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Pontryagin's Maximum Principle of Optimal Control Governed by A Convection Diffusion Equation

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Abstract In this paper we analyze an optimal control problem governed by a convection diffusion equation. This problem with state constraints is discussed by adding penalty arguments involving the application of Ekeland's variational principle and finite codimensionality of certain sets. Necessary conditions for optimal control is established by the method of spike variation.

1. Introduction and Main Results

Consider the following controlled convection diffusion equations:

$$\begin{cases} -\mu\triangle y + \beta\nabla y + \sigma y = f(x,u) & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where the diffusity constant $\mu > 0$ and the advective field $\beta \in (L^\infty(\Omega))^2$ with $\nabla\beta \in L^\infty(\Omega)$, the reaction $\sigma \in L^\infty(\Omega)$ with $\sigma \geq \sigma_0 > 0$. To simplify the notation throughout $\sigma = 1$. Here $\Omega \subseteq \mathbb{R}^2$ is a convex bounded polygonal domain with a smooth boundary $\partial\Omega$, $f : \Omega \times U \to \mathbb{R}$, with U being a separable metric space. Function $u(\cdot)$, called a control, is taken from the set

$$\mathscr{U} = \{w : \Omega \to U \mid w(\cdot) \text{ is measurable}\}.$$

Under some mild conditions, for any $u(\cdot) \in \mathscr{U}$, (1.1) admits a unique weak solution $y(\cdot) \equiv y(\cdot\,; u(\cdot))$, which is called the state (corresponding to the control $u(\cdot)$). The performance of the control is measured by the cost functional

$$J(u(\cdot)) = \int_\Omega f^0(x, y(x), u(x))dx, \tag{1.2}$$

for some given map $f^0 : \Omega \times \mathbb{R} \times U \to \mathbb{R}$.

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Our optimal control problem can be stated as follows.

Problem C. Find a $\bar{u}(\cdot)\in\mathscr{U}$ such that

$$J(\bar{u}(\cdot))=\inf_{u(\cdot)\in\mathscr{U}} J(u(\cdot)). \tag{1.3}$$

Any $\bar{u}(\cdot)\in\mathscr{U}$ satisfying the above is called an optimal control, and the corresponding $\bar{y}(\cdot)\equiv\bar{y}(\cdot\,;u(\cdot))$ is called an optimal state. The pair $(\bar{y}(\cdot),\bar{u}(\cdot))$ is called an optimal pair.

In this paper, we make the following assumptions.

(H1) Set $\Omega\subseteq\mathbb{R}^2$ is a convex bounded polygonal domain with a smooth boundary $\partial\Omega$.

(H2) Set U is a separable metric space.

(H3) The function $f:\Omega\times U\to\mathbb{R}$ has the following properties: $f(\cdot;u)$ is measurable on Ω, and $f(x,\cdot)$ continuous on $\mathbb{R}\times U$ and for any $R>0$, a constant $M_R>0$, such that $|f(x,u)|\le M_R$, for all $(x,u)\in\Omega\times U$.

(H4) Function $f^0(x,y,v)$ is measurable in x and continuous in $(y,v)\in\mathbb{R}\times U$ for almost all $x\in\Omega$. Moreover, for any $R>0$, there exists a $K_R>0$ such that

$$|f^0(x,y,v)|+|f^0_y(x,y,v)|\le K_R,\quad \text{a.e. } (x,v)\in\Omega\times U,\ |y|\le R. \tag{1.4}$$

(H5) $\mathscr{X}$ is a Banach space with strict convex dual $\mathscr{X}^*$, $F:W_0^{1,p}(\Omega)\to\mathscr{X}$ is continuously Fréchet differentiable, and $W\subset\mathscr{X}$ is closed and convex.

(H6) $F'(\bar{y})D_r-W$ has finite condimensionality in $\mathscr{X}$ for some $r>0$, where $D_r=\{z\in\mathscr{X};\|z\|_{\mathscr{X}}\le r\}$.

