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Research Article

FRW Viscous Dark Fluid with Non-linear Inhomogeneous Equation of State

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Abstract. In this paper, we considered a bulk viscosity described by non-linear inhomogeneous equation of state of the type $p = \omega(\rho) + f(\rho) + \Lambda(H)$, where $\omega(\rho) = b_0\rho^{\delta-1} - 1$, $f(\rho) = A\rho^{\alpha}$ and $\Lambda(H) = \Lambda_0 H$. We assume the bulk viscosity as a linear combination of two terms of the form $\zeta = \zeta_0 + \zeta_1 H$ i.e. one is constant and the other is proportional to Hubble parameter H. In the first part of the paper we find the solution of the field equations in terms of time-dependent dark energy density ρ, Hubble parameter H, scale factor a and also obtain the transition from non-phantom era to the phantom era by using exponential function method. In the second part of the paper, we again find the solutions of the field equations by using the simple integration method and again obtain ρ, H, and a for the particular case. Finally, we discuss the stability of the model.

Keywords. Dark energy, Viscous fluid, Non-linear equation of state

Mathematics Subject Classification (2020). 83A05; 83C05; 83C15; 83C56

1. Introduction

The discovery of accelerating universe has led to the appearance of a new theoretical model [8]. The cosmic acceleration can be explained via the introduction of dark energy [7] strange properties like negative pressure and negative entropy. It is well known that present universe

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is subject to acceleration, which can be explained in terms of an ideal fluid (dark energy) with usual matter, and which has uncommon equation of state.

Khadekar *et al.* [5], consider the effect of dark energy model with inhomogeneous equation of state of the form $p = (\gamma - 1)\rho + \Lambda(t)$ and proposed dynamical generalized scale factor for the universe in which they assume the bulk viscosity ζ and time dependent parameter Λ as a linear combination of two terms: one is proportional to constant and other is proportional to scalar expansion θ. In continuation of this work recently Khadekar and Deepti [3] obtained the solutions of the field equations by using the above inhomogeneous equation of state with equation of state parameter ω is constant and is a function of ρ. Similarly, Khadekar and Rupali [4] also describe the cosmological evolution by considering the above ζ and Λ and obtained the analytical solutions of the field equations using effective equation of state.

Brevik *et al.* [1] investigated the specific model for a dark fluid with a non-linear inhomogeneous equation of state of the type $p = \omega(\rho) + f(\rho) + \Lambda(t)$ and find the solutions of the field equation in terms of Hubble parameter $H(t)$, Scale factor $a(t)$ and also investigated the transition from non-phantom to the phantom era. In particular they studied the transition towards super acceleration, i.e., the case in which the third derivative of the scale factor $a(t)$ is positive.

In this paper, we use the exponential function method to solve the non-linear inhomogeneous equation of state and obtained the time dependent dark energy density ρ, Hubble parameter $H(t)$ and scale factor $a(t)$ and investigate the effect of viscosity to the evolution of the universe.

This paper is organized as follows. Section 2 deals with the model and field equations. In Section 3, we consider a non-linear inhomogeneous equation of state of the universe and obtained the solutions of the field equations by using exponential function method. We also discussed the time dependent dark energy density ρ, Hubble parameter $H(t)$ and the scale factor $a(t)$ for particular case $\delta = \frac{1}{2}$. In Section 4 we have the discuss sound speed and stability of the model. In last section, we present our conclusion.

2. Model and Field Equation

We consider the FRW metric of the form [5]

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2), \tag{2.1}$$

where a is the scalar factor.

The Einstein field equations takes the usual form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{2.2}$$

In the FRW cosmology with bulk viscosity the stress energy momentum tensor can be written as

$$T_{\mu\nu} = (p + \rho)U_\mu U_\nu + p g_{\mu\nu} - \zeta\theta H_{\mu\nu}, \tag{2.3}$$

where ζ is the bulk viscosity, θ the expansion factor defined by $\theta = 3\dot{a}/a$ and the projection tensor $H_{\mu\nu}$ is defined by $H_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ with $U_{\mu\nu}$ being the four velocity and fluid on the comoving coordinates p and ρ are pressure and density.

