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Research Article

The Representation of Duplication and Gluing of Almost Symmetric Numerical Semigroup With *RF*-matrices

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Abstract. In this paper, the duplication and gluing of almost numerical semigroup are given by using *RF*(*Row-Factorization*) matrices.

Keywords. Almost numerical semigroup; Ideals; Duplication; Gluing; RF-matrices

Mathematics Subject Classification (2020). 20M14; 20M25

1. Introduction

Almost symmetric numerical semigroups were studied by Barucci and Fröberg in 1997 ([1]). Goto *et al*. [5] gave the duplication of numerical semigroup. Rosales and García-Sánchez [8] calculated the half of the numerical semigroup. In 2012, Strazzanti calculated the type of the duplication of numerical semigroup with relative ideals. Moscariello introduced the *RF*(*Row-Factorization*) matrices which is very useful in the classification of almost symmetric numerical semigroups in 2016 ([7]). The gluing of numerical semigroups was given in the thesis of Rosales [9]. In this study, we present duplication and gluing of almost symmetric numerical semigroups with *RF*-matrices.

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2. Basic Definitions

Most of the definitions can also be found in [8] or [2].

Definition 2.1. A numerical semigroup S and a semigroup ideal $E \subseteq S$ produces a new numerical semigroup, denoted by $S \infty^b E$ (where b is any odd integer belonging to S), such that

$$S = \{x \in \mathbb{N} : 2x \in S \infty^b E\}$$

i.e.

$$S = \frac{S \infty^b E}{2}.$$

This new semigroup is called the numerical duplication of S with respect to E and b.

Definition 2.2. Let $\mathbb{N}$ is the set of non-negative integers and $S \subset \mathbb{N}$. If S is closed under the addition in $\mathbb{N}$ and $0 \in S$ and $\mathbb{N} \backslash S$ is finite then S is called a numerical semigroup, for all $s_1, s_2, \ldots, s_n \in S$ it is denoted by

$$S = \langle s_1, s_2, \ldots, s_n \rangle = \left\{ \sum_{i=1}^{n} s_i x_i : x_i \in \mathbb{N} \right\}$$

and the following is correct

$$(s_1, s_2, \ldots, s_n) = 1 \Leftrightarrow \mathbb{N} \backslash S \text{ is finite.}$$

Example 2.3. Let $S = \langle 3, 5 \rangle = \{3x_1 + 3x_2 : x_1, x_2 \in \mathbb{N}\} = \{0, 3, 5, 6, 8, 9, 10, \cdots\}$

(i) $0 \in S$,

(ii) for all $s_1, s_2 \in S$, there is $x_1, x_2, y_1, y_2 \in \mathbb{N}$ such that $s_1 = 3x_1 + 5x_2$, $s_2 = 3y_1 + 5y_2$ and $x + y = 3(x_1 + y_1) + 5(x_2 + y_2) \in S$,

(iii) $\mathbb{N} \backslash S = \{1, 2, 4, 7\}$ is finite so S is a numerical semigroup.

Definition 2.4. Let the numerical semigroup S is given by

$$S = \langle s_1, s_2, \ldots, s_n \rangle$$

then

(i) the number $m(S) = s_1$ is called the multiplicity of S,

(ii) the number $e(S) = n$ is called the embedding dimension of S [8].

Definition 2.5. Let S is a numerical semigroup the largest integer that is not in S is called the frobenius number of S and denoted by $F(S)$ ([8]), i.e.

$$F(S) = \max(\mathbb{N} \backslash S)$$

or

$$F(S) = \max\{x \in \mathbb{Z} : x \notin S\}.$$

Definition 2.6. The positive elements that is not in S and is denoted by $G(S)$. The elements of gaps is called genus of S and $g(S) = |G(S)|$.

Example 2.7. $S = \langle 8,13,15,19\rangle = \{0,8,13,15,16,19,21,23,24,26,27,28,29,30,31,32,34,\cdots\}$,

$G(S) = \{1,2,3,4,5,6,7,9,10,11,12,14,17,18,20,22,25,33\}$, the set of gaps,

$F(S) = 33$, the largest number that is not in S,

$m(S) = 8$, multiplicity,

$e(S) = 4$, embedding dimension,

$g(S) = 18$, the number of elements of gaps.

