \\ &\le rk(A) + rk(AB). \end{aligned}$$

Also,

$$\begin{aligned} rk(A) + rk(B) &= rk(M^2) \\ &\le rk(BA) + rk\begin{pmatrix} A^2 & AB \end{pmatrix} \\ &\le rk(BA) + rk\left(A\begin{pmatrix} A & B \end{pmatrix}\right) \\ &\le rk(BA) + rk(A). \end{aligned}$$

Then, $rk(B) \le rk(AB) \le rk(B)$ and $rk(B) \le rk(BA)$. Therefore,

$$rk(B) = rk(AB) = rk(BA).$$

From $rk(B) = rk(AB) \le rk(A)$ and $rk(A) = rk(AB) \le rk(B)$, we have

$$rk(A) = rk(B).$$

Since

$$rk(A)+rk(B)\le rk\begin{pmatrix}AB & A^2\end{pmatrix}+rk(BA)$$

and

$$rk\begin{pmatrix}AB & A^2\end{pmatrix}\le rk(A)\le rk\begin{pmatrix}AB & A^2\end{pmatrix},$$

we get

$$rk\begin{pmatrix}AB & A^2\end{pmatrix}=rk(A).$$

Thus,

$$rk\begin{pmatrix}AB & A^2\end{pmatrix}=rk(AB)$$

then there exists a matrix $U\in M_n(\mathbb{C})$ such that $ABU=A^2$. Thus,

$$rk(M^2)=rk\begin{pmatrix}AB & 0\\ 0 & BA\end{pmatrix}=rk(AB)+rk(BA).$$

So, we get $rk(A)=rk(B)=rk(AB)=rk(BA)$.

(ii) Let $X=\begin{pmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{pmatrix}$, we will prove that the matrix X satisfies the conditions of the core inverse. Firstly, we will compute.

$$MX=\begin{pmatrix}A & B\\ A & 0\end{pmatrix}\begin{pmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{pmatrix}=\begin{pmatrix}AZ_{11}+BZ_{21} & AZ_{12}+BZ_{22}\\ AZ_{11} & AZ_{12}\end{pmatrix},$$

$$XM=\begin{pmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{pmatrix}\begin{pmatrix}A & B\\ A & 0\end{pmatrix}=\begin{pmatrix}Z_{11}A+Z_{12}A & Z_{11}B\\ Z_{21}A+Z_{22}A & Z_{21}B\end{pmatrix}.$$

Applying Theorem 4.1 we have

$$\begin{aligned}AZ_{11}+BZ_{21}&=A(AB)^{\#}A-AB(AB)^{\#}A^2(BA)^{\#}+B(AB)^{\#}A\\&=A(AB)^{\#}A-A(BA)^{\#}A+B(AB)^{\#}A\\&=B(AB)^{\#}A,\end{aligned}$$

$$\begin{aligned}Z_{11}A+Z_{12}A&=(AB)^{\#}A^2-B(AB)^{\#}A^2(BA)^{\#}A+B(AB)^{\#}A-(AB)^{\#}A^2(BA)^{\#}BA\\&\quad+B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}BA\\&=(AB)^{\#}A^2-BA(BA)^{\#}(AB)^{\#}A+B(AB)^{\#}A-(AB)^{\#}A^2+BA(BA)^{\#}(AB)^{\#}A^2\\&=B(AB)^{\#}A,\end{aligned}$$

$$AZ_{11}+BZ_{21}=Z_{11}A+Z_{12}A,$$

$$\begin{aligned}AZ_{12}+BZ_{22}&=AB(AB)^{\#}-A(AB)^{\#}A^2(BA)^{\#}B+AB(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B\\&\quad-B(AB)^{\#}A^2(BA)^{\#}B\\&=AB(AB)^{\#}-A(AB)^{\#}A^2(BA)^{\#}B+A(AB)^{\#}A^2(BA)^{\#}B-B(AB)^{\#}A^2(BA)^{\#}B\\&=AB(AB)^{\#}-B(AB)^{\#}A^2(BA)^{\#}B,\end{aligned}$$

$$AZ_{11}+BZ_{21}=AB(AB)^{\#}-B(AB)^{\#}A^2(BA)^{\#}B,$$

$$Z_{11}B=(AB)^{\#}AB-B(AB)^{\#}A^2(BA)^{\#}B,$$

$$AZ_{12}+BZ_{22}=Z_{11}B,$$

$$\begin{aligned}
AZ_{11} &= A(AB)^{\#}A - AB(AB)^{\#}A^2(BA)^{\#} \\
&= A^2(AB)^{\#} - A^2(BA)^{\#} \\
&= 0, \\
Z_{21}A + Z_{22}A &= (AB)^{\#}A^2 - (AB)^{\#}A^2(BA)^{\#}BA \\
&= (AB)^{\#}A^2 - (AB)^{\#}A^2 \\
&= 0, \\
AZ_{11} &= Z_{21}A + Z_{22}A, \\
AZ_{12} &= AB(AB)^{\#} - A(AB)^{\#}A^2(BA)^{\#}B + AB(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B \\
&= AB(AB)^{\#} - A(AB)^{\#}A^2(BA)^{\#}B + A(AB)^{\#}A^2(BA)^{\#}B \\
&= AB(AB)^{\#}, \\
Z_{21}B &= (AB)^{\#}AB \\
&= AB(AB)^{\#}.
\end{aligned}$$

