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Research Article

Numerical Solution of 2nd Order Boundary Value Problems with Dirichlet, Neumann and Robin Boundary Conditions using FDM

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Abstract. In many fields of science and engineering, to determine the harmonic motion, damped and forced variation, current from electric circuit, 2nd order ODE is required to solve. Solving the ODE with complicated boundary condition that occur in engineering problems is a great challenges analytically. Therefore, numerical technique *finite difference method* (FDM) is very popular and important for solving the boundary value problems. In this article three different conditions as Dirichlet, Neumann and Robin (mixed) boundary conditions are applied in initial-boundary problem. FDM is used to solve ODE boundary value problems. Error calculation, stability, convergence are also explained. To test the accuracy numerical solutions are verified with analytical solution and error is calculated at each point for different mesh grid size as mesh grid size is decreased result will give the accuracy.

Keywords. Finite difference scheme, Dirichlet condition, Convergence, Neumann condition, Mixed condition, Stability

Mathematics Subject Classification (2020). 65M06, 65N06

1. Introduction

In science and engineering several problems are occurred in mathematically analyzing ordinary differential equations which are satisfying particular condition. Only a selected class of differential equations can be solved analytically. These problem consists differential equation and complicated conditions which not to be solved in practically, in general on can get some closed form solution, and hence can use numerical methods for solving such differential equations. Some researchers studied in this field like Balagurusamy [5], Hildebrand [6], Jain [7], Levy [9], Sastry [11, 12], Scheid [13] developed the FDM for determining the initial-boundary values problems. Second order, third order BVM with ordinary differential equation for Dirichlet boundary condition is solved by Lakshmi [8], Muhammad [9], Siddiqi [14], and Xu [15]. Adak [1–4] studied finite difference methods to solve partial differential equation with convergence of numerical techniques.

From previous research it is cleared that the most of boundary value problems with Dirichlet boundary condition are investigated. Hence, the main focus of this article, well-posed (Dirichlet boundary condition) as well as ill-posed (Neumann and Robin boundary condition) problems has been determined.

2. Boundary Value Problems

A *boundary value problem* (BVP) consists an ODE or PDE and specific *boundary condition* (BC) in physically.

We begin by discussing various types of boundary conditions that can be imposed. A BVP which only has one independent variable is an ODE but we consider BVP in dimensions we need to use PDE.

Throughout this article, linear non-homogeneous second order ODE is considered defined by

$$u'' + r(x)u' + s(x)u = t(x), \quad a < x < b\,. \tag{2.1}$$

Corresponding to ODE (2.1), there are three important boundary conditions. They are given by

Dirichlet Boundary Condition. If the values of the function are specified on the boundary in a BVP, this type of constraint is called Dirichlet boundary condition.

For example, $u(a) = \alpha$, $u(b) = \beta$ in domain $[a, b]$, where α and β are constant.

Neumann Boundary Condition. If the derivatives of the unknown function are specified on the boundary in a BVP, this type of constraint is called Neumann boundary condition.

For example, $u'(a) = \alpha$, $u'(b) = \beta$ in domain $[a, b]$, where α and β are constant.

Robin Boundary Condition. If a weighted combination at the function value and its derivative at the boundary is called Robin boundary condition or mixed boundary condition.

For example, $u'(a) + cu(a) = \alpha$, $u'(b) + du(b) = \beta$, where c and d are constants.

3. Finite Difference Approximation

In process of finite difference method to solve a BVP, derivatives are replaced by finite difference approximation in ODE with in the specific conditions. After simplifying finite difference approximations with initial-boundary conditions we get linear system of equations. Solving this system of equations desired solution of BVP is obtained.

The interval $[a,b]$ is discretized in x axis into n number of equally spaced subintervals with h for solving problem given by (2.1), so that

$$x_i = x_0 + ih, \quad i = 1,2,3,\dots,$$

where $x_0 = a$, $x_n = b$.
At these points the corresponding value of u are denoted by

$$u(x_i) = u_i = u(x_0 + ih), \quad i = 0,1,2,\dots,n.$$

Using Taylor's expansion, values of $u'(x)$ and $u''(x)$ at point $x = x_i$ is given by

$$u'_i = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2),$$

$$u''_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + O(h^2).$$

The *ordinary differential equation* (ODE) (2.1) at point $x = x_i$ is denoted by

$$u''_i + r_i u'_i + s_i u_i = t_i\,.$$

Substituting the expressions for u'_i and u''_i, we obtain

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + r_i \frac{u_{i+1} - u_{i-1}}{2h} + s_i u_i = t_i, \quad i = 1,2,\dots,n\,,$$

where $u_i = u(x_i)$, $s_i = s(x_i)$.

