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Research Article

Coincidence Point and Common Fixed Point Theorems for Generalized Kannan Contraction on Weakly Compatible Maps in Generalized Complex Valued Metric Spaces

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Abstract. In this work, we study the generalized complex valued metric space for some partial order relation and give some example. Then we established and proved a uniqueness of coincidence point and uniqueness of common fixed point theorems with satisfy weakly compatible for generalized some contraction. The results extend and improve some results of Elkouch and Marhrani [8], and Abbas and Jungck [1].

Keywords. General Kannan condition, Class of generalized complex valued metric space

Mathematics Subject Classification (2020). 46C05, 47D03, 47H09, 47H10, 47H20

1. Introduction

The axiomatic development of a metric space was essentially carried out by French mathematician Freehen in the year 1960. In the year 1922, Banach [5], introduce the Banach fixed point theorem in a complex valued metric space, has been generalized in many space.

In 2008, Abbas and Jungck [1], proved the existence of coincidence points and common fixed points for mappings satisfying certain contractive conditions, without appealing to continuity, in a cone metric space.

In recent years, this notion has been generalized in several directions and many notions of a metric-type space was introduced (b-metric, dislocated space, generalized metric space, quasi-metric space, symmetric space, etc.).

In 2015, Jleli and Samet [11], introduced a very interesting concept of a generalized metric space, which covers different well-known metric structures including classical metric spaces, b-metric spaces, dislocated metric spaces, modular spaces, and so on.

In 2017, Elkouch and Marhrani [8], they proved existence results for the Kannan contraction defined by (1.1), and they introduced the Chatterjea contraction in generalized metric space [13].

In 2011, Azam *et al*. [3], introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed point of a pair of mappings satisfying a contractive condition.

In 2019, Inchan and Deepan [10], they defined the generalized complex valued metric space for some partial order relation and give some example. Then we study and established a fixed point theorem for general Hardy-Rogers contraction.

In this paper, we are introduce by Abbas and Jungck [1], Jleli and Samet [11], Elkouch and Marhrani [8], and Inchan and Deepan [10], we establish some coincidence point and common fixed point in generalized complex valued metric spaces.

2. Preliminaries

In this section, we give some definitions and lemmas for this work.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0,\infty)$ is called a metric if for $x,y,z \in X$ the following conditions are satisfied:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \le d(x,y) + d(y,z)$.

The pair (X,d) is called a *metric space*, and d is called a metric on X.

Next, we suppose the definition of b-metric space, this space is generalized than metric spaces.

Definition 2.2 ([4])**.** Let X be a nonempty set and $s \ge 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is called a b-metric if for all $x,y,z \in X$ the following conditions are satisfied:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \le s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a *b-metric space*. The number $s \ge 1$ is called the coefficient of (X,d).

The following is some example for b-metric spaces.

Example 2.3 ([4]). Let (X,d) be a metric space. The function $\rho(x,y)$ is defined by $\rho(x,y) = (d(x,y))^2$. Then (X,ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

In 2017, Elkouch and Marhrani [8] defined a new class of metric space, let X be a nonempty set, and $D : X \times X \to [0,+\infty]$ be a given mapping. For every $x \in X$, define the set

$$C(D,X,x) = \left\{\{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n,x) = 0\right\}.$$

Definition 2.4 ([11]). A mapping D is called a generalized metric if it satisfies the following conditions:

1. For every $(x,y) \in X \times X$, we have
$$D(x,y) = 0 \Leftrightarrow x = y.$$
2. For every $(x,y) \in X \times X$, we have
$$D(x,y) = D(y,x).$$
3. There exists a real constant $C > 0$ such that for all $(x,y) \in X \times X$ and $\{x_n\} \in C(D,X,x)$, we have
$$D(x,y) \le C \limsup_{n\to\infty} D(x_n,y).$$

The pair (X,D) is called a *generalized metric space*.