For the FRW model eq. (2.1) Einstein field equations are given by

$$\frac{3}{X^2}H^2 = \rho, \tag{2.4}$$

$$\dot{H} + H^2 = -\frac{X}{6}(\rho + 3\bar{p}), \tag{2.5}$$

where $X = 8\pi G$, $\bar{p}$ is an equivalent pressure defined by $\bar{p} = p - \zeta\theta$ and dot $(\dot{})$ stands for differentiation with respect to time.

3. Non-linear Inhomogeneous Equation of State and Its Solutions

In this section, we assume the non-linear inhomogeneous equation of state depending on time as given by Brevik *et al.* [1]

$$p = \omega(\rho)\rho + f(\rho) + \Lambda(t), \tag{3.1}$$

where $\omega(\rho) = b_0\rho^{\delta-1} - 1$ given by Myrzakul *et al.* [6] $f(\rho) = A\rho^{\alpha}$ and $\Lambda = \Lambda_0 H$.

Energy of conservation for a complete dynamics system is given by

$$\dot{\rho} + (\rho + \bar{p})\theta = 0, \tag{3.2}$$

where $\bar{p} = p - \zeta\theta$ is the effective pressure, $\zeta = \zeta_0 + \zeta_1 H$ and $\theta = 3H$.

By using eq. (3.1) and considering $\alpha = \frac{1}{2}$, then conservation eq. (3.2) becomes

$$\dot{\rho} = \sqrt{3}\zeta_1 X^3 \rho^{\frac{3}{2}} - (\sqrt{3}AX + \Lambda_0 X^2 - 3\zeta_0 X^2)\rho - \sqrt{3}b_0 X \rho^{\delta+\frac{1}{2}}. \tag{3.3}$$

We are unable to find out the solutions of above differential equation due to its non-linear behavior. Hence to solve this equation we consider the change of variable $\rho = Y^2$ with $\delta = -m$ where $m > 0$, gives

$$\dot{Y} + b_1 Y^2 + b_2 Y^{-2m} + b_3 Y = 0, \tag{3.4}$$

where $b_1 = \frac{-\sqrt{3}\zeta_1 X^3}{2}$, $b_2 = \frac{\sqrt{3}b_0 X}{2}$ and $b_3 = \frac{\sqrt{3}AX - 3X^2\zeta_0 + \Lambda_0 X^2}{2}$.

By using exponential function method given by Ganji and Kachapi [2], eq. (3.4) yields

$$Y = \frac{c_1 e^{-kt} + c_2 + c_3 e^{kt}}{e^{-kt} + C + c_4 e^{kt}}, \tag{3.5}$$

where c_1, c_2, c_3, c_4 and b are constants given by Ganji and Kachapi [2].

We get the energy density parameter of the form

$$\rho = Y^2 = \left[\frac{c_1 e^{-kt} + c_2 + c_3 e^{kt}}{e^{-kt} + C + c_4 e^{kt}}\right]^2. \tag{3.6}$$

In this case, when $C = 0$ and $k = 0$ then any dependence on time vanishes and the energy density ρ reduces to a constant and will be conserved during the time. The behavior of this energy density ρ as shown in Figure 1.

In the following, we find out the solutions non-linear eq. (3.3) for particular case $\delta = \frac{1}{2}$. In this case, the differential eq. (3.3) becomes

$$\dot{\rho} = \sqrt{3}X\rho^{\frac{3}{2}}\left[\zeta_1 X^2 - \left(A + \frac{\Lambda_0 X}{\sqrt{3}} - \sqrt{3}\zeta_0 X + b_0\right)\rho^{\frac{-1}{2}}\right]. \tag{3.7}$$