Error Calculation. We first find the exact solution. Solve the given differential equation, that means, determine CF (Complementary Function) and PI (Particular Integral). Then different type of errors can be calculated by

$$\text{Absolute Error (AE)} = |\text{Exact solution} - \text{Numerical solution}|,$$

$$\text{Relative Error (RE)} = \frac{|\text{Exact solution} - \text{Numerical solution}|}{\text{Exact solution}},$$

$$\text{Percentage Error (PE)} = \frac{|\text{Exact solution} - \text{Numerical solution}|}{\text{Exact solution}} \times 100.$$

Stability of Numerical Method. If the difference value of the numerical result and the exact result tends to zero as number of iteration tends to infinity,then numerical method is said to be stable.

Convergence of Numerical Method. In iteration of numerical technique, if each successive iteration result is progressively closer to the exact result, it is known as *convergence*. A numerical method is not always given converging results. Convergence should be satisfied certain conditions. If these conditions are not satisfied, it is known as *divergence*.

Consistency of Numerical Method. If the finite difference representation converges to the differential equation to solve problem with mesh size tends to very small value, then a numerical scheme is called consistent.

We have explained the method with three types of boundary conditions. In several practical problems, derivative of boundary conditions may be specified, and this requires finite approximation in case of boundary condition which are described above. The following examples illustrate FDM to obtain the solution of BVP.

4. Test Problems and Verification

Problem 1 (Dirichlet Boundary Conditions). *Solve the heat transfer ODE boundary value problem for a rod of length 1 unit. The governing equation is defined by* $u'' + u + 1 = 0$, $0 \le x \le 1$, *with BC* $u(0) = 0$ *and* $u(1) = 0$. *Determine the value of* $u(0.5)$ *using finite difference method with sub-interval* $h = 0.5$ *and* $h = 0.25$. *Also, calculate error at each point using FDM.*

Solution. *Case* I: Consider $h = 0.5$.
Here interval is $0 \le x \le 1$, i.e., $[0,1]$. We discretize the interval into subintervals with mesh size $h = 0.5$.
So $x_0 = 0$, $x_1 = x_0 + h = 0.5 = \frac{1}{2}$, $x_2 = x_0 + 2h = 1$.
There are only three values of x, so we have three points.
Dirichlet's boundary conditions are given by $u(0) = 0$ and $u(1) = 0$.
That means, $u(0) = u(x_0) = u_0 = 0$ and $u(1) = u(x_2) = u_2 = 0$.
We have to find $u(0.5) = u_1$.
The given differential equation is approximated as

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + u_i + 1 = 0.$$

$$\Rightarrow \quad u_{i-1} - (2 - h^2)u_i + u_{i+1} = -h^2, \quad i = 1, 2, \ldots, n-1 \tag{4.1}$$

Using boundary conditions $u_0 = 0$ and $u_n = u_2 = 0$. Since number of sub-interval $n = 2$ and unknown point is 1, then take $i = 1$.
Eq. (4.1) becomes $u_0 - \left(2 - \frac{1}{4}\right)u_1 + u_2 = -\frac{1}{4}$.
Using BCS (*boundary conditions*), $u_0 = 0$ and $u_2 = 0$

$$u_1 = u(0.5) = \frac{1}{7} = 0.142854.$$

Exact Solution. To find exact solution of the given differential equation

$$u'' + u + 1 = 0.$$

$$\Rightarrow \quad (D^2 + 1)u = -1.$$

$$\text{AE} \quad m^2 + 1 = 0 \quad \text{or} \quad m = \pm i$$

$$\text{CF} \quad u_c = C_1 \cos x + C_2 \sin x$$

$$\text{PI} = \frac{1}{D^2 + 1}(-1) = -1$$

The general solution is

$$u(x) = C_1 \cos x + C_2 \sin x - 1. \tag{4.2}$$

Using BCS $u(0) = 0$ in eq. (4.2), we get

$$C_1 = 1.$$

Using BCS $u(1) = 0$ in eq. (4.2), we get

$$C_2 = 0.5463.$$

Therefore, the obtained exact solution is given by

$$u(x) = \cos x + (0.5463)\sin x - 1.$$

The value of y at $x = 0.5$ is

$$u(0.5) = 0.139493.$$

Error at $x = 0.5$ is $|0.139493 - 0.142854| = 0.003361$.

