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Research Article

Study on Total Irregularity in Totally Segregated ∞ Bicyclic Graphs

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Abstract. Many measures of irregularity were introduced and studied before. Among them, the most investigated one is the *Total Irregularity* of a graph defined by Abdo *et al*. (The total irregularity of a graph, *Discrete Mathematics and Theoretical Computer Science* **16**(1) (2014), 201 – 206). It is defined as sum of absolute values of difference of vertex degrees over all vertices of the graph. A graph in which every two adjacent vertices have distinct degrees is totally segregated. Here, we find the greatest *Total Irregularity* of *Totally Segregated* ∞ *bicyclic graphs* and identify the extremal graphs.

Keywords. Total irregularity, Totally segregated ∞ bicyclic graph

Mathematics Subject Classification (2020). 05C30

1. Introduction

In many fields of applications, especially in the case of manufacture of drugs, we can model chemical compounds by graphs. Topological index of a molecular graph (the graph which represents molecule) is a numeric quantity and it does not depend on its pictorial representation. It is proved that there is some connections between the characteristics of the compounds and their topological indices. Among different topological indices, the indices which based on degree of vertices have important application in chemical graph theory. In this paper we see one degree based topological index called *Total Irregularity* which is an important irregularity measure. In many applications it is very significant to know the measures of irregularity of a graph which

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is a model of some chemical substance. These measures of irregularity will lead to determine many chemical properties of the chemical substance. When doing experiments in the case manufacture of medicines, many chemical substance are very costly. In that cases, we can find its some properties using these measures of irregularity even in the absence of chemical substances. In literature, several measures of such *irregularity* were proposed [2], [5], [4] and [3]. Among them, the most investigated one is the *Total Irregularity* of a graph. It is found by Abdo *et al*. [1] as:

$$irr_t(H) = \frac{1}{2}\sum_{u,v\in V(H)} |d_H(u) - d_H(v)|. \tag{1.1}$$

In a graph H, if $d_H(u) \neq d_H(v)$, for every adjacent vertices u and $v \in V(H)$, the graph is said to be totally segregated graph. Jackson and Entringer [6] studied this class of graphs. Here, we use TS graph to represent totally segregated graph. Three types of bicyclic graphs are introduced and investigated by You *et al*. in [7]. In this paper, *Totally Segregated* ∞ *Bicyclic Graphs* are discussed and maximum *Total Irregularity* of the graphs is determined and the extremal graph is presented.

Here, we see ∞ bicyclic graphs which is given in [7]. Let C_e denotes a cycle of length e. A bicyclic graph is a connected graph in which the number of edges exceeds exactly one than the number of vertices. The basic bicycle ∞-graph denoted by $\infty(e,f,1)$ is got from two vertex-disjoint cycles C_e and C_f by pasting one vertex of C_e and one vertex of C_f (Figure 1) where $e,f \geq 3$.

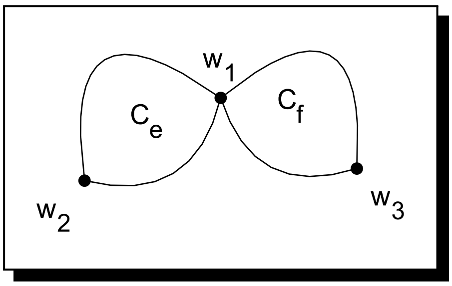

Figure 1. The graph $\infty(e,f,1)$ with $e \geq 3$ and $f \geq 3$

In Figure 1, let $w_1 = V(C_e)\cap V(C_f)$, $w_2 \in V(C_e)\setminus V(C_f)$ and $w_3 \in V(C_f)\setminus V(C_e)$.

Let $P^* = \{P : P$ is a rooted path with length at least one and the root is the initial vertex$\}$, $R^* = \{R : R$ is a rooted star and the root is its center$\}$, $PR^* = \{P+R : P \in P^*$ and $R \in R^*\}$. Here, we get the rooted graph $P+R$ by pasting the end vertex of $P \in P^*$ with the center of a star R. Note that the root of $P+R$ is the root of P. Here R_r denote the star on r vertices. Let H_1 and H_2 be two graphs with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. The graph $H = (H_1,v_1)\circ(H_2,v_2)$ denote the resultant graph obtained from pasting v_1 with v_2. Let $x \in V(\infty(e,f,1))$ and v be the starting vertex of the tree T. We denote $(\infty(e,f,1,x))\circ(T,v)$ by $\infty(e,f,1,x\circ T)$. In this case, we say that a tree T is affixed to the basic bicycle $\infty(e,f,1)$ at x. For example, see Figures 2 and 3.

