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Research Article

Some Results on Generalized Derivations of BH-Algebras

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Abstract. In this paper, we study the notion of generalized derivations on BH-algebras and investigate simple, interesting and elegant results.

Keywords. BH-algebras, BH-subalgebras, Generalized derivations on BH-algebras, Regular generalized derivations on BH-algebras

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1. Introduction

Imai and Iséki [6, 7] introduced the axiom system of propositional calculi and have been extensively investigated by many researchers. Iséki and Tanaka [8] introduced the theory of BCK-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Zhang *et al*. [15] introduced the notion of BH-algebras. They investigated several relations between BH-algebras and BCK-algebras. In 1957, Posner [13] introduced the notion of derivations in prime rings theory. Also, Lee and Lee [11] developed on derivations of prime rings.

The notion of derivations in ring theory is quite old and plays an important role in algebras. Al-Shehrie [2] introduced the notion of derivations of B-algebras. Many research papers have appeared on the derivations of BCI-algebras in different ways. Al-Roqi [1] introduced the notion

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of on generalized (α,β) derivations in BCI-algebras. Bawazeer *et al*. [4] introduced the notion of generalized derivations of BCC-algebras. Also, Kamali and Davvaz [3] developed the properties in generalized derivations of BCI-algebras. Jun and Xin [10] introduced the notion of derivations of BCI-algebras. Javed and Aslam [9] introduced the concept of f-derivations in BCI-algebras. Also, Zhan and Liu [14] developed the notion of f-derivations on BCI-algebras. Muhiuddin and Al-Roqi [12] introduced on t-derivations of BCI-algebras.

Recently, in the year 2019 Ganesan and Kandaraj [5] defined and studied the notion of various derivations such as derivations, compositions of derivations, f-derivations, composition of f-derivations, t-derivations and regular t-derivations of BH-algebras. Using the idea of regular derivations in BH-algebras and obtained some of its properties. The term algebra is used here to denote the algebraic structure defined on a non-empty set with a binary composition satisfying certain laws that resemble the algebras of logic but not the usual algebras.

The notion of the derivations is the same as that in ring theory and the usual algebraic theory. Motivated by a lot of work done on derivations of BH-algebras and on derivations of other related abstract algebraic structures such as TM-algebras and d-algebras. In this paper, we introduce the notion of generalized derivations of BH-algebras and investigate simple, interesting and elegant results.

2. Preliminaries

We review some basic definitions and properties that will be useful in our results,

Definition 2.1 ([15])**.** Let X be a set X with a binary operation $*$ and a constant 0. Then $(X,*,0)$ is called a BH-algebra, if it satisfies the following axioms:

(i) $x*x=0$,

(ii) $x*0=x$,

(iii) if $x*y=0$ and $y*x=0 \Rightarrow x=y$, for all $x,y\in X$.

Define a binary relation $\le$ on X by taking $x\le y$ if and only if $x*y=0$. In this case, $(X,\le)$ is a partially ordered set [4].
Let $(X,*,0)$ be a BH-algebra and $x\in X$. Define $x*X=\{x*y \mid y\in X\}$.
Then X is said to be edge BH-algebra if for any $x\in X$, $x*X=\{x,0\}$.

Definition 2.2 ([15])**.** Let S be a nonempty subset of a BH-algebra X. Then S is called subalgebra of X, if $x*y\in S$, for all $x,y\in S$.

Definition 2.3 ([15])**.** Let X be a BH-algebra and I $(\neq 0)\subseteq X$. Then I is called a BH-ideal of X if

(a) $0\in I$,

(b) $x*y\in I$ and $y\in I\Rightarrow x\in I$, for all $x,y\in I$.

In BH-algebra X, for all $x,y,z\in X$, the following property hold [15]

(i) $((x*y)*(x*z))*(z*y)=0$,

(ii) $(x*y)*x=0$,

(iii) $(x*(x*y))=y$.

Theorem 2.4 ([15]). *Every BH-algebra satisfying the condition* (i) *is a BCI-algebra and satisfying the conditions* (i) *and* (ii) *is a BCK-algebra.*

Theorem 2.5 ([15]). *Every BH-algebra satisfying the condition*

$$(x*y)*z=(x*z)*y, \quad \text{for all } x,y,z\in X$$

is a BCH-algebra.
For a BH-algebra X, we denote $x\wedge y$ for $y*(y*x)$, for all $x,y\in X$.