Case II: $h = 0.25$.

Here interval is $0 \le x \le 1$, i.e., $[0,1]$. We discretize the interval into subintervals with mesh size $h = 0.25$.

So $x_0 = 0$, $x_1 = x_0 + h = 0.25$, $x_2 = x_0 + 2h = 0.5$, $x_3 = x_0 + 3h = 0.75$, $x_4 = x_0 + 4h = 1$.

Using $h = 0.25$, eq. (4.1) becomes

$$u_{i-1} - \frac{31}{16}u_i + u_{i+1} = -\frac{1}{16}, \quad i = 1,2,\dots,n-1. \tag{4.3}$$

Put $i = 1,2,3$ (unknown points) in eq. (4.3), system of linear equations is

$$u_0 - \frac{31}{16}u_1 + u_2 = -\frac{1}{16},$$
$$u_1 - \frac{31}{16}u_2 + u_3 = -\frac{1}{16},$$
$$u_2 - \frac{31}{16}u_3 + u_4 = -\frac{1}{16},$$

where $u_0 = 0$ and $u_4 = 0$ (due to BCS).

The above equations become

$$-\frac{31}{16}u_1 + u_2 = -\frac{1}{16},$$
$$u_1 - \frac{31}{16}u_2 + u_3 = -\frac{1}{16},$$
$$u_2 - \frac{31}{16}u_3 = -\frac{1}{16}.$$

Solving system of equations, we get

$$u_1 = y(0.25) = 0.10476,$$

$$u_2 = y(0.5) = 0.14031,$$

$$u_3 = y(0.75) = 0.10476.$$

Using eq. (4.3), exact solutions at each point are

$$u_1 = y(0.25) = 0.1041,$$

$$u_2 = y(0.5) = 0.1395,$$

$$u_3 = y(0.75) = 0.1041.$$

Errors are 0.0006, 0.0008, 0.0006 at $x = 0.25, 0.5, 0.75$.

Since the ratio of two errors in *Case* I and *Case* II is about 4, it follows that the order of convergence is h^2. These results show that the accuracy of numerical method depends upon the mesh size h which is selected. As h is decreased, the accuracy increases but the number of equation to be increased for solving.

Problem 2 (Neumann Boundary Conditions)**.** *Solve the ill-posed BVP defined by* $u''(x)-(1-x)u'(x)+xu(x)=x$*,* $u'(0)=0$ *and* $u(1)=0$*, using finite difference method with mesh size* $h=\frac{1}{3}$*.*

Solution. Here interval is $0\le x\le 1$, i.e., $[0,1]$.
Points are given by $x_0=0$, $x_1=x_0+h=\frac{1}{3}$, $x_2=x_0+2h=\frac{2}{3}$, $x_3=x_0+3h=\frac{3}{3}=1$.
Therefore, $h=\frac{1}{3}$.
To solve the equation $u_i''+(1-x_i)u_i'+x_iu_i=x_i$ use finite difference approximation

$$\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}+(1-x_i)\frac{u_{i+1}-u_{i-1}}{2h}+x_iu_i=x_i$$

$$\Rightarrow \quad (2-h+x_ih)u_{i-1}-(4-2h^2x_i)u_i+(2+1-x_ih)u_{i+1}=x_ih^2$$

$$\Rightarrow \quad \left(\frac{5}{3}+\frac{x_i}{3}\right)u_{i-1}-\left(4-\frac{2}{9}x_i\right)u_i+\left(3-\frac{x_i}{3}\right)u_{i+1}=\frac{x_i}{9} \tag{4.4}$$