It is not difficult to observe that metric d in Definition 2.1 satisfies all the conditions (i)-(iii) with $C = 1$. In 2015, Jleli and Samet [11] prove that any b-metric on X is a generalized metric on X.

In this work we will study the generalized metric space in a complex form. Let $\mathbf{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbf{C}$. Define a partial order relation $\precsim$ on $\mathbf{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \le Re(z_2) \text{ and } Im(z_1) \le Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.

(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.

(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e. one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ only (4) is satisfied.

Remark 2.5. We can easily to check the following:

(i) If $a, b \in \mathbf{R}$, $0 \le a \le b$ and $z_1 \precsim z_2$ then $az_1 \precsim bz_2$, for all $z_1, z_2 \in \mathbf{C}$.

(ii) $0 \precsim z_1 \precsim z_2 \Rightarrow |z_1| < |z_2|$.

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Azam *et al*. [3] defined the complex valued metric space in the following way:

Definition 2.6 ([3])**.** Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbf{C}$ satisfies the following conditions:

(C1) $0 \preceq d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(C2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(C3) $d(x,y) \preceq d(x,z) + d(z,y)$, for all $x, y \in X$.

Then d is called a complex valued metric on X and (X,d) is called a *complex valued metric space*.

In this work, we consider a nonempty set X, and $D : X \times X \to \mathbf{C}$ be a given mapping. For every $x \in X$, we define the set

$$C(D,X,x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} |D(x_n,x)| = 0 \right\}.$$

Definition 2.7. Let X be a nonempty set, a mapping $D : X \times X \to \mathbf{C}$ is called a generalized complex value metric if it satisfies the following conditions:

1. For every $x, y \in X$, we have
$$0 \preceq D(x,y).$$
2. For every $x, y \in X$, we have
$$D(x,y) = 0 \Rightarrow x = y.$$
3. For all $x, y \in X$, we have
$$D(x,y) = D(y,x).$$
4. There exists a complex constant $0 \prec r$ such that for all $x, y \in X$ and $\{x_n\} \in C(D,X,x)$, we have
$$D(x,y) \preceq r \limsup_{n\to\infty} |D(x_n,y)|.$$

Then a pair (X,D) is called a *generalized complex valued metric space*.

Definition 2.8. Let (X,D) be a generalized complex valued metric space, let $\{x_n\}$ be a sequence in X, and let $x \in X$. We say that $\{x_n\}$ is converge to x in X, if $\{x_n\} \in C(D,X,x)$. We denote by $\lim_{n\to\infty} x_n = x$.

Example 2.9. Let $X = [0,1]$ and let $D : X \times X \to \mathbf{C}$ be the mapping define by for any $x, y \in X$

$$\begin{cases} D(x,y) = (x+y)i; & x \neq 0 \text{ and } y \neq 0 \\ D(x,0) = D(0,x) = \frac{x}{2} i. \end{cases}$$

Proof. Let $x, y \in X$, we have $x \geq 0$ and $y \geq 0$, thus $x + y \geq 0$.
If $D(x,y) = (x+y)i = 0 + (x+y)i \succeq 0 + 0i = 0$.
If $D(x,0) = \frac{x}{2} i = 0 + \frac{x}{2} i \succeq 0 + 0i = 0$.

Hence $D(x,y) \succeq 0$.
If $D(x,y)=0$, then $(x+y)i=0$. Hence, $x=0=y$.
If $x \neq 0$ and $y \neq 0$, $D(x,y)=(x+y)i=(y+x)i=D(y,x)$ and $D(x,0)=D(0,x)$.
Let $\{x_n\}=\{\frac{(n-1)x}{n}\} \subseteq X$, we see that $\limsup_{n\to\infty}|D(x_n,x)|=0$ and put $r=i$, then we have

$$D(0,y)=\frac{y}{2}i \text{ and } \limsup_{n\to\infty}|D(x_n,y)|=\limsup_{n\to\infty}\sqrt{\left(\frac{(n-1)x}{n}+y\right)^2}=x+y.$$

