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Research Article

Idempotents and Ideals of Regular Rings

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Abstract. Multiplicative semigroups of rings form an important class of semigroups and one theme in the study of semigroups is how the structure of this semigroup affects the structure of the ring. An important tool in analyzing the structure of a semigroup is the Green's relations. We study some properties of these relations on the multiplicative semigroup of a regular ring with unity. This also gives easier proofs of some known results on ring theory.

Keywords. Idempotents; Green's relations; Ideals

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1. Introduction

All rings defined in this article are rings with unity [2]. The concept of Green's equivalences was introduced for semigroups and this can be extended to rings. If S is a semigroup, then the principal left(right) ideal of S generated by a is $S^1a(aS^1)$ where $S^1a = Sa \cup \{a\}(aS^1 = aS \cup \{a\})$. The principal two sided ideal [10] of S generated by a is S^1aS^1. J.A. Green (1951) [4] defined five equivalence relations on S, called Green's relations on S, by

$$a \mathscr{L} b \text{ iff } S^1a = S^1b$$

$$a \mathscr{R} b \text{ iff } aS^1 = bS^1$$

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$$a \mathscr{J} b \text{ iff } S^1as^1 = S^1bS^1$$
$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}$$
$$\mathscr{D} = \mathscr{L} \vee \mathscr{R} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$$

The respective equivalence classes [8] containing a are written as L_a, R_a, J_a, H_a and D_a. The elements in a $\mathscr{D}$-class D of a semigroup S are arranged in a rectangular pattern like an egg-box. The rows corresponds to $\mathscr{R}$-classes, the columns to $\mathscr{L}$-classes and each cell to an $\mathscr{H}$-class contained in D. An element $a \in S$ is regular if there exists an element $x \in S$ such that $axa = a$. If every element of S is regular, then S is called a regular semigroup [5]. A ring R is regular if every element of R is regular.

Let R be a ring with unity. Then R is a semigroup with identity[1] and so the Green's relations [9] on R can be defined by replacing S^1 in equations by R as

$$a \mathscr{L} b \text{ iff } Ra = Rb$$
$$a \mathscr{R} b \text{ iff } aR = bR$$
$$a \mathscr{J} b \text{ iff } RaR = RbR$$

An element $e \in R$ is an idempotent if $e^2 = e$. If $e, f \in R$ be two idempotents with $e \mathscr{R} f$, then $f = e + ex(1-e)$ for some $x \in R$ and if $e \mathscr{L} f$, then $f = e + (1-e)xe$ for some $x \in R$ (see [15, Lemma 2.7]). An idempotent f in an $\mathscr{R}$-class R_e is a left identity in R_e and an idempotent g in an $\mathscr{L}$-class L_e is a right identity in L_e (see [3, Lemma 2.14]).

2. Idempotents and Ideals

An extensive study of the class of principal left ideals and the class of principal right ideals of a regular ring is done in [15]. In this section, we show that many of these ideas can be described using ideas from the theory of semigroups, especially, the structure of the idempotents of a regular semigroup, as developed in [13]. Throughout this section, we will be mostly considering principal left ideals. All these ideas have obvious analogues for principal right ideals also.

The basic idea is that in a ring R with multiplicative identity, every principal left ideal is of the form Rx for some element x and if the ring is regular, then x can be replaced by an idempotent, as in the case of a regular semigroup. This correpondence between the class of principal left ideals of R and the set of idempotents in R allows us to translate many of the properties of principal left ideals to that of idempotents and vice versa [6]. Thus for idempotents in R, a regular ring with multiplicative identity 1 [7], we define the relations ω^l and ω^r by

$$e\,\omega^l\, f \iff ef = e$$
$$e\,\omega^r\, f \iff fe = e$$

Also, the Green's relations $\mathscr{L}$ and $\mathscr{R}$ on the (multiplicative semigroup of) R, applied to idempotents e and f, can be described by

$$e \mathscr{L} f \iff ef = e \text{ and } fe = f$$

$$e\,\mathscr{R}\,f \iff ef = f \text{ and } fe = e$$

(see [3, Lemma 2.14]). In other words, for idempotents e and f of R, $e\,\mathscr{L}\,f$ iff $e\,\omega^l\,f$ and $f\,\omega^l\,e$, and $e\,\mathscr{R}\,f$ iff $e\,\omega^r\,f$ and $f\,\omega^r\,e$. These describe the partial order on principal left ideals and principal right ideals of R.