Therefore,

$$AZ_{12} = Z_{21}B.$$

Thus,

$$MX = \begin{pmatrix} B(AB)^{\#}A & AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B \\ 0 & AB(AB)^{\#} \end{pmatrix},$$

$$XM = \begin{pmatrix} B(AB)^{\#}A & AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B \\ 0 & AB(AB)^{\#} \end{pmatrix}.$$

Now,

$$\begin{aligned}
(1)\quad MXM &= \begin{pmatrix} A & B \\ A & 0 \end{pmatrix}\begin{pmatrix} B(AB)^{\#}A & AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B \\ 0 & AB(AB)^{\#} \end{pmatrix} \\
&= \begin{pmatrix} AB(AB)^{\#}A & AAB(AB)^{\#} - AB(AB)^{\#}A^2(BA)^{\#}B + BAB(AB)^{\#} \\ AB(AB)^{\#}A & AAB(AB)^{\#} - AB(AB)^{\#}A^2(BA)^{\#}B \end{pmatrix} \\
&= \begin{pmatrix} A & A - A + B \\ A & A - A \end{pmatrix} \\
&= \begin{pmatrix} A & B \\ A & 0 \end{pmatrix} \\
&= M, \\
(2)\quad XMX &= \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} B(AB)^{\#}A & AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B \\ 0 & AB(AB)^{\#} \end{pmatrix} \\
&= \begin{pmatrix} Z_{11}B(AB)^{\#}A & Z_{11}(AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B) + Z_{12}AB(AB)^{\#} \\ Z_{21}B(AB)^{\#}A & Z_{21}(AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B) + Z_{22}AB(AB)^{\#} \end{pmatrix} \\
&= \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},
\end{aligned}$$

$$Y_{11} = (AB)^{\#}AB(AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B(AB)^{\#}A$$

$$
\begin{aligned}
&= (AB)^{\#}AB(AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}\\
&= Z_{11},\\
Y_{12} &= Z_{11}(AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B) + Z_{12}AB(AB)^{\#}\\
&= (AB)^{\#}AAB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}AB(AB)^{\#}\\
&\quad - (AB)^{\#}AB(AB)^{\#}A^2(BA)^{\#}B + B(AB)^{\#}A^2(BA)^{\#}B(AB)^{\#}A^2(BA)^{\#}B\\
&\quad + B(AB)^{\#}AB(AB)^{\#} - (AB)^{\#}A^2(BA)^{\#}BAB(AB)^{\#}\\
&\quad + B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}BAB(AB)^{\#}\\
&= (AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#} - (AB)^{\#}A^2(BA)^{\#}B\\
&\quad + B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B + B(AB)^{\#} - (AB)^{\#}A^2(BA)^{\#}B\\
&\quad + B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B\\
&= B(AB)^{\#} - (AB)^{\#}A^2(BA)^{\#}B + B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B\\
&= Z_{12},\\
Y_{21} &= (AB)^{\#}AB(AB)^{\#}A\\
&= (AB)^{\#}A\\
&= Z_{21},\\
Y_{22} &= Z_{21}(AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B) + Z_{22}AB(AB)^{\#}\\
&= (AB)^{\#}AAB(AB)^{\#} - (AB)^{\#}AB(AB)^{\#}A^2(BA)^{\#}B\\
&\quad - (AB)^{\#}A^2(BA)^{\#}BAB(AB)^{\#}\\
&= (AB)^{\#}AAB(AB)^{\#} - (AB)^{\#}A^2(BA)^{\#}B - (AB)^{\#}A^2(BA)^{\#}B\\
&= -(AB)^{\#}A^2(BA)^{\#}B\\
&= Z_{22},\\
XMX &= \begin{pmatrix} Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{pmatrix},
\end{aligned}
$$

(3) $(MX)^* = MX$.