Given boundary conditions are

$$u'(0)=0$$
$$\Rightarrow \quad u_0'=0$$
$$\Rightarrow \quad \frac{u_1-u_{-1}}{2h}=0$$
$$\Rightarrow \quad u_1=u_{-1}$$
$$u(1)=0$$
$$\Rightarrow \quad u_3=0$$

Putting $i=0,1,2$ (unknown points) in eq. (4.4), we get

$$\frac{5}{3}u_{-1}-4u_0+3u_1=0$$
$$6u_0-7u_1=0 \quad (\text{because } u_1=u_{-1})$$
$$16u_0-\frac{106}{3}u_1+26u_2=\frac{1}{3}$$
$$17u_1-\frac{104}{3}u_2=\frac{2}{3}$$

Solving above three equations, we get the numerical approximations

$$u_0=-8.342,\ u_1=-7.15u_2=-3.986.$$

Problem 3 (Robin Boundary Conditions)**.** *Determine the temperature distribution from mixed BVP defined by* $u''(x)-xu(x)=0$*,* $0\le x\le 1$*, for* $u(x_i)$*,* $x_i=0,\ \frac{1}{3},\ \frac{2}{3}$*, with BC* $u(0)+u'(0)=1$ *and* $u(1)=1$*, using finite difference method.*

Solution. Here interval is $0\le x\le 1$, i.e., $[0,1]$.
Points are given by $x_0=0$, $x_1=x_0+h=\frac{1}{3}$, $x_2=x_0+2h=\frac{2}{3}$, $x_3=x_0+3h=\frac{3}{3}=1$.
Therefore, $h=\frac{1}{3}$.

Mixed boundary conditions are given by $u(0)+u'(0)=1$ and $y(1)=1$.
That means, $u(0)+u'(0)=u(x_0)+u'(x_0)=1$ and $u(1)=u(x_3)=u_3=1$.
We have to find u_0, u_1, u_2.
The given differential equation is approximated as

$$\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}=x_iu_i\,,$$

$$u_{i-1}-(2+h^2x_i)u_i+u_{i+1}=0, \quad i=0,1,2. \tag{4.5}$$

Putting $i=0,1,2$ (unknown points) in eq. (4.5), we get

$$\begin{aligned}
&u_{-1}-2u_0+u_1=0\,, \quad (\text{because } x_0=0)\\
&u_0-\frac{55}{27}u_1+u_2=0\,, \quad \left(\text{because } x_1=\frac{1}{3}\right)\\
&u_1-\frac{56}{27}u_2+u_3=0\,. \quad \left(\text{because } x_2=\frac{2}{3}\right)
\end{aligned} \tag{4.6}$$

The first boundary condition is

$$\begin{aligned}
&u(0)+u'(0)=1\\
\Rightarrow\quad &u_0+u_0'=1\\
\Rightarrow\quad &u_0+\frac{u_1-u_{-1}}{2h}=1 \quad (\text{use centre difference approximation})\\
\Rightarrow\quad &2u_0+3(u_1-u_{-1})=1\\
\Rightarrow\quad &u_{-1}=\frac{2u_0+3u_1-2}{3}
\end{aligned}$$

The second boundary condition is

$$u(1)=u_3=1\,.$$

Using BCS, eqs. (4.6) become the system of linear equations given by

$$\begin{aligned}
&-2u_0+3u_1=1\,,\\
&u_0-\frac{52}{27}u_1+u_2=0\,,\\
&u_1-\frac{55}{27}u_2=-1\,.
\end{aligned}$$

Solving the above equations, we get the numerical approximations

$$\begin{aligned}
&u_0=u(0)=\frac{13}{55}=0.23636\,,\\
&u_1=u\,(1/3)=\frac{27}{55}=0.49091\,,\\
&u_2=u\,(2/3)=\frac{39}{55}=0.70909\,.
\end{aligned}$$

5. Conclusion

This study focused in various boundary conditions to obtain solutions of 2nd order initial-boundary value problem with ODE. Finite difference method is used for predicting harmonic motion, damped and forced variation, current from electric circuit. In this study, three problems are consider in various boundary conditions. The numerical results are examined with analytical results with Dirichlet condition for checking accuracy of numerical method. From the results it is cleared that the numerical results are very closer with exact results. Results also show that if the mesh size is reduced finite difference method will give the better accuracy. Hence, this technique can be successfully applied in more complicated domain in future work.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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