Definition 2.8. $PF(S)$ is the set of Pseudo-Frobenius number of S,

$$\begin{aligned} PF(S) &= \{x \notin S \mid x+s \in S, \text{ for every } s \in S\backslash\{0\}\} \\ &= \{x \notin S \mid x+n_i \in S, \text{ for every } i = 1,2,\dots,e\}. \end{aligned}$$

Definition 2.9. A numerical semigroup S is almost symmetric ([6,7]) if for every $x \in \mathbb{Z}\backslash S$ such that $F(S)-x \notin S$ we have $\{x, F(S)-x\} \subseteq PF(S)$.

Definition 2.10. If $f \in PF(S)$, then $f+n_i \in S$, for every $i = 1,\dots,e$, hence there exist $a_{i1}, a_{i2}, \dots, a_{ie} \in \mathbb{N}$ such that

$$f + n_t = \sum_{j=1}^{e} \alpha_{ij} n_j .$$

Nevertheless, $\alpha_{ii} > 0$ would imply $f \in S$; thus $\alpha_{ii} = 0$. Thus, for every i, there exist $a_{i1}, a_{i2}, \dots, a_{ie} \in \mathbb{N}$ such that

$$f = \sum_{j=1}^{e} \alpha_{ij} n_j \quad \text{and} \quad a_{ii} = -1 .$$

Let $S = \langle n_1, n_2, \dots, n_e\rangle$ be a numerical semigroup and $f \in PF(S)$.

We say that $A = (a_{ij}) \in M_e(\mathbb{Z})$ is an *RF*-matrix (*Row-Factorization* matrix) for f if $a_{ii} = -1$ for every $i = 1,2,\dots,e$, $a_{ij} \in \mathbb{N}$ if $i \neq j$ and for every $i = 1,2,\dots,e$

$$\sum_{j=1}^{e} \alpha_{ij} n_j = f .$$

If S is almost symmetric and $f \in PF(S)\backslash\{F(S)\}$, there exists an *RF*-matrix for both f and $F(S)-f$. In general, this matrix is not unique.

Example 2.11. Let $S = \langle 6,7,9,10\rangle = \{0,6,7,9,10,12,13,14,15,16,17,\cdots\}$,

$$F(S) = 11,$$

$$G(S) = N\backslash S = \{1,2,3,4,5,8,11\},$$

$$n_1 = 6,\ n_2 = 7,\ n_3 = 9,\ n_4 = 10.$$

Let take $f = 8 \in PF(S)$ and let try to write *RF*-matrices of f. Firstly, we find the numbers $a_{12}, a_{13}, a_{14} \in \mathbb{N}$ such that,

$$8 = a_{11}6 + a_{12}7 + a_{13}9 + a_{14}10,$$

where $a_{11} = -1$ from the equality,

$$8 = (-1).6 + 2.7 + 0.9 + 0.10$$

We find $a_{12} = 2$, $a_{13} = 0$, $a_{14} = 0$.
Hence, first row of the RF-matrices is found for $f = 8PF(S)$.
In a similar manner, the numbers is found $a_{21} = 1$, $a_{23} = 1$, $a_{24} = 0$ such that

$$8 = 1.6 + (-1).7 + 1.9 + 0.10,$$

where $a_{21} = -1$.
This gives the second row of the matrix go on in this way, RF-matrix of $f = 8 \in PF(S)$

$$\begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \end{pmatrix}.$$

Also, if we consider,

$$8 = 3.6 + 0.7 + 0.9 + (-1).10.$$

Then alternative the RF-matrices of f turns to

$$\begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 3 & 0 & 0 & -1 \end{pmatrix}.$$

This proves that RF-matrices are not unique.

Proposition 2.12. *Let $f, f' \notin PF(S)$ and $f + f' \notin S$. Take $RF(f) = A = (a_{ij})$ and $RF(f') = B = (b_{ij})$ in this case, for all $i \neq j$, $a_{ij} = 0$ or $b_{ij} = 0$ in particular if $RF\left(\frac{F(S)}{2}\right) = (a_{ij})$ then, for all $i \neq j$, $a_{ij} = 0$ or $a_{ji} = 0$.*

Proof. See [2]. □

Proposition 2.13. *A set $E \subseteq \mathbb{Z}$ is said to be a relative ideal of S if $S + E \subseteq E$ (i.e. $s + t \in E$ for every $s \in S$ and $t \in E$) and there exist $s \in S$ such that $S + E = \{s + t : t \in E\} \subseteq S$. Relative ideals contained in S are simply called ideals.*
If E and F are relative ideals of S then

$$E + F = \{t + u : t \in E, u \in F\} \text{ and } E - F = \{z \in \mathbb{Z} : z + u \in E, \text{ for all } u \in F\}$$

are both relative ideals of S.
One can always shift a relative ideal E adding to it an integer x

$$x + E = \{x + e : e \in E\}.$$

Definition 2.14. The basic ideal is generated by are ideal element if $Z \in \mathbb{Z}$, then

$$z + s = \{z + s : s \in S\}$$

is the basic ideal generated by the element $\mathbb{Z}$.