3. Generalized Derivations of BH-Algebras

Definition 3.1. Let U be a BH-algebra. A mapping $G:U\to U$ is called a generalized left-right (briefly (l,r)-derivation) if there exists an left-right derivation $g:U\to U$ such that

$$G(u*v)=(G(u)*v)\wedge(u*g(v)), \quad \text{for all } u,v\in U.$$

If there exists an right-left derivation (briefly (r,l)-derivation) $g:U\to U$ such that $G(u*v)=(u*G(v))\wedge(g(u)*v)$, for all $u,v\in U$. A mapping $G:U\to U$ is called a generalized (r,l)-derivation. Moreover, if G is both a generalized (l,r) and (r,l)-derivation, then G is called generalized derivation.

Example 3.2. Let $U=\{0,a,b,c\}$ be a BH-algebra with the following Cayley Table 1.

Table 1

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	a	0

Define a map $g:U\to U$ such that $g(u)=\begin{cases}0 & \text{if } u=0,a,c,\\ b & \text{if } u=b.\end{cases}$

Then g is a derivation of U.

Define a map $G:U\to U$ such that $G(u)=\begin{cases}0 & \text{if } u=0,b,\\ b & \text{if } u=a,c.\end{cases}$

It is easily checked that G is a generalized right-left derivation of U.

Example 3.3. Let $U=\{0,a,b,c\}$ be a BH-algebra with Table 1 in Example 3.2.

Define a self-map $g(u)=\begin{cases}u & \text{if } u=a,b,\\ 0 & \text{if } u=0,c.\end{cases}$

Then g is a left-right derivation of U.

Define a map $G:U\to U$ such that $G(u)=\begin{cases}0 & \text{if } u=0,a,b,\\ c & \text{if } u=c.\end{cases}$

It is easily verify that the condition $(u*v)=(G(u)*v)\wedge(u*g(v))$, for all $u,v\in U$.
Therefore, G is generalized (l,r)-derivation of U.

Proposition 3.4. *Let G be a self-map on BH-algebra U, then*

(i) *if G is a generalized left-right derivation on U, then $G(u) = G(u) \wedge u$, for all $u \in U$,*

(ii) *if G is a generalized right-left derivation on U, then $G(u) = u \wedge g(u)$, for all $u \in U$.*

Proof. (i) Since G is a generalized left-right derivation of U, then there exists an (l,r)-derivation g such that

$$G(u * v) = (G(u) * v) \wedge (u * g(v)), \quad \text{for all } u, v \in U.$$

Now

$$\begin{aligned} G(u) &= G(u * 0) \\ &= (G(u) * 0) \wedge (u * g(0)) \\ &= G(u) \wedge (u * g(0)) \\ &= (u * g(0)) * ((u * g(0)) * G(u)) \\ &= (u * g(0)) * ((u * G(u)) * g(0)) \\ &= u * (u * G(u)) \\ &= G(u) \wedge u. \end{aligned}$$

Hence $G(u) = G(u) \wedge u$.

(ii) Since G is generalized right-left derivation on U, then there exists an (r,l)-derivation g such that

$$(u * v) = (u * G(v)) \wedge (g(u) * v), \quad \text{for all } u, v \in U.$$

We have

$$\begin{aligned} G(u) &= G(u * 0) \\ &= (u * G(0)) \wedge (g(u) * 0) \\ &= (u * G(0)) \wedge g(u) \\ &= g(u) * (g(u) * (u * G(0))) \\ &= g(u) * (g(u) * u) \\ &= u \wedge g(u). \end{aligned}$$

□

Proposition 3.5. *Let U be a BH-algebra with partial order $\leq$, and let G be a generalized derivation of U, then the following results are hold for all $u, v \in U$.*

(i) $G(u) \leq g(u) \leq u$,

(ii) $G(u * v) \leq u * g(v)$,

(iii) $G(u * v) \leq g(u) * v$,

(iv) $G(u * G(u) = 0$,

(v) $G(G(u) * u) = 0$,

(vi) $G(g(u) * u) = 0$,

(vii) $G(u * g(u)) = 0$,

(viii) $G(G(u)) \leq u$.