Hence, $D(0,y)=\frac{y}{2}i \preceq (x+y)i$, and we see that

$$D(x,y)=(x+y)i \text{ and } \limsup_{n\to\infty}|D(x_n,y)|=\limsup_{n\to\infty}\sqrt{\left(\frac{(n-1)x}{n}+y\right)^2}=x+y.$$

Hence, $D(x,y)=(x+y)i \preceq r\limsup_{n\to\infty}|D(x_n,y)|$. □

Definition 2.10. Let (X,D) be a generalized complex valued metric space. Then a sequence $\{x_n\}$ in X is said to Cauchy sequence in X, if $\lim_{n\to\infty}|D(x_n,x_{n+m})|=0$.

Definition 2.11. Let (X,D) be a generalized complex valued metric space. If every Cauchy sequence is convergent in X then (X,D) is called a complete complex valued metric space.

Definition 2.12. Let AA and B be two nonempty subsets of a complex valued rectangular b-metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if, for any $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$ such that

$$d(x_1,y_1)=d(A,B) \text{ and } d(x_2,y_2)=d(A,B) \Rightarrow d(x_1,x_2)=d(y_1,y_2).$$

Definition 2.13. Let f and g be self maps of a set X. If $w=fx=gx$ for some x in X, then x is called a *coincidence point* of f and g, and w is called a point of coincidence of f and g.

Definition 2.14. [12] Let A and S be mappings from a metric space (X,d) into itself. Then A and S are said to be *weakly compatible* if they commute at their coincident point x, that is, $Ax=Sx$ implies $ASx=SAx$.

Proposition 2.15 ([1]). *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w=fx=gx$, then w is the unique common fixed point of f and g.*

3. Main Results

In this section, we study some common fixed point of contractive conditions. First, we can prove some proposition for uses.

Proposition 3.1. *$C(D,X,x)$ is nonempty set if and only if $D(x,x)=0$*

Proof. Let $C(D,X,x)\neq\emptyset$, thus there exists sequence $\{x_n\}$ in $C(D,X,x)$ such that

$$\lim_{n\to\infty}|D(x_n,x)|=0.$$

From property 4 in Definition 2.7, there exists $0 \prec r$ such that

$$D(x,x) \preceq r \limsup_{n\to\infty} |D(x_n,x)| = 0.$$

Hence, $D(x,x) = 0$.
Assume that $D(x,x) = 0$. Then the sequence $\{x_n\}$ in X with $x_n = x$ for any $n \in \mathbf{N}$ such that $\{x_n\}$ converges to x. It follows that $C(D,X,x) \neq \emptyset$. This proof is complete. □

Proposition 3.2. *Let (X,D) be a generalized complex valued metric space. Let $\{x_n\}$ be a sequence in X and $(x,y) \in X \times X$. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.*

Proof. Suppose that $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, by Definition 2.8 we have $\{x_n\} \in C(D,X,x)$ and $\{x_n\} \in C(D,X,y)$, it follows that:

$$|D(x_n,x)| \to 0, \text{ and } |D(x_n,y)| \to 0, \text{ as } n \to \infty.$$

Using the property 4 in Definition 2.7, we have there exists a complex constant $0 \prec r$ such that for all $x,y \in X$ and since $\{x_n\} \in C(D,X,x)$ such that

$$D(x,y) \preceq r \limsup_{n\to\infty} |D(x_n,y)|.$$

Hence, $D(x,y) = 0$. Using Property 2 in Definition 2.7, we have $x = y$. This proof is complete. □

Let (X,D) be a generalized complex valued metric space and $T,S : X \to X$ be mappings. We define the contraction as follows:

Definition 3.3. Let $k \in [0,1)$, T and S be two self-mappings on X satisfy

$$D(Tx,Ty) \preceq kD(Sx,Sy) \tag{3.1}$$

for all $x,y \in X$.