Definition 1.1 ([2, 6]). Let X is a Banach space and X_0 is a subspace of X. We say that X_0 is finite codimensional in X if there exists $x_1,x_2,\cdots,x_n\in X$ such that

$$\text{span}\{X_0,x_1,\cdots,x_n\}=\text{the space spanned by }\{X_0,x_1,\cdots,x_n\}=X.$$

A subset S of X is said to be finite codimensional in X if for some $x_0\in S$, span$(S-\{x_0\})$ the closed subspace spanned by $\{x-x_0\mid x\in S\}$ is a finite codimensional subspace of X and $\bar{\text{co}}\,S$ the closed convex hull of $S-\{x_0\}$ has a nonempty interior in this subspace.

Lemma 1.2. *Let* (H1)-(H4) *hold. Then, for any* $u(\cdot)\in\mathscr{U}$, (1.1) *admits a unique weak solution* $y(\cdot)\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. *Furthermore, there exists a constant* $K>0$, *independent of* $u(\cdot)\in\mathscr{U}$,

$$\|y(\cdot;u)\|_{W_0^{1,p}(\Omega)\cap L^\infty(\Omega)}\le K. \tag{1.5}$$

The weak solution $y\in V=H_0^1(\Omega)$ of the state equation (1.1) is determined by

$$a(y,v)=(f,v),\quad \text{for all } v\in V,$$

using the bilinear form $a : V \times V \to \mathbb{R}$ given by

$$a(y,v) = \int_\Omega \mu \nabla y \nabla v dx + \int_\Omega \beta \nabla y v dx + \int_\Omega \sigma y v dx, \quad \text{for all } v \in V.$$

Existence and uniqueness of the solution to (1.1) follow from the above hypotheses on the problem data (see [1, 3]). Thus, a state constraint of the following type makes sense:

$$F(y) \in W. \tag{1.6}$$

Let $\mathscr{A}_{ad}$ be the set of all pairs $(y(\cdot), u(\cdot))$ satisfying (1.1) and (1.6) is called an admissible set. Any $(y,u) \in \mathscr{A}_{ad}$ is called an admissible pair. We refer to such a pair $(\bar{y}, \bar{u})$, if it exists, as an optimal pair and refer to $\bar{y}$ and $\bar{u}$ as an optimal state and control, respectively.

Now, let $(\bar{y}, \bar{u})$ be an optimal pair of Problem C. Let $z = z(\cdot\,; u(\cdot)) \in W_0^{1,p}(\Omega)$ be the unique solution of the following problem:

$$\begin{cases} -\mu \triangle z + \beta \nabla z + \sigma z = f(x,u) - f(x,\bar{u}) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

And define the reachable set of variational system (1.7)

$$\mathscr{R} = \{z(\cdot; u(\cdot)) \mid u(\cdot) \in \mathscr{U}\}. \tag{1.8}$$

Now, let us state the necessary conditions of an optimal control to Problem C as follows.

Theorem 1.3 (*Pontryagin's Maximum Principle*). *Let* (H1)-(H6) *hold. Let* $(\bar{y}(\cdot), \bar{u}(\cdot))$ *be an optimal pair of* Problem C. *Then there exists a triplet* $(\psi^0, \psi, \varphi^0) \in \mathbb{R} \times W_0^{1,p'} \times \mathscr{X}^*$ *with* $(\psi^0, \varphi^0) \neq 0$ *such that*

$$\langle \psi^0, \eta - F(\bar{y}) \rangle_{\mathscr{X}^*, \mathscr{X}} \leq 0, \quad \text{for all } \eta \in W, \tag{1.9}$$

$$\begin{cases} -\mu \triangle \bar{\psi} - \beta \nabla \bar{\psi} + (\sigma - \nabla \beta)\bar{\psi} = \psi^0 f_y^0(x, \bar{y}, \bar{u}) - F'(\bar{y})^* \varphi & \text{in } \Omega, \\ \bar{\psi} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

$$H(x, \bar{y}(x), \bar{u}(x), \psi^0, \psi(x)) = \max_{u(\cdot) \in \mathscr{U}} H(x, \bar{y}(x), u(x), \psi^0, \psi(x)) \ a.e.\ x \in \Omega. \tag{1.11}$$

In the above, (1.9), (1.10), and (1.11) are called the transversality condition, the adjoint system (along the given optimal pair), and the maximum condition, respectively.