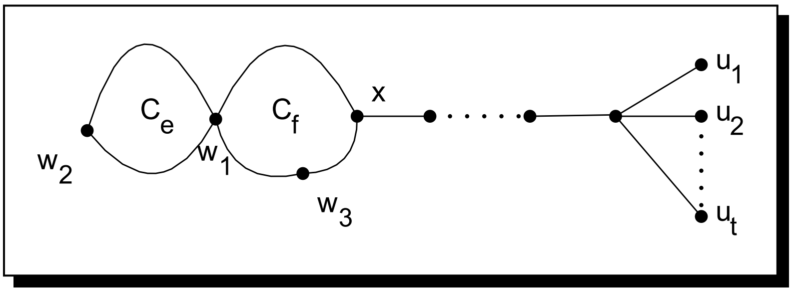

Figure 2. The graph $\infty(e,f,1,x\circ T)$ where $T\in PR^*$

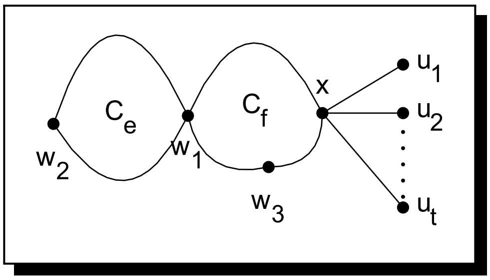

Figure 3. The graph $\infty(e,f,1,x\circ R_{t+1})$

Note that we get a bicyclic graph H from a basic bicycle ∞-graph ($\infty(e,f,1)$) (possibly) by affixing trees to some vertices of the basic bicycle. We say H as bicyclic graph with basic bicycle $\infty(e,f,1)$ if we get H by affixing trees to some of the vertices of basic bicycle $\infty(e,f,1)$. The ∞-bicyclic graph is a bicyclic graph with a basic bicycle $\infty(e,f,1)$. The bicyclic graph $\infty_n(e,f,1,w_1\circ T_1,w_2\circ T_2)$ denote ∞-bicyclic graph on n vertices, with basic bicycle $\infty(e,f,1)$ with affixed trees T_1 and T_2, such that root of the rooted tree T_1 and T_2 is affixed at w_1 and w_2 of basic bicycle, respectively.

A *Totally Segregated Bicyclic* (*TSB*) *Graph* is a bicyclic graph which is totally segregated. The set denoted by $B_n(C_e\circ T_1,C_f\circ T_2)$ is the set of those graphs each of which is an ∞-bicyclic graph such that a tree is affixed to at least one vertex (say w_2) in $V(C_e)\setminus\{w_1\}$ and a tree is affixed to at least one vertex (say w_3) in $V(C_f)\setminus\{w_1\}$, where w_1,w_2,w_3 are as defined in Figure 1.

Remark 1.1. If H is a *Totally Segregated ∞-Bicyclic Graph*, then $H\in B_n(C_e\circ T_1,C_f\circ T_2)$.

Remark 1.2. For $n\le 6$, a *Totally Segregated ∞-Bicyclic Graph* of order n does not exist.

2. Maximum Total Irregularity of Totally Segregated ∞ Bicyclic Graphs

Definition 2.1 (*Type 1*-Transformation, [7])**.** Let $H=(V,E)$ be a bicyclic graph with basic bicycle $\infty(e,f,1)$ with rooted trees $T_1,\cdots,T_s$ ($s\ge 1$) affixed and let $u\in V$ be one of its vertices with maximum degree and let b be any vertex of H of degree 1, and is adjacent to vertex $a(a\ne u)$. Let H' be the graph obtained from H by deleting the pendant edge ab and adding a pendant edge ub. The transformation from H to H' is a *Type 1* - Transformation on H (see Figure 4).