Generally, the generation of an ideal E by $e_1, e_2, \ldots, e_n$ means that

$$E = (e_1, e_2, \ldots, e_n) = (e_1 + S) \cup (e_2 + S) \cup \ldots \cup (e_n + S). \qquad ([3])$$

Definition 2.15. Set $2S = \{2s : s \in S\}$ and $2E = \{2t : t \in E\}$. Let $b \in S$ be an add integer. Then, we define the numerical duplication $S \infty^b E$ of S with respect to E and b as the following subset of $\mathbb{N}$

$$S \infty^b E = 2S \cup (2E + b).$$

Proposition 2.16. *The following properties hold for $S \infty^b E$;*

(i) $f(S \infty^b E) = 2f(E) + b$,

(ii) $g(S \infty^b E) = g + g(E) + m(E) + \dfrac{b-1}{2}$.

Proof. See [3]. □

3. The Duplication of Numerical Semigroups With *RF*-matrices

The numerical duplication of a numerical semigroup has been studied by Anna and Strazzanti [3]. In this section, the duplication of numerical semigroups is given by RF-matrices.

Remark 3.1. If the numerical semigroup S is generated minimally by $s_1, s_2, \ldots, s_n$ and ideal E is generated minimally by $e_1, e_2, \ldots, e_m$ then, the numerical duplication of S with respect to E and b is generated by $2s_1, 2s_2, \ldots, 2s_n, 2e_1 + b, \ldots, 2e_m + b$.

Example 3.2. Let $S = \langle 3, 7, 11 \rangle = \{0, 3, 6, 7, 9, \cdots\}$,

$$1 + S = \{1, 4, 7, 8, 10, \cdots\},$$

$$2 + S = \{2, 5, 8, 9, 11, \cdots\},$$

$$E = (1 + S) \cup (2 + S) = \{1, 2, 4, 5, 7, \cdots\},$$

$$2E = \{2, 4, 8, 10, 14, 16, 18, \cdots\}.$$

Let $b = 3 \in S$

$$2E + b = \{5, 7, 11, 13, 17, 19, 21, \cdots\},$$

$$2S = \{0, 6, 12, 14, 18, 20, 22, \cdots\},$$

$$S \infty^b E = 2S \cup (2E + b) - \{0, 5, 6, 7, 11, 12, 13, 14, 17, \cdots\}$$

Generators are $6, 14, 22, 5, 7$.
Let write the RF-matrix of $PF(S \infty^b E) = \{15\}$.
By giving the -1 value to the variables x, y, z, w and q in the equation $6x + 5y + 7z + 14w + 22q = 15$, separately. For instance, give -1 to the x and the other numbers satisfy the equation and this gives the first row of the RF-matrix.

As a result, RF-matrix for $PF(S\infty^b E) = \{15\}$ is the following:

$$A = \begin{pmatrix} -1 & 1 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 3 & 0 & 3 & 0 & -1 \end{pmatrix}.$$

Example 3.3. Let $S = \langle 4,7,13\rangle$ and $E = (0,2)$, $b = 7$,

$$S\infty^b E = \{0,7,8,11,14,15,16,19,21,22,23,\cdots\}.$$

Generators are $8,14,26,7$ and RF-matrix for $PF(S\infty^b E) = \{20\}$ is

$$\begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 2 & 0 & 1 & -1 \end{pmatrix}.$$

4. The Gluing of Numerical Semigroups With *RF*-matrices

In this section, it will be studied the gluing of numerical semigroups via RF-matrices. The concept of gluing has been given by Rosales and García-Sánchez in ([8]).

Definition 4.1. Let S_1, S_2 be numerical semigroups, $d_1 \in S_2$ and $d_2 \in S_1$ if d_1 and d_2 are coprime then,

$$S = d_1S_1 + d_2S_2 = \{d_1x + d_2y : x \in S_1, y \in S_2\}$$

is a numerical semigroup. We say that S is glued by S_1 and S_2.
Assume that S is glued by S_1 and S_2 then,

$$F(S) = d_1F(S_1) + d_2F(S_2) + d_1d_2.$$

holds.