Proposition 2.1. *Let e and f be idempotents in R. Then $Re \subseteq Rf$ if and only if $e\,\omega^l\,f$ and $eR \subseteq fR$ if and only if $e\,\omega^r\,f$. In particular, $e\,\mathscr{L}\,f$ if and only if $Re = Rf$ and $e\,\mathscr{R}\,f$ if and only if $eR = fR$.*

In the following, we describe some properties of the idempotents of R, and rephrase in terms of ideals those that lead to significant results.

One important fact which distinguishes the set of idempotents of a ring with multiplicative identity, from that of a semigroup, is that every idempotent e has an associated idempotent $1-e$ [12], since

$$(1-e)^2 = 1-2e+e^2 = 1-2e+e = 1-e.$$

We note that the map $e \mapsto 1-e$ reverses and dualizes the relations above (cf. [11, Corollary 2.2.5]).

Proposition 2.2. *Let e and f be idempotents in R. Then we have the following:*

(i) *$e\,\omega^l\,f$ iff $(1-f)\,\omega^r\,(1-e)$,*

(ii) *$e\,\omega^r\,f$ iff $(1-f)\,\omega^l\,(1-e)$.*

Proof. Let $e\,\omega^l\,f$, hence $ef = e$, so that

$$(1-e)(1-f) = 1-e-f+ef = 1-f$$

and so $1-f\,\omega^r\,1-e$. Again, if $e\,\omega^r\,f$, then $fe = e$, and therefore

$$(1-f)(1-e) = 1-f-e+fe = 1-f$$

and so $1-f\,\omega^l\,1-e$. The reverse implications follow from the fact that $1-(1-e) = e$ for every idempotent e in R □

The following corollary is immediate:

Corollary 2.3. *Let e and f be idempotents in R. Then we have the following:*

(i) *$e\,\mathscr{L}\,f$ iff $(1-e)\,\mathscr{R}\,(1-f)$,*

(ii) *$e\,\mathscr{R}\,f$ iff $(1-e)\,\mathscr{L}\,(1-f)$.*

Note that $e(1-e) = (1-e)e = 0$, for each idempotent e. In the following, an idempotent e of R is said to be *orthogonal* to an idempotent f, written $e \perp f$, if $ef = fe = 0$. Thus for each idempotent e of R, we have $(1-e) \perp e$.

Also, 1 is the largest idempotent of R in the following sense. The relation $\omega = \omega^r \cap \omega^l$ is a partial order on the set of idempotents of any semigroup (see [13]) and in R, we have $e\,\omega\,1$.

We next see, how these ideas lead to the following generalization of the above result:

Proposition 2.4. *If e and f are idempotents in R with $e\,\omega\, f$, then $f-e$ is an idempotent in R with $(f-e)\,\omega\, f$ and $(f-e)\perp e$.*

Proof. Let e and f be idempotents in R with $e\,\omega\, f$, so that $ef = fe = e$. Then

$$(f-e)^2 = f - fe - ef + e = f - e - e + e = f - e$$

and hence $f-e$ is an idempotent. Also,

$$f(f-e) = f - fe = f - e$$

and

$$(f-e)f = f - ef = f - e.$$

Hence $(f-e)\,\omega\, f$. Again,

$$e(f-e) = ef - e = 0$$

and

$$(f-e)e = fe - e = 0$$

so that $f-e$ is orthogonal to e. □

Note that in the above result, we have $f = e + (f-e)$ so that f is decomposed as the sum of a pair orthogonal idempotents less than f under the partial order ω. The next result shows that the sum of any pair of orthogonal idempotents gives a larger idempotent.

Proposition 2.5. *If e and f are idempotents in R with $e\perp f$, then $e+f$ is an idempotent with $e\,\omega\,(e+f)$ and $f\,\omega\,(e+f)$.*

Proof. Direct computation gives

$$(e+f)^2 = (e+f)(e+f) = e + ef + fe + f$$

Therefore, if $ef = fe = 0$, then $e+f$ is also an idempotent. Also, $e(e+f) = e + ef = e$ and $(e+f)e = e + fe = e$, gives $e\,\omega\,(e+f)$ and $f(e+f) = fe + f = f$ and $(e+f)f = ef + f = f$, imply $f\,\omega\,(e+f)$. □

Either of the two equations defining orthogonality can be characterized in terms of the biorder relations:

Proposition 2.6. *Let e and f be idempotents in R. Then the following are equivalent:*

(i) $ef = 0$,

(ii) $e\,\omega^l\,(1-f)$,

(iii) $f\,\omega^r\,(1-e)$.