In Figure 4, the edge $ab \in E(T_i)$ and $u \in V(T_i)$. In fact, $ab \in E(T_j)$ for any $j \in \{1,2,\cdots,s\}$.

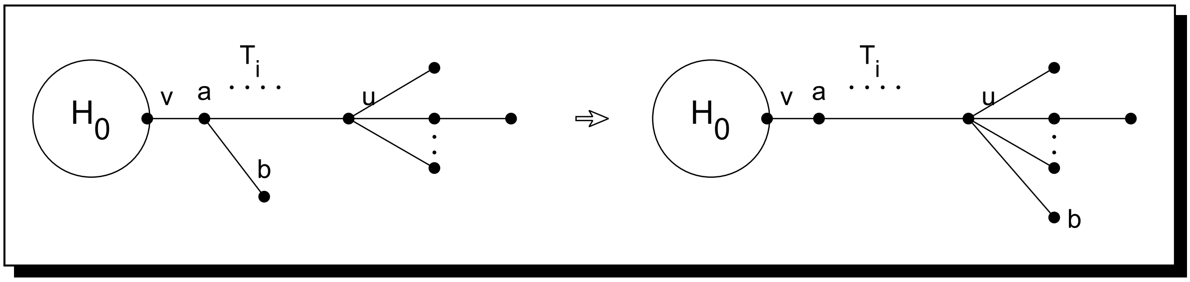

Figure 4. *Type 1* - Transformation

Lemma 2.1 ([7])**.** *Let $H = (V,E)$ be a ∞ bicyclic graph with $s(\geq 1)$ rooted trees $T_1,T_2,\cdots,T_s$ affixed and let H' be the graph obtained from H by Type 1 - Transformation. Then $irr_t(H) < irr_t(H')$.*

By Lemma 2.1 and by the Definition 2.1, we get the result as follows.

Lemma 2.2. *Let $H = (V,E)$ be a Totally Segregated ∞-Bicyclic Graph on n vertices.*

(a) *If the graph $H_1 = (V,E') \in B_n(C_e \circ T_1, C_f \circ T_2)$, which is obtained from H by repeating Type 1 - Transformation until it is not possible to get a new graph which belongs to $B_n(C_e \circ T_1, C_f \circ T_2)$ from H_1, then we can find some rooted tree T such that $H_1 \cong \infty(e,f,1,w_1 \circ T, w_2 \circ R_2, w_3 \circ R_2)$ or $H_1 \cong \infty(e,f,1,w_2 \circ T, w_3 \circ R_2)$ or $H_1 \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ T)$ and $irr_t(H) < irr_t(H_1)$, where $T \in R^* \cup \mathrm{PR}^*$; $e \geq 3, f \geq 3$; w_1, w_2, w_3 are as defined in Figure 1.*

(b) *In Case* (a)*, let u be a vertex of T and let $u_1,u_2,\cdots,u_t$ be the pendant vertices belongs to neighbourhood of u. Then $d_{H_1}(u) \geq d_{H_1}(x)$ $\forall$ $x \in V$.*

Definition 2.2 (*Type 2* and *Type 3* - Transformation)**.** Let $H = (V,E)$ be a ∞ bicyclic graph, such that all trees affixed to the basic bicycle are R_2 (star on two vertices) except T, where $T \in R^* \cup PR^*$ and R_2-s are affixed to vertex x, $x \in V(H) \setminus \{w_1\}$.
Let $u \in V$ be one of the vertices of greatest degree and let $u_1,u_2,\cdots,u_t (t \geq 1)$ be the vertices of degree one adjacent to u. We get the graph H' from H by removing the leaves of H, say $uu_1,uu_2,\cdots,uu_t$ and adding the pendant edges $w_1u_1,w_1u_2,\cdots,w_1u_t$. We call the transformation from H to H' a *Type 2* - Transformation on H.
Let $u \in V$ be one vertex with maximum degree which is the root of the rooted star R^* $(=R_{t+1})$ and $u_1,u_2,\cdots,u_t$ $(t \geq 2)$ be the vertices of degree one adjacent to u. Suppose we get H'' from H by removing the leaves $uu_2,\cdots,uu_t$ and adding the leaves $w_1u_2,\cdots,w_1u_t$. The transformation from H to H'' is said to be a *Type 3* - Transformation on H (refer Figures 5 and 6).