Theorem 4.2. *Let $S_1 = \langle n_1,\dots,n_{s_1}\rangle$, $S_2 = \langle n'_1,\dots,n'_{s_2}\rangle$ be numerical semigroups and put $S = d_1S_1 + d_2S_2$, where $d_1 \in S_2$, $d_2 \in S_1$. Assume $f_1 \notin S_1$ (resp. $f_2 \notin S_2$) and $f_1 + d_2 \in S_1$ (resp. $f_2 + d_1 \in S_2$), $f = d_1f_1 + d_2f_2 \notin S$. Further, if we write $f_1 + d_2 = \sum_i a_in_i$ and $f_2 + d_1 = \sum_i a'_in'_i$ then we have*

$$RF(F) = \begin{pmatrix} RF(f_1) & N_2 \\ N_1 & RF(f_2) \end{pmatrix},$$

where N_1 (resp. N_2) is an $s_2 \times s_1$-matrix (resp. $s_1 \times s_2$-matrix) whose ij-entry is a_i (resp. a'_i) for each i,j. Further,

$$\det RF(f) = \frac{-f(\det RF(f_1))\cdot(\det RF(f))}{f_1f_2}.$$

Therefore, if $RF(f_1) = (-1)^{s_1+1}f_1$ and $RF(f_2) = (-1)^{s_2+1}f_2$ then $RF(f) = (-1)^{s_1+s_2+1}f$.

Proof. See [4]. □

Example 4.3. Take $S_1 = \langle 6,7\rangle$ and $S_2 = \langle 8,9\rangle$, let examine the gluing of these two numerical semigroups.
For $S_1 = \langle 6,7\rangle$, $F(S_1) = 29$.
For $S_2 = \langle 8,9\rangle$, $F(S_2) = 55$.
Now, let us find the gluing of S_1 and S_2. Choose p and q such that,

$$p = 13 \in S_1 - \langle 6,7\rangle \quad \text{and} \quad q = 17 \in S_2 - \langle 8,9\rangle\,.$$

Therefore, it is clear that $p = 13 = 1.6 + 1.7$ and $q = 17 = 1.8 + 1.9$.
Thus, it is obtained

$$\begin{aligned} S &= \langle 17.6, 17.7, 13.8, 13.9\rangle \\ &= \langle 102, 117, 104, 117\rangle\,. \end{aligned}$$

From Definition 4.1, Frobenius number is $F(S) = 1429$.
Now, let us find the RF-matrices of S_1 and S_2.
For $S_1 = \langle 6,7\rangle$, take $n_1 = 6$, $n_2 = 7, 29 \in PF(S_1)$.
Write $a_{ii} = -1$ and $\sum_{j=1}^{2} a_{ij} n_j = 29$, i.e.,

$$a_{11}n_1 + a_{12}n_2 = 29\,,$$

$$a_{21}n_1 + a_{22}n_2 = 29$$

in particular

$$(-1)\cdot 6 + (5)\cdot 7 = 29,$$

$$(6)\cdot 6 + (-1)\cdot 7 = 29\,.$$

So, the RF-matrix of $PF(S_1) = 29$ is

$$\begin{pmatrix} -1 & 5 \\ 6 & -1 \end{pmatrix}.$$

For $S_2 = \langle 8,9\rangle$, let us take $n_1 = 8, n_2 = 9$ are generators. $55 \in PF(S_2)$ and setting $a_{ii} = -1$ and $\sum_{j=1}^{2} a_{ij} n_j = 55$ the equations, similarly, we obtain the RF-matrix of $55 \in PF(S_2)$ is

$$\begin{pmatrix} -1 & 7 \\ 8 & -1 \end{pmatrix}.$$

Therefore, from Theorem 4.2, the RF-matrix of S is

$$RF(1429) = \begin{pmatrix} RF(29) & M_1 \\ M_2 & RF(55) \end{pmatrix},$$

$$RF(185) = \begin{pmatrix} -1 & 5 & 1 & 1 \\ 6 & -1 & 1 & 1 \\ 1 & 1 & -1 & 7 \\ 1 & 1 & 8 & -1 \end{pmatrix}.$$

5. Conclusion

In this study, we represent the numerical duplication of a numerical semigroups that was examined by Anna and Strazzanti in 2012 [3] with the help of RF-matrices. Also, we give the gluing of the numerical semigroups via RF-matrices this was firstly introduced by A. Moscariello [7]. He use it to prove that the upper bound of the type of almost symmetric semigroup of embedding dimension four is three.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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