Problem of the types studied here arise in originally from the optimal control of linear convection diffusion equation (cf. [3, 4, 5]). In the work mentioned above, the control set is convex. However, in many practical cases, the control set can not convex. This stimulates us to study Problem C. Necessary conditions for optimal control is established by the method of spike variation.

In the next section, we will prove Pontryagin's maximum principle of optimal control of Problem C.

2. Proof of the Maximum Principle

This section is devoted to the proof of the maximum principle.

***Proof of Theorem* 1.3.** First, let $\bar{d}(u(\cdot),\bar{u}(\cdot)) = |\{x \in \Omega \mid u(\cdot) \neq \bar{u}(\cdot)\}|$, where $|D|$ is the Lebesgue measure of $D \subseteq \Omega$. We can easily prove that (u,ρ) is a complete metric space. Let $(\bar{y},\bar{u})$ be an optimal pair of Problem C. For any $y(\cdot;u)$ be the corresponding state, emphasizing the dependence on the control. Without loss of generality, we may assume that $J(\bar{u})=0$. For any $\epsilon>0$, define

$$J_\epsilon(u)=\{[(J(u)+\epsilon)^+]^2+d_Q^2(F(y(\cdot;u)))\}^{1/2}. \tag{2.1}$$

Where $d_Q=\inf\limits_{\bar{x}\in Q}|x-\bar{x}|$, and $\bar{u}$ is an optimal control. Clearly, this function is continuous on the (complete) metric space $(\mathscr{U},\bar{d})$. Also, we have

$$\begin{cases} J_\epsilon(u)>0, \quad \text{for all } u\in\mathscr{U}, \\ J_\epsilon(\bar{u})=\epsilon\le \inf\limits_{u\in\mathscr{U}} J_\epsilon(u)+\epsilon. \end{cases} \tag{2.2}$$

Hence, by Ekeland's variational principle, we can find a $u^\epsilon\in\mathscr{U}$, such that

$$\begin{cases} \bar{d}(u,u^\epsilon)\le\sqrt{\epsilon}, \\ J_\epsilon(\widehat{u})-J_\epsilon(u^\epsilon)\ge-\sqrt{\epsilon}\bar{d}(\widehat{u},u^\epsilon), \text{ for all } \widehat{u}\in\mathscr{U}. \end{cases} \tag{2.3}$$

We let $v\in\mathscr{U}$ and $\epsilon>0$ be fixed and let $y^\epsilon=y(\cdot\,;u^\epsilon)$, we know that for any $\rho\in(0,1)$, there exists a measurable set $E_\rho^\epsilon\subset\Omega$ with the property $|E_\rho^\epsilon|=\rho|\Omega|$, such that if we define

$$u_\rho^\epsilon(x)=\begin{cases} u^\epsilon(x), & \text{if } x\in\Omega\backslash E_\rho^\epsilon, \\ v(x), & \text{if } x\in E_\rho^\epsilon \end{cases}$$

and let $y_\rho^\epsilon=y(\cdot;u_\rho^\epsilon)$ be the corresponding state, then

$$\begin{cases} y_\rho^\epsilon=y^\epsilon+\rho z^\epsilon+r_\rho^\epsilon, \\ J_\epsilon(u_\rho^\epsilon)=J(u^\epsilon)+\rho z^{0,\epsilon}+r_\rho^{0,\epsilon}, \quad \text{for all } \widehat{u}\in\mathscr{U}, \end{cases} \tag{2.4}$$

where z^ϵ and $z^{0,\epsilon}$ satisfying the following

$$\begin{cases} -\mu\triangle z^\epsilon+\beta\nabla z^\epsilon+\sigma z^\epsilon=f(x,v)-f(x,u^\epsilon) & \text{in } \Omega, \\ z^\epsilon=0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

$$z_\rho^{0,\epsilon}=\int_\Omega [f_y^0(x,y^\epsilon,u^\epsilon)z^\epsilon+h^{0,\epsilon}(x)]\,dx, \tag{2.6}$$

with

$$\begin{cases} h^{0,\epsilon}(x)=f^0(x,y^\epsilon,v)-f^0(x,y^\epsilon,u^\epsilon), \\ \lim\limits_{\rho\to0}\frac{1}{\rho}\|r_\rho^\epsilon\|_{W_0^{1,p}}=\lim\limits_{\rho\to0}\frac{1}{\rho}|r_\rho^{0,\epsilon}|=0. \end{cases} \tag{2.7}$$