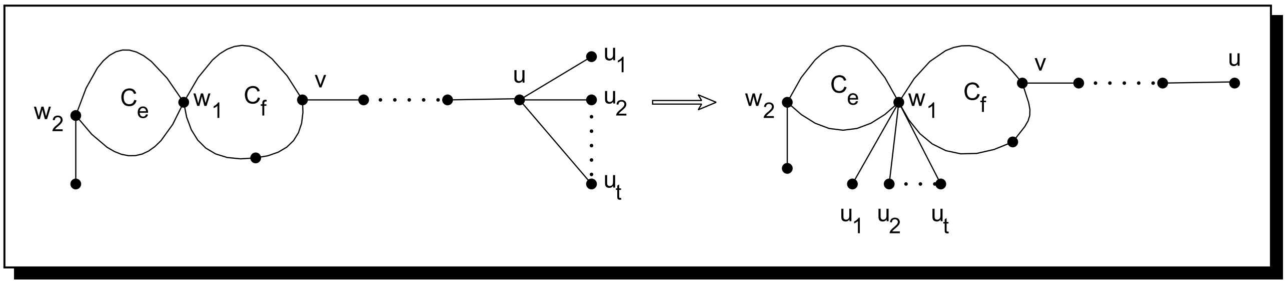

Figure 5. *Type 2* - Transformation on ∞-bicyclic graph with one R_2 and $T \in PR^*$ are affixed

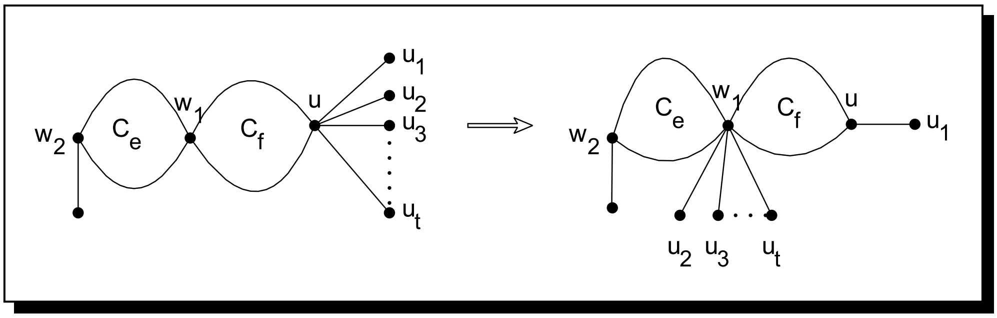

Figure 6. *Type 3* - Transformation on ∞-bicyclic graph with one R_2 and $T \in R^*$ are affixed

Lemma 2.3. *Let H be a ∞-bicyclic graph on n vertices obtained as in Lemma 2.2, i.e., H is ∞-bicyclic graph and $H \cong \infty(e,f,1,w_1 \circ T, w_2 \circ R_2, w_3 \circ R_2)$ or $H \cong \infty(e,f,1,w_2 \circ T, w_3 \circ R_2)$ or $H \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ T)$.*
Let $T \in \mathrm{PR}^*$ and let v be the root of the rooted tree T. Let u be a vertex of maximum degree and let $u_1, u_2, \cdots, u_t (t \geq 1)$ be the vertices of degree one and belongs to neighborhood of u. If we get the graph H' from H by Type 2 - Transformation (Figure 5) then, $irr_t(H) < irr_t(H')$.

Proof. Let $H = (V,E)$ be the graph. Let v be the root of the rooted tree and it is not compulsory that v is different from w_1. It is clear that only the degrees of vertices u and w_1 is changed after the *Type 2* - Transformation, then $d_{H'}(u) = 1$, $d_{H'}(w_1) = d_H(w_1) + d_H(u) - 1$ and $d_{H'}(x) = d_H(x)$ for any vertex $x \in V \setminus \{u, w_1\}$. Let $U = V \setminus \{u, w_1\}$.
It is given that the vertex u is one of the vertices of H with greatest degree, that is, $d_H(u) \geq d_H(x)$ for any vertex $x \in V$. Then

$$|d_{H'}(u) - d_{H'}(w_1)| - |d_H(u) - d_H(w_1)| = 2d_H(w_1) - 2, \tag{2.1}$$

$$\sum_{x \in U} |d_{H'}(u) - d_{H'}(x)| - \sum_{x \in U} |d_H(u) - d_H(x)| = 2\sum_{x \in U} d_H(x) - (n-2)(d_H(u)+1). \tag{2.2}$$