By using Lemma 2.6, we get

$$MX = \begin{pmatrix} B(AB)^{\#}A & AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B\\ 0 & AB(AB)^{\#}\end{pmatrix}.$$

Since $AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B = 0$

$$
\begin{aligned}
(MX)^* &= \begin{pmatrix} B(AB)^{\#}A & AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B\\ 0 & AB(AB)^{\#}\end{pmatrix}^*\\
&= \begin{pmatrix} (B(AB)^{\#}A)^* & 0\\ AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B & (AB(AB)^{\#})^*\end{pmatrix}\\
&= \begin{pmatrix} B(AB)^{\#}A & 0\\ AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B & AB(AB)^{\#}\end{pmatrix}
\end{aligned}
$$

$$(MX)^* = MX,$$

(6) $$XM^2 = \begin{pmatrix} B(AB)^{\#}A & AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B \\ 0 & AB(AB)^{\#} \end{pmatrix}\begin{pmatrix} A & B \\ A & 0 \end{pmatrix}$$

$$= \begin{pmatrix} B(AB)^{\#}A^2 + AB(AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}BA & B(AB)^{\#}AB \\ AB(AB)^{\#}A & 0 \end{pmatrix}$$

$$= \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

$$X_{11} = B(AB)^{\#}A^2 + AB(AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}BA$$
$$= B(AB)^{\#}A^2 + AB(AB)^{\#}A - B(AB)^{\#}A^2$$
$$= A,$$
$$X_{12} = B(AB)^{\#}AB$$
$$= B,$$
$$X_{21} = AB(AB)^{\#}A$$
$$= A.$$

Therefore, $XM^2 = M$.

(7) $$MX^2 = \begin{pmatrix} B(AB)^{\#}A & AB(AB)^{\#} - B(AB)^{\#}A^2(BA)^{\#}B \\ 0 & AB(AB)^{\#} \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

$$= \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

$$Y_{11} = B(AB)^{\#}AZ_{11} + AB(AB)^{\#}Z_{21} - B(AB)^{\#}A^2(BA)^{\#}BZ_{21}$$
$$= B(AB)^{\#}A(AB)^{\#}A - B(AB)^{\#}AB(AB)^{\#}A^2(BA)^{\#}$$
$$+ AB(AB)^{\#}(AB)^{\#}A - B(AB)^{\#}A^2(BA)^{\#}B(AB)^{\#}A$$
$$= B(AB)^{\#}A(AB)^{\#}A - B(AB)^{\#}AB(AB)^{\#}A^2(BA)^{\#},$$
$$= Z_{11},$$
$$Y_{12} = B(AB)^{\#}AZ_{12} + AB(AB)^{\#}Z_{22} - B(AB)^{\#}A^2(BA)^{\#}BZ_{22}$$
$$= B(AB)^{\#}A(B(AB)^{\#} - (AB)^{\#}A^2(BA)^{\#}B + B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B)$$
$$+ AB(AB)^{\#}(-(AB)^{\#}A^2(BA)^{\#}B) - B(AB)^{\#}A^2(BA)^{\#}B(-(AB)^{\#}A^2(BA)^{\#}B)$$
$$= B(AB)^{\#}A(B(AB)^{\#} - (AB)^{\#}A^2(BA)^{\#}B$$
$$+ B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B) - (AB)^{\#}A + (AB)^{\#}A$$
$$= B(AB)^{\#} - (AB)^{\#}A^2(BA)^{\#}B + B(AB)^{\#}A^2(BA)^{\#}A(BA)^{\#}B$$
$$= Z_{12},$$
$$Y_{21} = AB(AB)^{\#}(AB)^{\#}A$$
$$= (AB)^{\#}A$$
$$= Z_{21},$$

$$\begin{aligned} Y_{22} &= AB(AB)^{\#}(-(AB)^{\#}A^2(BA)^{\#}B) \\ &= -(AB)^{\#}A^2(BA)^{\#}B \\ &= Z_{22}, \end{aligned}$$

$$MX^2 = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}.$$

So we have $X = M^{\#}$. □

Theorem 4.3. Let $A, B \in M_{n\times n}(\mathbb{C})$, if $rk(B) = rk(AB) = rk(BA)$. Then AB and BA are similar.

Proof. Suppose $rk(A) = r$, using Lemma 3.1, there are invertible matrices $P, Q \in M_{n\times n}(\mathbb{C})$ such that

$$A = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q, \quad B = Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}P^{-1},$$

where $B_1 \in M_{r\times r}(\mathbb{C})$, $X \in M_{r\times(n-r)}(\mathbb{C})$, $Y \in M_{(n-r)\times r}$. Hence

$$\begin{aligned} AB &= P\begin{pmatrix} B_1 & B_1X \\ 0 & 0 \end{pmatrix}P^{-1} \\ &= P\begin{pmatrix} I_r & -X \\ 0 & I_{n-r} \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & X \\ 0 & I_{n-r} \end{pmatrix}P^{-1}, \\ BA &= Q^{-1}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix}Q \\ &= Q^{-1}\begin{pmatrix} I_r & 0 \\ Y & I_{n-r} \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ -Y & I_{n-r} \end{pmatrix}Q. \end{aligned}$$

So AB and BA are similar. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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