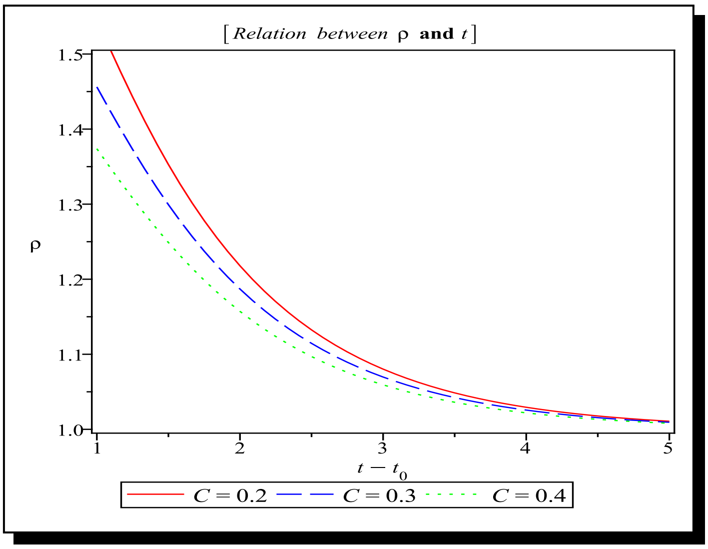

Figure 1. Time-dependent dark energy density with $C = 0.2$ and $c_1 = c_2 = c_3 = c_4 = 1$

After integrating eq. (3.7) we get the density parameter $\rho(t)$ as

$$\rho = \left[\frac{D_0}{\zeta_1 X^2 - \exp\left[D_0 \frac{\sqrt{3}Xt + D}{2}\right]}\right]^2, \tag{3.8}$$

where $D_0 = B - \sqrt{3}X\zeta_0 - A + \frac{\Lambda_0 X}{\sqrt{3}}$.

By using the field eq. (2.4) we get the Hubble parameter $H(t)$ as

$$H = \frac{X}{\sqrt{3}}\left[\frac{D_0}{\zeta_1 X^2 - \exp\left[D_0 \frac{\sqrt{3}Xt + D}{2}\right]}\right]. \tag{3.9}$$

The time derivative of $H(t)$ becomes

$$\dot{H} = \frac{X^2 D_0^3 \exp\left[D_0 \frac{\sqrt{3}Xt + D}{2}\right]}{2\left(\zeta_1 X^2 - \exp\left[D_0 \frac{\sqrt{3}Xt + D}{2}\right]\right)^2}. \tag{3.10}$$

If $\dot{H} > 0$, then the universe is accelerating and if $\dot{H} < 0$, then the universe is decelerating. In our case, we get $\dot{H} > 0$ i.e. universe is accelerating. It is well known that in the phantom phase if $\dot{\rho} > 0$, then the energy density grows and the universe is expanding and in non-phantom phase $\dot{\rho} < 0$ i.e. energy density decreases. In our case, if $t \to \infty$ then $H(t) \to 0$ and $\rho(t) \to 0$ so that phantom energy decreases. In this case, the cosmology singularity does not appear.

The scale factor $a(t)$ can take the following form

$$a(t) = \left[X^2\zeta_1 \exp\left[-\frac{D_0}{2}(\sqrt{3}Xt + D)\right] - 1\right]^{\frac{-2}{3X^2\zeta_1}}. \tag{3.11}$$

The derivative of scale factor is given by

$$\dot{a}(t) = \frac{D_0}{\sqrt{3}} X \exp\left[-\frac{D_0}{2}(\sqrt{3}Xt + D)\right]\left[X^2\zeta_1 \exp\left[-\frac{D_0}{2}(\sqrt{3}Xt + D)\right] - 1\right]^{-\left(\frac{2}{3X^2\zeta_1} + 1\right)}. \tag{3.12}$$

The second derivative of scale factor is

$$\ddot{a}(t)=\dot{a}(t)\frac{D_0\sqrt{3}X}{2}\left[\frac{\frac{2}{3}\exp\left[-\frac{D_0}{2}(\sqrt{3}Xt+D)\right]+1}{X^2\zeta_1\exp\left[-\frac{D_0}{2}(\sqrt{3}Xt+D)\right]-1}\right]. \tag{3.13}$$

Here $\ddot{a}(t)=0$ for $t_1=\frac{2}{\sqrt{3}XD_0}\log\left(\frac{2}{3D}\right)$.

For this, it is observed that the for $t<t_1$, first and second derivative of scale factor $a(t)$ are both positive i.e. the universe expands with acceleration. While for $t>t_1$, first and second derivative of scale factor $a(t)$ are both negative i.e. universe expands with deceleration. The behaviour of energy density ρ, Hubble parameter $H(t)$ and the scale factor $a(t)$ are shown graphically in Figure 2, Figure 3 and Figure 4, respectively.