Now, we discuss $\sum_{x \in U} |d_{H'}(w_1) - d_{H'}(x)| - \sum_{x \in U} |d_H(w_1) - d_H(x)|$ as follows:

Here $t \geq 3$ since u is one of the maximal degree vertices of H.
As $d_H(w_1) \geq d_H(x)$ for any $x \in U$,

$$\sum_{x \in U} |d_{H'}(w_1) - d_{H'}(x)| - \sum_{x \in U} |d_H(w_1) - d_H(x)|$$

$$= \sum_{x\in U}(d_H(w_1)+d_H(u)-1-d_H(x)) - \sum_{x\in U}(d_H(w_1)-d_H(x))$$
$$= (n-2)(d_H(u)-1).$$

Then we have

$$\sum_{x\in U}|d_{H'}(w_1)-d_{H'}(x)| - \sum_{x\in U}|d_H(w_1)-d_H(x)| \geq (n-2)(d_H(u)-1)-2. \tag{2.3}$$

By equations (2.1), (2.2), (2.3) and since $d_H(w_1)\geq 3$ and $d_H(x)\geq 1$ for any $x\in U$, we have

$$\begin{aligned} irr_t(H')-irr_t(H) &= |d_{H'}(u)-d_{H'}(w_1)| + \sum_{x\in U}|d_{H'}(u)-d_{H'}(x)| + \sum_{x\in U}|d_{H'}(w_1)-d_{H'}(x)| \\ &\quad - \left[|d_H(u)-d_H(w_1)| + \sum_{x\in U}|d_H(u)-d_H(x)| + \sum_{x\in U}|d_H(w_1)-d_H(x)|\right] \\ &\geq 2d_H(w_1)-2+2\sum_{x\in U}d_H(x)-(n-2)(d_H(u)+1)+(n-2)(d_H(u)-1)-2 \\ &\geq 2\sum_{x\in U}d_H(x) \\ &> 0. \end{aligned} \tag{2.4}$$

It follows the result. □

Lemma 2.4. *Let H be ∞-bicyclic graph and $H\cong\infty(e,f,1,w_2\circ T,w_3\circ R_2)$ or $H\cong\infty(e,f,1,w_2\circ R_2,w_3\circ T)$, where $T\in R^*$. Let $T=R_{t+1}$ and $u\in\{w_2,w_3\}$ be a vertex of maximal degree which is the root of the rooted tree T and $u_1,u_2,\cdots,u_t$ $(t\geq 2)$ be the vertices of degree one adjacent to u. If we get the graph H' from H by Type 3 - Transformation (Figure 6) then*

(1) $H'\cong\infty(e,f,1,w_1\circ R_t,w_2\circ R_2,w_3\circ R_2)$.

(2) $irr_t(H)<irr_t(H')$.

Proof. By the definition of *Type 3* - Transformation (Definition 2.2) result (1) is obvious. Now, we show that result (2) holds. Here only the degrees of vertices u and w_1 is changed after the *Type 3* - Transformation. Then $d_{H'}(u)=3$, $d_{H'}(w_1)=d_H(w_1)+d_H(u)-3$, and $d_{H'}(x)=d_H(x)$ for any vertex $x\in V\setminus\{u,w_1\}$. Let $U=V\setminus\{u,w_1\}$. Note that $t\geq 2$.
The vertex u is one of the vertices of H with greatest degree. Then $d_H(u)\geq d_H(x)$ for any vertex $x\in V$ and $d_H(w_1)\geq d_H(x)$ for any $x\in U$, $d_{H'}(u)\geq d_{H'}(x)$ for any vertex $x\in U$ and $d_{H'}(w_1)\geq d_{H'}(x)$ for any $x\in U$. Then

$$|d_{H'}(u)-d_{H'}(w_1)| - |d_H(u)-d_H(w_1)| = 2d_H(w_1)-6, \tag{2.5}$$
$$\sum_{x\in U}|d_{H'}(w_1)-d_{H'}(x)| - \sum_{x\in U}|d_H(w_1)-d_H(x)| = (n-2)(t-1), \tag{2.6}$$
$$\sum_{x\in U}|d_{H'}(u)-d_{H'}(x)| - \sum_{x\in U}|d_H(u)-d_H(x)| = (n-2)(1-t). \tag{2.7}$$