Proof. If $ef = 0$, then $e(1-f) = e - ef = e$. This prove that (i) implies (ii). From Proposition 2.2, it follows that (ii) implies (iii). Finally, if $f\,\omega^r\,(1-e)$, then $(1-e)f = f$ from which we get $ef = 0$. Thus (iii) implies (i). □

The following characterization of orthogonality is immediate:

Corollary 2.7. *Let e and f be idempotents in R. Then the following are equivalent:*

(i) $e \perp f$,

(ii) $e\,\omega\,(1-f)$,

(iii) $f\,\omega\,(1-e)$.

We also note the following simple result which will be useful in the sequel:

Lemma 2.8. *If e and f are idempotents in R with $ef = 0$, then $Re \bigcap Rf = \{0\}$.*

Proof. For any idempotents e and f of R if $x \in Re \bigcap Rf$, then $x = ye = zf$ for some y and z in R, so that $xe = ye = x$ and $xf = zf = x$. If $ef = 0$, then this gives

$$x = xf = (xe)f = x(ef) = 0.$$

Thus $Re \bigcap Rf = \{0\}$. □

Next, we see how the intersection and sum of principal left ideals of R (the meet and join in the lattice of principal left ideals) can be described in terms of idempotents. For this, we make use of some of the ideas in [13]. We first prove the following:

Proposition 2.9. *For idempotents e and f of a semigroup,*

(i) *if $e\,\omega^l\,f$, then fe is an idempotent with $fe\,\mathscr{L}\,e$ and $fe\,\omega\,f$.*

(ii) *if $e\,\omega^r\,f$, then ef is an idempotent with $ef\,\mathscr{R}\,e$ and $ef\,\omega\,f$.*

Proof. Note that for idempotents e and f of a semigroup, if $e\,\omega^l\,f$, then $ef = e$, by definition, hence

$$(fe)(fe) = f(ef)e = fee = fe$$

and so fe is an idempotent. Also, $(fe)e = fe$ and $e(fe) = (ef)e = e$, so that $fe\,\mathscr{L}\,e$; and $(fe)f = f(ef) = fe$ and $f(fe) = fe$, imply $fe\,\omega\,f$. Dually, we can show that if $e\,\omega^r\,f$, then ef is an idempotent with $ef\,\mathscr{R}\,e$ and $ef\,\omega\,f$. □

In [13] we have the sets $\mathsf{M}(e,f) = \omega^l(e) \cap \omega^r(f)$ and

$$\mathsf{S}_1(e,f) = \{h \in \mathsf{M}(e,f) : fhe = h \text{ and } ehf = ef\}$$

for idempotents e and f of a semigroup. Since (the multiplicative semigroup of) R is regular, the set $\mathsf{S}_1(e,f)$ is non-empty for every pair of idempotents e and f (see [13, Corollary 4.5]). Using this, we can express the intersection and sum of principal left ideals of R in terms of idempotents:

Proposition 2.10. *Let e and f be idempotents in R and let $g \in \mathsf{S}_1(e, 1-f)$. Define*

$$h = e(1-g) \ \textit{and} \ k = g + (1-g)f.$$

Then h and k are idempotents in R with

$$Re \bigcap Rf = Rh \ \textit{and} \ Re + Rf = Rk.$$

Proof. We first prove that h is an idempotent and that $Re \bigcap Rf = Rh$. Since $g \in \mathsf{S}_1(e, 1-f)$, we have $g\,\omega^l\,e$ so that eg is an idempotent with $eg\,\omega\,e$, by Proposition 2.9. So, $h = e(1-g) = e - eg$ is an idempotent with $h\,\omega\,e$, by Proposition 2.4. Hence $Rh \subseteq Re$. To see that $Rh \subseteq Rf$ also, note that since $g \in \mathsf{S}_1(e, 1-f)$, we have $eg(1-f) = e(1-f)$, which gives $eg - egf = e - ef$. So,

$$hf = e(1-g)f = ef - egf = e - eg = e(1-g) = h$$

which gives $Rh \subseteq Rf$. Thus $Rh \subseteq Re \bigcap Rf$.

To prove the reverse inclusion, let $x \in Re \bigcap Rf$. Then

$$xe = xf = x$$

and hence

$$xh = xe(1-g) = x(1-g) = x - xg.$$

Again since $g\,\omega^r\,(1-f)$, we have $fg = 0$, by Proposition 2.6, we get

$$xg = (xf)g = x(fg) = 0.$$

So $xh = x$ and hence $x = xh \in Rh$. Thus $Re \bigcap Rf = Rh$.