Proof. (i) Now $G(u) = G(u * 0) = (u * G(0)) \wedge (g(u) * 0) = u \wedge g(u) \leq g(u)$.
From proposition of BH-algebra $g(u) \leq u$ (since $d(x) \leq x$).
(ii) $G(u * v) = (G(u) * v) \wedge (u * g(v)) \leq u * g(v))$.
(iii) $G(u * v) = (u * G(v)) \wedge (g(u) * v) \leq g(u) * v$.
(iv) $G(u * G(u)) = (G(u) * G(u)) \wedge (u * g(G(u))) = 0 \wedge (u * g(G(u))) = 0$ (since $y * (y * x) = x$).
(v) $(G(u) * u) = (G(u) * G(u)) \wedge (g(G(u)) * u) = 0 \wedge (g(G(u)) * u) = 0$.
(vi) $G(g(u) * u) = (G(g(u)) * u) \wedge (g(u) * g(u)) = (G(g(u)) * u) \wedge 0 = 0 * (0 * (G(g(u)) * u)) = 0$.
(vii) $G(u * g(u)) = (u * G(g(u))) \wedge (g(u) * g(u)) = (u * G(g(u))) \wedge 0 = 0 * (0 * (u * G(g(u)))) = 0$.
(viii) We have

$$\begin{aligned} G(G(u)) &= G(u \wedge g(u)) \\ &= G(g(u) * (g(u) * u)) \\ &= (g(u) * G(g(u) * u)) \wedge (g(g(u)) * (g(u) * u)) \\ &= (g(u) * G(0)) \wedge (g(g(u)) * 0) \\ &= (g(u) * 0) \wedge g(g(u)) \\ &= g(u) \wedge g(g(u)) \\ &\leq g(g(u)) \leq u. \end{aligned}$$

□

Similarly, we can prove the following theorem.

Theorem 3.6. *Let U be a BH-algebra. Then $G_n(G_{n-1}(\dots(G_2(G_1(u)))\dots)) \leq u$ for $n \in N$, where $G_1, G_2, \dots, G_n$ are generalized derivations of U.*

Definition 3.7. Let G be a generalized derivation of BH-algebra U. An ideal I of U is said to be G-invariant if $G(I) \subseteq I$, where $G(I) = \{G(u) \mid u \in I\}$.

Theorem 3.8. *Let$G$ be a generalized derivation of a BH-algebra U, then every ideal I of U is G-invariant.*

Proof. Let I be an ideal of a BH-algebra U.
Let $v \in G(I)$.
Then $v = G(u)$ for some $u \in I$.
This implies that

$$\begin{aligned} v * u &= G(u) * u \\ &= 0 \in I \\ &\Rightarrow v \in I. \end{aligned}$$

Hence $G(I) \subseteq I$.
Therefore, I is G-invariant. □

Definition 3.9. Let U be a BH-algebra and let G be a generalized derivation. Define a $\ker G$ by $\ker G = \{u \in U \mid G(u) = 0\}$.

Theorem 3.10. *Let U be a BH-algebra and let G be a generalized derivation. If $v \in \ker(G)$ and $u \in U$, then $u \wedge v \in \ker(G)$.*

Proof. Let $v \in \ker(G)$. Therefore, $G(v) = 0$.
Now

$$\begin{aligned} G(u * v) &= G(v * (v * u)) \\ &= (G(v) * (v * u)) \wedge (v * g(v * u)) \\ &= (0 * (v * u)) \wedge (v * g(v * u)) \\ &= 0 \wedge (v * g(v * u)) \\ &= 0 \end{aligned}$$

that is, $G(u \wedge v) = 0$.
Therefore, $u \wedge v$ belongs to $\ker(G)$. □

Theorem 3.11. *Let U be a commutative BH-algebra and let G be a generalized derivation. If $u \leq v$ and $v \in \ker(G)$, then $u \in \ker(G)$.*

Proof. Let $u \leq v$ and $v \in \ker(G)$. We have $u * v = 0$ and $G(v) = 0$ and so

$$\begin{aligned} G(u) &= G(u * 0) \\ &= G(u * (u * v)) \\ &= G(v * (v * u)) \\ &= (G(v) * (v * u)) \wedge (v * g(v * u)) \\ &= (0 * (v * u)) \wedge (v * g(v * u)) \\ &= 0 \wedge (v * g(v * u)) = 0. \end{aligned}$$

Hence $u \in \ker(G)$. □

Theorem 3.12. *Let U be a commutative BH-algebra and let G be a generalized derivation. If $u \in \ker(G)$, one has then $u * v \in \ker(G)$, for all $v \in U$.*

Proof. Let $u \in \ker(G)$. Then $G(u) = 0$. By definition

$$\begin{aligned} G(u * v) &= (G(u) * v) \wedge (u * g(v)) \\ &= (0 * v) \wedge (u * g(v)) \\ &= 0 \wedge (u * g(v)) = 0. \end{aligned}$$

Hence $G(u * v) = 0 \Rightarrow u * v \in \ker(G)$. □

Therefore, we have following theorem.