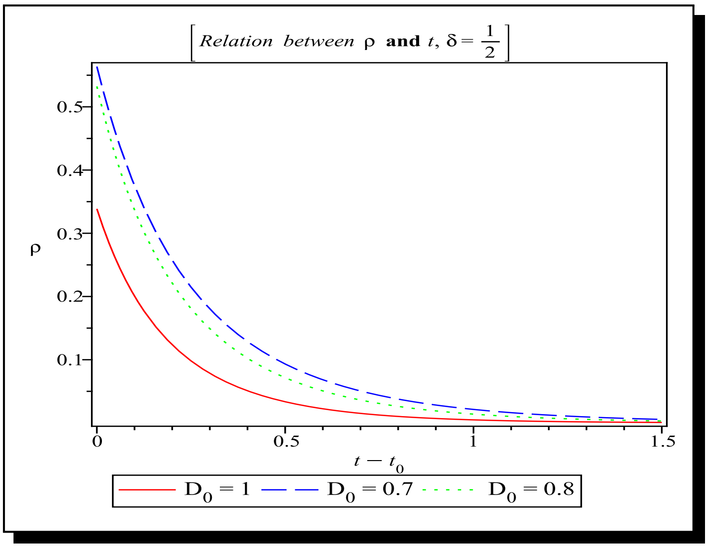

Figure 2. Time-dependent dark energy density with $D_0=\zeta_1=D=X=1$

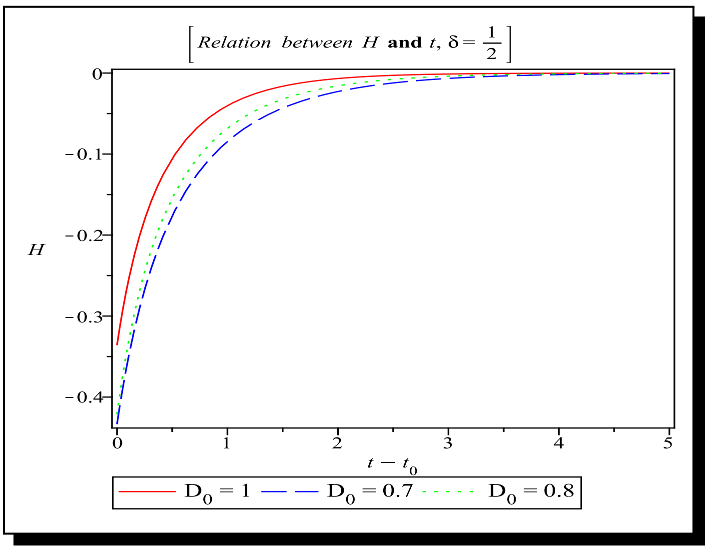

Figure 3. Hubble expansion parameter with $D_0=\zeta_1=D=X=1$

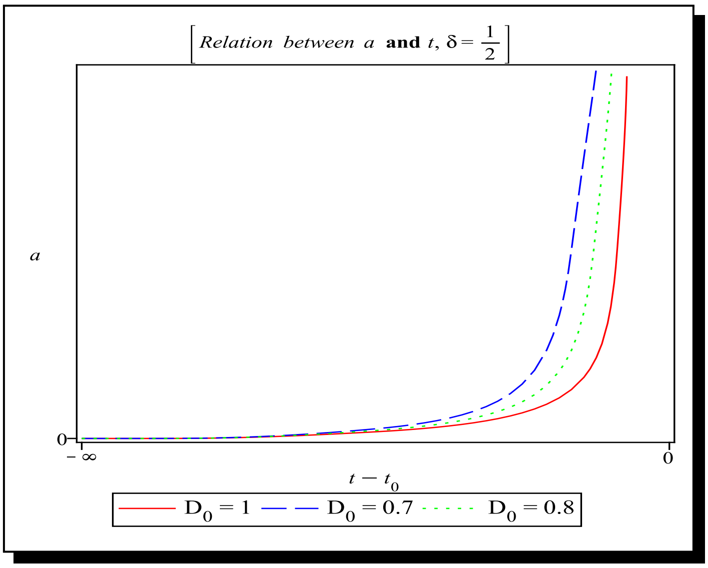

Figure 4. Scale factor with $D_0 = \zeta_1 = D = X = 1$

4. Sound Speed

There are several ways to discuss the stability of the model. In an acceptable model, the sound speed $c_s^2 = \frac{\partial \bar{p}}{\partial \rho}$ should be bounded by a constant, like the upper bound of light speed or approaches to a constant to a late time of the universe. In our case, the value of $\bar{p}$ is