Proposition 3.4. *Suppose that $T,S : X \to X$ satisfy contractive condition in Definition 3.3. Then any common fixed point $p \in X$ of T and S satisfies*

$$|D(p,p)| < \infty \Rightarrow D(p,p) = 0.$$

Proof. Let $p \in X$ be a common fixed point $p \in X$ of T and S such that $|D(p,p)| < \infty$. From Definition 3.3, we have

$$\begin{aligned} D(p,p) = D(Tp,Tp) &\preceq k(D(Sp,Sp) \\ &= kD(p,p). \end{aligned}$$

From Remark 2.5(ii), we have

$$|D(p,p)| \leq 2k|D(p,p)|.$$

Since $k \in [0,1)$, we get $D(p,p) = 0$. This proof is complete. □

Next, let T and S be two self-mappings on X such that $T(X) \subseteq S(X)$. if $x_0 \in X$ is arbitrary, we can choose a point x_1 in X such that $Tx_0 = Sx_1$. Continuing in this process for $x_n \in X$, we have $x_{n+1} \in X$ such that

$$Tx_n = Sx_{n+1}, \quad n = 0,1,2,\cdots.$$

Now, we define

$$\delta(D,S,T,x_0) = \sup\left\{|D(Sx_p,Sx_1)| : p \geq 2\right\}.$$

Theorem 3.5. *Let (X,D) be a generalized complex valued metric space. Suppose mappings $T,S : X \to X$ such that $T(X) \subseteq S(X)$ and satisfy contractive condition*

$$D(Tx,Ty) \preceq kD(Sx,Sy), \tag{3.2}$$

where $k \in \left[0,\inf\left\{1,\frac{1}{|r|}\right\}\right)$. Assume that $S(X)$ is a complete subspace of X and $\delta(D,S,T,x_0) < \infty$, then the sequence $\{Sx_n\}$ converge to $u = Sa$ with $a \in X$. Moreover,

if $|D(Sa,Ta)| < \infty$ then u is a point of coincidence of S and T in X;

if T and S are weakly compatible, T and S have a unique common fixed point.

Proof. For any $n \geq m \geq 2$ from eq. (3.2), we have

$$\begin{aligned} D(Sx_n,Sx_{n-1}) &= D(Tx_{n-1},Tx_{n-2}) \\ &\preceq kD(Sx_{n-1},Sx_{n-2}). \end{aligned}$$

Then, by induction, we get

$$D(Sx_n,Sx_{n-1}) \preceq k^{n-2}D(Sx_2,Sx_1). \tag{3.3}$$

From eq. (3.2) again, and $n \geq m$, we have

$$\begin{aligned} D(Sx_n,Sx_m) &= D(Tx_{n-1},Tx_{m-1}) \\ &\preceq kD(Sx_{n-1},Sx_{m-1}) \\ &= kD(Tx_{n-2},Tx_{m-2}) \\ &\preceq k\left[kD(Sx_{n-2},Sx_{m-2})\right] \\ &= k^2D(Sx_{n-2},Sx_{m-2}) \\ &\vdots \\ &= k^{m-1}D(Sx_{n-(m-1)},Sx_1). \end{aligned}$$

From Remark 2.5, we get

$$|D(Sx_n,Sx_m)| \leq k^{m-1}|D(Sx_{n-(m-1)},Sx_1)| \leq k^{m-1}\delta(D,S,T,x_0).$$

Since, $k < 1$ and $\delta(D,S,T,x_0) < \infty$, then we have $|D(Sx_n,Sx_m)| \to 0$ as $m \to \infty$, it follows that $\{Sx_n\}$ is Cauchy in $S(X)$. Since $S(X)$ is complete subspace of X then the sequence $\{Sx_n\}$ is converge to $u \in S(X)$, which implies that there exists $a \in X$ such that $u = Sa$. Consider,

$$\begin{aligned} D(Sx_n,Ta) &= D(Tx_{n-1},Ta) \\ &\preceq kD(Sx_{n-1},Sa). \end{aligned}$$

From Remark 2.5 again, we have

$$|D(Sx_n,Ta)| \leq k|D(Sx_{n-1},Sa)|. \tag{3.4}$$

Since, $|D(Sx_n,Sa)| \to 0$, then $|D(Sx_n,Ta)| \to 0$ as $n \to \infty$. By Proposition 3.2, we have $Ta = u = Sa$. Thus u is a point of coincidence of T and S.