Theorem 3.13. *Let U be a BH-algebra and let G be a generalized derivation. Then $\ker(G)$ is subalgebra of U.*

Definition 3.14. Let U be a BH-algebra and let G be a generalized derivation on U. Define $\mathrm{Fix}_G(U) = \{u \in U \mid G(u) = u\}$.

Proposition 3.15. *Let U be a BH-algebra and let G be a generalized derivation on U. If $u \in \mathrm{Fix}_G(U)$, then $g(u) = u$.*

Proof. Since $u \in \mathrm{Fix}_G(U)$, then by definition $G(u) = u$.
From Proposition 3.5, we have $G(u) \le g(u) \le u$, this implies that $u \le g(u) \le u$.
Hence $g(u) = u$. □

Proposition 3.16. *Let U be a BH-algebra and let G be a generalized derivation on U. Then $\mathrm{Fix}_G(U)$ is subalgebra of U.*

Proof. If $u, v \in \mathrm{Fix}_G(U)$ we get $G(u) = u$ and $g(v) = v$. Now

$$\begin{aligned} G(u * v) &= (G(u) * v) \wedge (u * g(v)) \\ &= (u * v) \wedge (u * v) \\ &= (u * v). \end{aligned}$$

Therefore, $(u * v) \in \mathrm{Fix}_G(U)$. □

Proposition 3.17. *Let U be a BH-algebra and let G be a generalized derivation on U. If $u, v \in \mathrm{Fix}_G(U)$ we have $u \wedge y \in \mathrm{Fix}_G(U)$.*

Proof. Let $u, v \in \mathrm{Fix}_G(U)$. Then $G(u) = u$ and $G(v) = v$.
From Proposition 3.17, we have $v * u \in \mathrm{Fix}_G(U)$ and so $g(v * u) = v * u$.
Now, we have

$$\begin{aligned} G(u \wedge v) &= G(v * (v * u)) \\ &= (G(v) * (v * u)) \wedge (v * g(v * u)) \\ &= (v * (v * u)) \wedge (v * (v * u)) \\ &= v * (v * u) \\ &= u * v. \end{aligned}$$

Therefore, $u \wedge v \in \mathrm{Fix}_G(U)$. □

Theorem 3.18. *Let G be a generalized derivation of a BH-algebra U. Then*

$$G(0) = G(u) * u, \quad \text{for all } u \in U.$$

Proof. Let G be a generalized derivation of a BH-algebra. Now

$$\begin{aligned} G(0) &= G(u * u) \\ &= (G(u) * u) \wedge (u * g(u)) \\ &= (u * g(u)) * ((u * g(u)) * (G(u) * u)) \\ &= (G(u) * u). \end{aligned}$$

Therefore $G(0) = G(u) * u$. □

Theorem 3.19. *Let U be a BH-algebra and let G be a generalized derivation on U. If $u \le v$ and $G(u * v) = G(u) * G(v)$ for all $u, v \in U$. Then $G(u) = G(v)$.*

Proof. Since $u \le v$ and $G(u * v) = G(u) * G(v)$

$$u \le v \Rightarrow u * v = 0.$$

Now

$$\begin{aligned} G(u) &= G(u * 0) \\ &= G(u * (u * v)) \\ &= G(u) * G(u * v) \\ &= G(u) * (G(u) * G(v)) \\ &= G(v). \end{aligned}$$

Therefore $G(u) = G(v)$. □

4. Conclusion

An algebraic structure that arises from the study of algebraic formulations of propositional logic. Taking different theorems or statements of propositional logic, different algebraic structures could be obtained. The BH-algebras is one such algebras. The derivations concepts are an important and very interesting area of research in the theory of algebraic structures in mathematics. The deep theory has been developed for derivations through various algebras. It plays an important role in algebras, algebraic geometry and linear differential equations.

We have considered the concepts of generalized derivations in BH-algebras. Finally, we investigated the notion of the some results on generalized derivations in BH-algebras. In our opinion these definitions and main results may be similarly extended to some other algebra such as BCI-algebras, d-algebras and B-algebras so forth. In future any Researcher can study the notion of generalized derivations in different algebraic structures which may have a lot of applications in various fields. This work is a foundation for the further study of the researcher on derivations of algebras

The future study of derivations on BH-algebras may be the following topics should be covered.

(a) To find the generalized derivations on d-algebras.

(b) To find the generalized derivations of Q-algebras, B-algebras and so on so.

(c) To find more results and its applications in generalized derivations on BH-algebras.

(d) To find to investigate how these concepts could be applied to the field of computers for processing information.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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