$$\bar{p} = b_0\rho^\delta - (1-\zeta_1 X^2)\rho + \left(\frac{A+\Lambda_0-3\zeta_0}{\sqrt{3}}\right)X_0\rho^{\frac{1}{2}}. \tag{4.1}$$

Hence, the sound speed is given by

$$c_s^2 = \delta b_0\rho^{\delta-1} - (1-\zeta_1 X^2) + \frac{1}{2}\left(\frac{A+\Lambda_0-3\zeta_0}{\sqrt{3}}\right)X_0\rho^{\frac{-1}{2}}. \tag{4.2}$$

Using eq. (3.6) we get the sound speed as

$$c_s^2 = \delta b_0\left[\frac{c_1e^{-kt}+c_2+c_3e^{kt}}{e^{-kt}+C+c_4e^{kt}}\right]^{2\delta-2} - (1-\zeta_1X^2) + \frac{1}{2}\left(\frac{A+\Lambda_0-3\zeta_0}{\sqrt{3}}\right)X_0\left[\frac{c_1e^{-kt}+c_2+c_3e^{kt}}{e^{-kt}+C+c_4e^{kt}}\right]^{-1}. \tag{4.3}$$

For $\delta = \frac{1}{2}$, using eq. (3.8) and eq. (4.1), we get the sound speed as

$$c_s^2 = \frac{1}{2}\left[b_0 + \left(\frac{A+\Lambda_0-3\zeta_0}{\sqrt{3}}\right)X_0\right]\left[\frac{D_0}{\zeta_1X^2 - \exp\left[D_0\frac{\sqrt{3}Xt+D}{2}\right]}\right]^{-1} - (1-\zeta_1X^2). \tag{4.4}$$

Figure 5 and Figure 6 shows the behaviour of sound speed throughout the universe. From Figure 5 it is observed that the sound speed begins with positive value and approaches to a constant value. Thus, we get the stability throughout the universe. Figure 6 is the plot of square of sound speed versus cosmic time and shows that there is instability in future time.

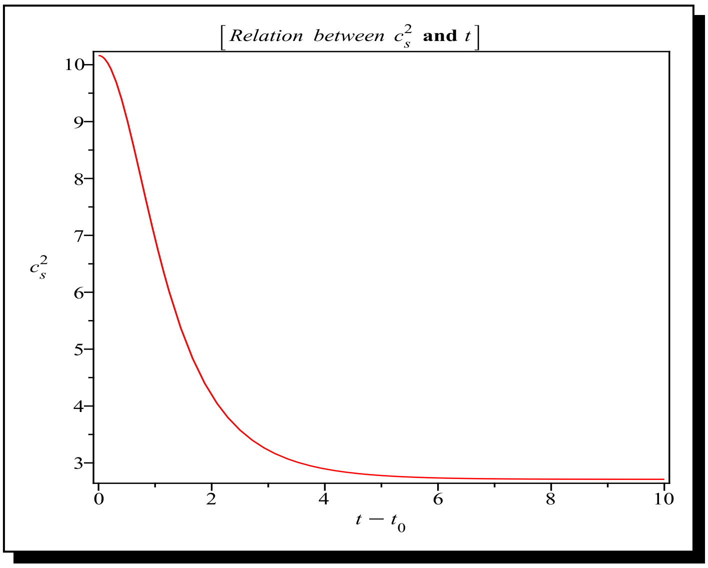

Figure 5. c_s^2 versus time t with $\delta = 3$, $b_0 = c_1 = c_2 = c_3 = c_4 = A = \zeta_1 = \Lambda_0 = \zeta_0 = X_0 = 1$ and $C = 0.2$