Next, assume there exists another point of coincidence of T and S, that is $v \in S(X)$ and $b \in X$ such that $Tb = v = Sb$. Now consider,

$$D(Sa,Sb) = D(Ta,Tb) \preceq kD(Sa,Sb).$$

From Remark 2.5, we have

$$|D(Sa,Sb)| \leq k|D(Sa,Sb)|.$$

Since $k < 1$, it follows that $|D(Sa,Sb)| = 0$ and then $D(Sa,Sb) = 0$. From Definition 2.7(2), $Sa = Sb$ and then $Ta = Tb$. Hence, T and S have a unique coincidence point X. From Proposition 2.15, we get T and S have a unique common fixed point in X. This proof is complete. □

Next, we extended the contractive condition to study common fixed point of T and S.

Definition 3.6. Let $k \in [0,\frac{1}{2})$, T and S be two self-mappings on X satisfy

$$D(Tx,Ty) \preceq k(D(Tx,Sx) + D(Sy,Ty)) \tag{3.5}$$

for all $x,y \in X$.

Proposition 3.7. *Suppose that $T,S : X \to X$ satisfy contractive condition in Definition 3.6. Then any common fixed point $p \in X$ of T and S satisfies*

$$|D(p,p)| < \infty \;\Rightarrow\; D(p,p) = 0.$$

Proof. Let $p \in X$ be a common fixed point $p \in X$ of T and S such that $|D(p,p)| < \infty$. From Definition 3.6, we have

$$\begin{aligned} D(p,p) = D(Tp,Tp) &\preceq k(D(Tp,Sp) + D(Sp,Tp)) \\ &= 2kD(p,p). \end{aligned}$$

From Remark 2.5(ii), we have

$$|D(p,p)| \leq 2k|D(p,p)|.$$

Since $k \in [0,\frac{1}{2})$, we get $D(p,p) = 0$. This proof is complete. □

Theorem 3.8. *Let (X,D) be a generalized complex valued metric space. Suppose mappings $T,S : X \to X$ such that $T(X) \subseteq S(X)$ and satisfy contractive condition*

$$D(Tx,Ty) \preceq k(D(Tx,Sx) + D(Sy,Ty)) \tag{3.6}$$

where $k \in \left[0, \inf\{\frac{1}{2}, \frac{1}{|r|}\}\right)$. Assume that $S(X)$ is a complete subspace of X and $\delta(D,S,T,x_0) < \infty$, then the sequence $\{Sx_n\}$ converge to $u = Sa$ with $a \in X$. Moreover,

if $|D(Sa,Ta)| < \infty$ then u is a point of coincidence of S and T in X;

if T and S are weakly compatible, T and S have a unique common fixed point.

Proof. For any $n \geq m \geq 2$ from eq. (3.6), we have

$$\begin{aligned} D(Sx_n, Sx_{n-1}) &= D(Tx_{n-1}, Tx_{n-2}) \\ &\preceq k\left(D(Tx_{n-1}, Sx_{n-1}) + D(Sx_{n-2}, Tx_{n-2})\right) \\ &= k(D(Sx_n, Sx_{n-1}) + D(Sx_{n-2}, Sx_{n-1})) \end{aligned}$$

which implies that

$$D(Sx_n, Sx_{n-1}) \preceq \frac{k}{1-k} D(Sx_{n-2}, Sx_{n-1}). \tag{3.7}$$

Then, by induction, we get

$$D(Sx_n, Sx_{n-1}) \preceq \left(\frac{k}{1-k}\right)^{n-2} D(Sx_2, Sx_1). \tag{3.8}$$