We take $\widehat{u}=u^{\epsilon}_{\rho}$. It follows that

$$
\begin{aligned}
-\sqrt{\epsilon}|\Omega| \leq & \frac{J_{\epsilon}(u^{\epsilon}_{\rho})-J_{\epsilon}(u^{\epsilon})}{\rho} \\
= & \frac{1}{J_{\epsilon}(u^{\epsilon}_{\rho})+J_{\epsilon}(u^{\epsilon})}\left\{\frac{[(J(u^{\epsilon}_{\rho})+\epsilon)^{+}]^{2}-[(J(u^{\epsilon})+\epsilon)^{+}]^{2}}{\rho}\right. \\
& \left. +\frac{d_{Q}^{2}(F(y^{\epsilon}_{\rho}))-d_{Q}^{2}(F(y^{\epsilon}))}{\rho}\right\} \\
\to & \frac{(J(u^{\epsilon})+\epsilon)^{+}}{J_{\epsilon}(u^{\epsilon})}z^{0,\epsilon}+\left\langle\frac{d_{Q}(F(y^{\epsilon}))\xi_{\epsilon}}{J_{\epsilon}(u^{\epsilon})},F'(y^{\epsilon})z^{\epsilon}\right\rangle, \quad (\rho\to 0), \qquad (2.8)
\end{aligned}
$$

where

$$
\xi_{\epsilon}=\begin{cases}\nabla d_{Q}(F(y^{\epsilon})), & \text{if } F(y^{\epsilon})\notin Q, \\ 0, & \text{if } F(y^{\epsilon})\in Q.\end{cases}
$$

$\nabla d_{Q}(\cdot)$ denotes the subdifferential of $d_{Q}(\cdot)$. Next, we define $(\varphi^{0,\epsilon},\varphi^{\epsilon})\in[0,1]\times\mathscr{X}$ as follows:

$$
\varphi^{0,\epsilon}=\frac{(J(u^{\epsilon})+\epsilon)^{+}}{J_{\epsilon}(u^{\epsilon})}, \quad \varphi^{\epsilon}=\frac{d_{Q}(F(y^{\epsilon}))\xi_{\epsilon}}{J_{\epsilon}(u^{\epsilon})}. \qquad (2.9)
$$

Then

$$
-\sqrt{\epsilon}|\Omega|\leq\varphi^{0,\epsilon}z^{0,\epsilon}+\langle\varphi^{\epsilon},F'(y^{\epsilon})z^{\epsilon}\rangle, \qquad (2.10)
$$

$$
|\varphi^{0,\epsilon}|^{2}+\|\varphi^{\epsilon}\|^{2}_{\mathscr{X}*}=1. \qquad (2.11)
$$

On the other hand, we have

$$
\langle\varphi^{\epsilon},\eta-F(y^{\epsilon})\rangle_{X^{*},X}\leq 0, \quad \text{for all } \eta\in W. \qquad (2.12)
$$

Then

$$
\|y^{\epsilon}-\bar{y}\|_{W_{0}^{1,p}(\Omega)}\to 0, \quad (\epsilon\to 0). \qquad (2.13)
$$

Consequently,

$$
\lim_{\epsilon\to 0}\|F'(y^{\epsilon})-F'(\bar{y})\|_{\mathscr{L}(W_{0}^{1,p}(\Omega),\mathscr{X})}=0 \qquad (2.14)
$$

then

$$
\langle\varphi^{\epsilon},\eta-F(y^{\epsilon})\rangle_{X^{*},X}\leq\langle\varphi^{\epsilon},F(y^{\epsilon})-F(\bar{y})\rangle_{X^{*},X}, \quad \text{for all } \eta\in W. \qquad (2.15)
$$