By equations (2.5), (2.6), (2.7) and since $d_H(w_1)\geq 3$, $t\geq 2$ and $d_H(u)-3\geq 1$, we have

$$\begin{aligned} irr_t(H')-irr_t(H) &= |d_{H'}(u)-d_{H'}(w_1)| + \sum_{x\in U}|d_{H'}(u)-d_{H'}(x)| + \sum_{x\in U}|d_{H'}(w_1)-d_{H'}(x)| \\ &\quad - \left[|d_H(u)-d_H(w_1)| + \sum_{x\in U}|d_H(u)-d_H(x)| + \sum_{x\in U}|d_H(w_1)-d_H(x)|\right] \end{aligned}$$

$$= 2d_H(w_1) - 6 + (n-2)(t-1) + (n-2)(1-t) \geq 0. \tag{2.8}$$

Hence $irr_t(H') - irr_t(H) = 2$ since $d_H(w_1) = 4$. It follows the result. □

By Lemma 2.1, 2.2 and 2.3 we have:
If $e, f (\geq 3)$ are given, then $\max\{irr_t(H) : H \in B_n(C_e \circ T_1, C_f \circ T_2)\} = irr_t(\infty(e, f, 1, w_1 \circ R_r, w_2 \circ R_2, w_3 \circ R_2))$, where $r = n - (e + f)$.

In the following theorem, the *Totally Segregated ∞-Bicyclic Graph* with the maximum *Total Irregularity* is found.

Let n, e, f, r be positive integers with $e, f \geq 3$ and $e + f + r = n$ and

$$H \cong \infty_n(e, f, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r).$$

Then the degree sequence of H is $((r+3)^1, 3^2, 2^{e+f-4}, 1^{r+1})$ and by simple calculation, using equation (1.1), we have

$$irr_t(H) = (e + f - 4)(2r + 4) + (r + 2)(r + 1) + 6r + 4. \tag{2.9}$$

Theorem 2.1. *If $n \geq 7$ is a positive integer and H is a Totally Segregated ∞-Bicyclic Graph with basic bicycle $\infty(e, f, 1)$, $e \geq 3$, $f \geq 3$, on n vertices, then $irr_t(H) \leq n^2 + n - 28$ and the equality holds if $H \cong \infty_n(3, 3, 1, w_1 \circ R_{n-6}, w_2 \circ R_2, w_3 \circ R_2)$.*

Proof. Let H be a *Totally Segregated ∞-Bicyclic Graph* with basic bicycle $\infty(e, f, 1)$ on n vertices where $e + f + r = n$.
Since H is totally segregated, by Remark 1.1 there exists a vertex $w_2 \in V(C_e) \setminus \{w_1\}$ with $d(w_2) \geq 3$ and a vertex $w_3 \in V(C_f) \setminus \{w_1\}$ with $d(w_3) \geq 3$. Then $H \in B_n(C_e \circ T_1, C_f \circ T_2)$.

We prove this theorem in two stages: In Stage 1, we obtain ∞-bicyclic graph $H' = \infty_n(e, f, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)$ from H such that $irr_t(H) < irr_t(H')$ by repeating *Type 1*, *Type 2*, *Type 3* - Transformations until it is not possible to get a new graph, which belongs to $B_n(C_e \circ T_1, C_f \circ T_2)$, from H' by any of these transformations.

In Stage 2,we obtain *Totally Segregated ∞-Bicyclic Graph* $H'' = \infty_n(3, 3, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_{n-6})$ from H' such that $irr_t(H') < irr_t(H'')$ by repeating replacement of edges so that one can not find a new *Totally Segregated ∞-Bicyclic Graph* from H'' by the replacement of edges.