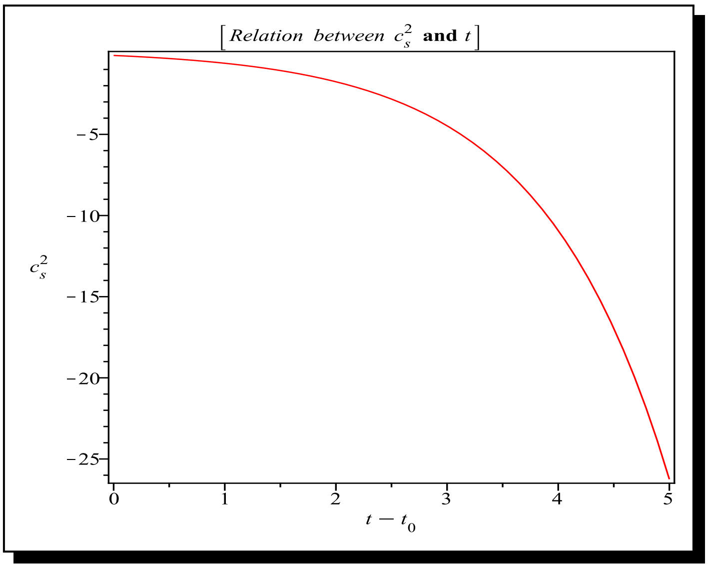

Figure 6. c_s^2 versus time t with $b_0 = A = \Lambda_0 = \zeta_0 = X_0 = D_0 = \zeta_0 = D = 1$

5. Conclusion

We have studied the cosmological model of the universe in which there is a non-linear inhomogeneous equation of state with equation of state parameter ω is of the form $\omega = b_0 \rho^{\delta-1} - 1$. We find the complete description of evolutionary transition according to the value of scale factor $a(t)$ and its first, second derivative which characterized different types accelerating (for $\ddot{a} > 0$) and decelerating (for $\ddot{a} < 0$) expansion of the universe. The introduction of viscosity to the system yields to a non-linear differential equation which gives the time-dependent dark energy density. We solved this equation by using the exponential function method, and studied the behavior of dark energy density ρ, Hubble parameter $H(t)$ and the scale factor $a(t)$.

From eq. (3.6), it is observed that when $C = 0$ and $k = 0$ then any dependence on time vanishes and the energy density ρ reduces to a constant and it will be conserved during the time. On the other hand if $k > 0$ and $C = 1$, eq. (3.6) becomes

$$\rho = \left[\frac{c_1 e^{-t} + c_2 + c_3 e^t}{e^{-t} + 1 + c_4 e^t}\right]^2. \tag{5.1}$$

From this equation it is observed that when $t \to \infty$ then $\rho \to \infty$. Numerically, we draw energy density ρ in terms of time t as shown in Figure 1. In this case, the dark energy density is a decreasing function of time which agrees with expansion of universe. For different values of C we can show that the increase in parameter C, decreases the value of dark energy density.

For the case $\delta = \frac{1}{2}$, we get the energy density in the form of eq. (3.8). We have seen from Figure 2, that energy density ρ is decreasing function of time which agrees with expansion of the universe. We also noted that it begins with a positive value and recovers asymptotically to a constant value. The Hubble parameter of the model from eq. (3.9) starts at relatively small value and rapidly decreases to a constant value, followed by smooth evolution between -0.1 and 0 as plotted in Figure 3. From eq. (3.11), we draw the behavior of scale factor $a(t)$ verses time t. It is observed that there are two phases during the evolution: an exponentially inflationary scenario at the beginning of the universe followed by decelerating phase as shown in Figure 4. Similarly, from eq. (3.12) and eq. (3.13), it is seen that for $t < t_1$, first and second derivative of scale factor $a(t)$ both are positive i.e. the universe expands with acceleration, while for $t > t_1$, $\dot{a}(t) < 0$ and $\ddot{a}(t) < 0$ i.e. universe expands with deceleration.

From eq. (4.3) and eq. (4.4) we draw the behaviour of sound speed throughout the universe as shown in Figure 5 and Figure 6, respectively. From Figure 5 it is observed that the sound speed begins with positive value and approaches to a constant value. Thus, we get the stability throughout the universe. From Figure 6, we observed that there is instability.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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