Put $\alpha = \frac{k}{1-k}$. From from eq. (3.6) again, we have

$$\begin{aligned} D(Sx_n, Sx_m) &= D(Tx_{n-1}, Tx_{m-1}) \\ &\preceq k\left(D(Tx_{n-1}, Sx_{n-1}) + D(Sx_{m-1}, Tx_{m-1})\right) \\ &= k(D(Sx_n, Sx_{n-1}) + D(Sx_m, Sx_{m-1})) \\ &\preceq k\alpha^{n-2} D(Sx_2, Sx_1) + k\alpha^{m-2} D(Sx_2, Sx_1) \\ &= k\left(\alpha^{n-2} + \alpha^{m-2}\right) D(Sx_2, Sx_1). \end{aligned}$$

From Remark 2.5, we get

$$|D(Sx_n, Sx_m)| \leq k\left(\alpha^{n-2} + \alpha^{m-2}\right) |D(Sx_2, Sx_1)|.$$

Since, $|D(Sx_2, Sx_1)| < \infty$ and $(\alpha^n + \alpha^m) \to 0$ as $n, m \to \infty$, we have $|D(Sx_n, Sx_m)| \to 0$ as $n, m \to \infty$, it follows that $\{Sx_n\}$ is Cauchy in $S(X)$. Since $S(X)$ is complete subspace of X then the sequence $\{Sx_n\}$ is converge to $u \in S(X)$, which implies that there exists $a \in X$ such that $u = Sa$. Consider,

$$\begin{aligned} D(Sx_n, Ta) &= D(Tx_{n-1}, Ta) \\ &\preceq k\left(D(Tx_{n-1}, Sx_{n-1}) + D(Sa, Ta)\right) \\ &= k(D(Sx_n, Sx_{n-1}) + D(Sa, Ta)) \\ &\preceq k(\alpha^{n-2} D(Sx_2, Sx_1) + D(Sa, Ta)) \\ &\preceq k\alpha^{n-2} \delta(D, S, T, x_0) + kD(Sa, Ta). \end{aligned}$$

From Remark 2.5, we have

$$|D(Sx_n, Ta)| \leq k\alpha^{n-2} \delta(D, S, T, x_0) + k|D(Sa, Ta)|. \tag{3.9}$$

By Definition 2.7, there exists complex constant $r \succ 0$ such that

$$D(Sa, Ta) \preceq r \limsup_{n\to\infty} |D(Sx_n, Ta)| \tag{3.10}$$

From (3.9) and (3.10), we have

$$|D(Sa, Ta)| \leq |r| k |D(Sa, Ta)|. \tag{3.11}$$

It follows that $Ta = u = Sa$. Thus u is a point of coincidence of S and T.
Finally, assume there exists another point of coincidence of T and S, that is $v \in S(X)$ and $b \in X$ such that $Tb = v = Sb$ and $C(D,X,v) \neq \emptyset$. Now consider,

$$\begin{aligned} D(Sa,Sb) = D(Ta,Tb) &\preceq k[D(Ta,Sa) + D(Sb,Tb)] \\ &= k[D(Sa,Sa) + D(Sb,Sb)]. \end{aligned}$$

By Proposition 3.1, we get

$$D(Sa,Sb) \preceq 0.$$

From Remark 2.5, we have

$$|D(Sa,Sb)| \leq 0.$$

From Definition 2.7(1), then $D(Sa,Sb) = 0$. From Definition 2.7(2), implies that $Sa = Sb$ and then $Ta = Tb$. Hence, T and S have a unique coincidence point X. From Proposition 2.15, we get T and S have a unique common fixed point in X. This proof is complete. □

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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