Next, we show that k is an idempotent and $Re + Rf = Rk$. Since $g \in \mathsf{S}_1(e, 1-f)$, we have $g\,\omega^r\,(1-f)$, then $f\,\omega^l\,(1-g)$, by Proposition 2.2 and so $(1-g)f$ is an idempotent with $(1-g)f\,\omega\,(1-g)$, by Proposition 2.9. Hence $(1-g)f \perp g$, by Proposition 2.7 and so $k = g + (1-g)f$ is an idempotent, by Proposition 2.5.

To prove that $Re + Rf = Rk$, first note that $g\,\omega^l\,e$, and so $g \in Re$ and $(1-g)f \in Rf$, and so $k = g + (1-g)f \in Re + Rf$. Hence $Rk \subseteq Re + Rf$. To prove the reverse implication, we write

$$k = g + (1-g)f = g + f - gf = f + g(1-f)$$

and note that

$$ek = ef + eg(1-f) = ef + e(1-f) = e$$

since $eg(1-f) = e(1-f)$; and also

$$fk = f + fg(1-f) = f.$$

Using the fact that $g\,\omega^r\,(1-f)$, and hence $fg = 0$, by Proposition 2.6. Since $e = ek$ and $f = fk$, it follows that that $Re \subseteq Rk$ and $Rf \subseteq Rk$ and hence $Re + Rf \subseteq Rk$. Thus $Re + Rf = Rk$. □

If A and B are principal right ideals of R with $A \bigcap B = \{0\}$, then we write $A + B$ as $A \oplus B$ and call it the *direct sum* of A and B. In particular if $R = A \oplus B$, then A and B are said to be *complements* of each other. It is easily seen that for any idempotent e in R, we have the direct sum decomposition [14] $R = Re \oplus R(1-e)$: for $e(1-e) = 0$, which implies $Re \bigcap R(1-e) = \{0\}$, by Proposition 2.6; and any element x of R can be written $x = xe + x(1-e)$, which means $R = Re + R(1-e)$. It is shown in [15], that any direct sum decomposition of R into principal left ideals arises in this manner (Theorem 2.1 of Part II). We derive this from a slightly more general result, proved using biorder relations. First we prove a couple of lemmas. The first one below characterizes principal left ideals with trivial intersection:

Lemma 2.11. *Let e and f be idempotents in R. Then the following are equivalent:*

(i) $Re \bigcap Rf = \{0\}$,

(ii) $\omega^l(e) \bigcap \omega^l(f) = \{0\}$,

(iii) $\mathsf{S}_1(e, 1-f) \subseteq L_e$,

(iv) $e(1-g) = 0$, *for every g in* $\mathsf{S}_1(e, 1-f)$,

(v) $e(1-g) = 0$, *for some g in* $\mathsf{S}_1(e, 1-f)$.

Proof. Let $Re \bigcap Rf = \{0\}$ and $g \in \omega^l(e) \bigcap \omega^l(f)$. Then $g \in Re \bigcap Rf$, so that $g = 0$ by (i). Thus (i) implies (ii).

Next, suppose that (ii) holds and let $g \in \mathsf{S}_1(e, 1-f)$. Then $g\, \omega^l\, e$, therefore eg is an idempotent with $eg\, \omega\, e$, by Proposition 2.9, and so $h = e - eg$ is an idempotent with $h\, \omega^l\, e$, by Proposition 2.4. Also, since $g \in \mathsf{S}_1(e, 1-f)$, we have $eg(1-f) = e(1-f)$ which gives $eg - egf = e - ef$ and so $hf = ef - egf = e - eg = h$. Thus $h \in \omega^l(e) \bigcap \omega^l(f)$ hence $h = 0$, by (ii). Hence $eg = e$ and so $e\, \omega^l\, g$. Since $g\, \omega^l\, e$ also, we have $g\, \mathscr{L}\, e$. This gives (iii).

Now, assume (iii) and let $g \in \mathsf{S}_1(e, 1-f)$. Then $g\, \mathscr{L}\, e$, by (iii) so that $eg = e$ and so $e(1-g) = 0$, which gives (iv). That (iv) implies (v) is obvious.

Finally, assume (v) and let $x \in Re \bigcap Rf$. Since R is regular, x has a generalized inverse x' in R. Then $h = x'x$ is an idempotent with $h\, \mathscr{L}\, x$, which imply $Rh = Rx$ by [3], and $Rx \subseteq Re \bigcap Rf$, by the choice of x. Hence $h \in Re \bigcap Rf$ and so $he = hf = h$. Now since $e(1-g) = 0$, we have $eg = e$ thus

$$hg = (he)g = h(eg) = he = h$$

and since $g\, \omega^r\, (1-f)$, we have $(1-f)g = g$, which gives $fg = 0$ so

$$hg = (hf)g = h(fg) = 0.$$

Thus $h = 0$ and hence $x = xx'x = xh = 0$. This gives (i). □

The next result shows that $\mathscr{L}$-classes of a pair of idempotents satisfying any of the conditions of the last lemma can be represented by orthogonal idempotents:

Lemma 2.12. *If e and f are idempotents in R with $\omega^l(e) \cap \omega^l(f) = \{0\}$, then there exist idempotents g and h such that $e\, \mathscr{L}\, g \perp h\, \mathscr{L}\, f$.*

Proof. Let e and f be idempotents with $\omega^l(e) \cap \omega^l(f) = 0$ and let $p \in \mathsf{S}_1(e, 1-f)$ and $q \in \mathsf{S}_1(f, 1-e)$. Then $q\, \omega^r\, (1-e)$ and $p\, \omega^r\, (1-f)$, so that $e\, \omega^l\, (1-q)$ and $f\, \omega^l\, (1-p)$, by Proposition 2.2, and so $g = (1-q)e$ and $h = (1-p)f$ are idempotents with $g\, \mathscr{L}\, e$ and $h\, \mathscr{L}\, f$, by Proposition 2.9.

Also, since $\omega^l(e)\bigcap\omega^l(f)=0$ and $p\in \mathsf{S}_1(e,1-f)$, we have $e(1-p)=0$; and since $\omega^l(f)\bigcap\omega^l(e)=0$ and $q\in \mathsf{S}_1(f,1-e)$, we have $f(1-q)=0$, by the last lemma. Hence $gh=(1-q)(e(1-p))f=0$ and $hg=(1-p)(f(1-q))e=0$. Thus $g\perp h$ as well. □

Now, if e and f are idempotents in R with $Re\bigcap Rf=\{0\}$, then $\omega^l(e)\bigcap\omega^l(f)=\{0\}$, by Lemma 2.11 and hence there exist orthogonal idempotents g and h with $Rg=Re$ and $Rh=Rf$, by Lemma 2.11. Since every principal ideal in R is generated by an idempotent, this gives the following:

Proposition 2.13. *If A and B are principal left ideals of R with $A\bigcap B=\{0\}$, then there exist orthogonal idempotents e and f in R such that $A=Re$ and $B=Rf$.*

Again if e and f are orthogonal idempotents in R, then $Re\bigcap Rf=\{0\}$ by Lemma 2.8, and thus $Re+Rf=Re\oplus Rf$. Also, in this case, $e+f$ is an idempotent with $e\,\omega^l\,(e+f)$ and $f\,\omega^l\,(e+f)$, by Proposition 2.5, so that $Re\subseteq R(e+f)$ and $Rf\subseteq R(e+f)$ and so $Re+Rf\subseteq R(e+f)$. Conversely, $e+f\in Re+Rf$, which gives $R(e+f)\subseteq Re+Rf$. Thus, we have the following:

Proposition 2.14. *If e and f are orthogonal idempotents in R, then $Re+Rf=Re\oplus Rf=R(e+f)$.*

From these results, we can derive the characterization of direct sum decompositions of R, proved in [15], as a special case of Proposition 2.13:

Proposition 2.15. *If A and B are principal left ideals of R such that $R=A\oplus B$, then there exists an idempotent e of R such that $A=Re$ and $B=R(1-e)$.*

Proof. If $R=A\oplus B$, then by Proposition 2.13, there exist orthogonal idempotents e and f in R such that $A=Re$ and $B=Rf$, so by Proposition 2.14, we have $R=A\oplus B=Re\oplus Rf=R(e+f)$. Hence $1=x(e+f)$ for some x in R, we get $e+f=x(e+f)(e+f)=x(e+f)x=1$ as $e+f$ is an idempotent. Thus $f=1-e$ and so $B=R(1-e)$. □

Now for any two idempotents e and f of R, we have $Re+Rf=R(g+(1-g)f)$, by Proposition 2.10, and in the course of the proof of this result, we have seen that $g\perp(1-g)f$. Hence $Re+Rf=Rg\oplus R((1-g)f)$, by Proposition 2.14. Thus we have the following:

Proposition 2.16. *For any two principal left ideals A and B in R, there exist idempotents e and f such that $e\perp f$ and $A+B=Re\oplus Rf$.*

3. Conclusion

In a regular semigroup and hence in a regular ring all the principal left(right) ideals are idempotent generated. Join and meet of two principal ideals A and B can be expressed as principal ideals generated by idempotents that can be obtained from the idempotent generators of A and B. Sandwich sets play an important role in identifying these idempotents. Hence the relation between idempotents and principal ideals in regular rings gives more insights into the study of regular rings.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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