By taking the limit for $\epsilon\to 0$ in (2.15), we get (1.9). From (2.5) and (2.6), we have

$$
\begin{cases} z^{\epsilon}\to z, & \text{in } W_{0}^{1,p}(\Omega), \\ z^{0,\epsilon}\to z^{0}, & (\epsilon\to 0)\end{cases}
$$

where z is the solution of system (1.6) and

$$z^0=\int_\Omega f_y^0(x,\bar y,\bar u)z(x)dx+\int_\Omega [f^0(x,\bar y,v)-f^0(x,\bar y,\bar u)]dx. \tag{2.16}$$

From (2.10) and (2.12), we have

$$\varphi^{0,\epsilon}z^{0,\epsilon}(v)+\langle\varphi^\epsilon,F'(y^\epsilon)z^\epsilon(\cdot;v)-\eta+F(y^\epsilon)\rangle\geq-\delta_\epsilon,\quad\text{for all } v\in\mathscr{U},\eta\in W, \tag{2.17}$$

with $\delta_\epsilon\to 0$ as $\epsilon\to 0$. Because $F'(\bar y)D_r-W$ has finite condimensionality in $\mathscr{X}$, we can extract some subsequence, still denoted by itself, such that

$$(\varphi^{0,\epsilon},\varphi^\epsilon)\to(\varphi^0,\varphi)\neq 0.$$

From (2.17), we have

$$\varphi^0 z^0(v)+\langle\varphi,F'(\bar y)z(\cdot;v)-\eta+F(\bar y)\rangle\geq 0,\quad\text{for all }\ v\in\mathscr{U},\ \eta\in Q. \tag{2.18}$$

Now, let $\psi^0=-\varphi^0\in[-1,0]$. Then $(\psi^0,\varphi)\neq 0$. Then we have

$$\psi^0z^0(v)+\langle\varphi,\eta-F(\bar y)\rangle-\langle F'(\bar y)^*\varphi,z(\cdot;v)\rangle\leq 0,\quad\text{for all }\ u\in\mathscr{U},\ \eta\in W. \tag{2.19}$$

Take $v=\bar u$, we obtain (1.9). Next, we let $\eta=F(\bar y)$ to get

$$\psi^0z^0(v)-\langle F'(\bar y)^*\varphi,z(\cdot\,;v)\rangle\leq 0,\quad\text{for all } v\in\mathscr{U}. \tag{2.20}$$

Because $F'(\bar y)^*\varphi\in W^{-1,p'}(\Omega)$, for the given ψ^0, there exists a unique solution $\psi\in W^{1,p'}(\Omega)$ of the adjoint equation (1.10). Then, from (1.6), (2.16) and (2.2), we have

$$\begin{aligned}
0&\geq\psi^0z^0(v)-\langle F'(\bar y)^*\varphi,z(\cdot\,;v)\rangle\\
&=\psi^0\int_\Omega f_y^0(x,\bar y,\bar u)z(x)dx+\psi^0\int_\Omega[f^0(x,\bar y,v)-f^0(x,\bar y,\bar u)]dx\\
&\quad+\langle-\mu\triangle\bar\psi-\beta\nabla\bar\psi+(\sigma-\nabla\beta)\bar\psi-\psi^0f_y^0(x,\bar y,\bar u),z\rangle\\
&=\int_\Omega\{\psi^0[f^0(x,\bar y,v)-f^0(x,\bar y,\bar u)]+\langle\psi,f(x,v)-f(x,\bar u)\rangle\}dx\\
&=\int_\Omega\{H(x,\bar y(x),v(x),\psi^0,\psi(x))-H(x,\bar y(x),\bar u(x),\psi^0,\psi(x))\}dx.
\end{aligned} \tag{2.21}$$

There, (1.11) follows. Finally, by (1.10), if $(\psi^0,\psi)=0$, then $F'(\bar y)^*\varphi=0$. Thus, in the case where $\mathscr{N}(F'(\bar y)^*)=\{0\}$, we must have $(\psi^0,\psi)\neq 0$, because $(\psi^0,\varphi)\neq 0$. □

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