Stage 1. If $r = 1$, then $n = e + f + 1$ and the rooted trees are R_2 affixed at w_2, R_2 affixed at w_3 and all other affixed trees T_i are trivial; namely, $|V(T_i)| = 1$. Then,

$$H \cong \infty_n(e, f, 1, w_2 \circ R_2, w_3 \circ R_2)$$

and

$$irr_t(H) = (e + f - 4)(2r + 4) + (r + 2)(r + 1) + 6r + 4.$$

Let $r \geq 2$. We get H_1 from H by repeating *Type 1* - Transformation. Then $H_1 \cong \infty(e, f, 1, w_2 \circ R_2, w_3 \circ T)$ or $H_1 \cong \infty(e, f, 1, w_2 \circ T, w_3 \circ R_2)$ or $H_1 \cong \infty(e, f, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ T)$, where $T \in R^* \cup PR^*$ and $irr_t(H) < irr_t(H_1)$ by Lemma 2.1.

Case 1: $H_1 \cong \infty(e, f, 1, w_2 \circ R_2, w_3 \circ T$, where $T \in R^*)$.

In this case we can get a new graph $H_2 \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)$ by *Type 3*-Transformation on H_1. Thus $irr_t(H) < irr_t(H_1) \leq irr_t(H_2) = (e+f-4)(2r+4)+(r+2)(r+1)+6r+4$ by Lemma 2.4 and equation (2.9).

Case 2: $H_1 \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ T)$ where $T \in PR^*$.
Let H_2 be the graph obtained from H_1 by *Type 2*-Transformation. Then $irr_t(H_1) < irr_t(H_2)$ by Lemma 2.3. We get H_3 from H_2 by repeating *Type 1*-Transformation until it is not possible to get a new graph which belongs to $B_n(C_e \circ T_1, C_f \circ T_2)$ from H_3 by *Type 1*-Transformation. Then $H_3 \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)$. Then by Lemma 2.1, $irr_t(H_2) < irr_t(H_3)$ and thus

$$irr_t(H) < irr_t(H_1) < irr_t(H_2) < irr_t(H_3) = (e+f-4)(2r+4)+(r+2)(r+1)+6r+4.$$

Case 3: $H_1 \cong \infty(e,f,1,w_2 \circ T, w_3 \circ R_2)$, where $T \in PR^* \cup R^*$.
The proof is as in the proof of Cases 1 and 2.

Case 4: $H_1 \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ T)$ where $T \in PR^*$.
Let v be vertex of T such that pendant vertices are adjacent to v and u be the starting vertex of the rooted tree. If $d_{H_1}(u,v) = 1$, suppose we get H_2 from H_1 by *Type 2*-Transformation; then

$$H_2 \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)$$

and

$$irr_t(H) < irr_t(H_1) < irr_t(H_2) = (e+f-4)(2r+4)+(r+2)(r+1)+6r+4$$

by Lemma 2.1.
If $d_{H_1}(u,v) > 1$, suppose we obtain H_2 from H_1 by *Type 2*-Transformation and get H_3 from H_2 by doing successively *Type 1*-Transformation until it is not possible to get a new graph which belongs to $B_n(C_e \circ T_1, C_f \circ T_2)$ from H_3 by *Type 1*-Transformation. Then $H_3 \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)$. By Lemma 2.3,

$$irr_t(H) < irr_t(H_1) < irr_t(H_2) < irr_t(H_3) = (e+f-4)(2r+4)+(r+2)(r+1)+6r+4.$$

Case 5: $H_1 \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ T)$ where $T \in R^*$.
Then

$$H_1 \cong \infty(e,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)$$

and

$$irr_t(H) < irr_t(H_1) = (e+f-4)(2r+4)+(r+2)(r+1)+6r+4$$

by Lemma 2.1.
From the above arguments, we get the proof of Stage 1.

Stage 2. Let e,f,r be positive integers with $e,f \geq 3$ and $e+f+r = n$.
In this stage, we prove:

(1) If $e \geq 4$, then

$$irr_t(\infty(e,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)) < irr_t(\infty(e-1,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_{r+1})).$$

(2) If $f \geq 4$, then

$$irr_t(\infty(e,f,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)) < irr_t(\infty(e,f-1,1,w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_{r+1})).$$

Let $n = e + f + r$; by equation (2.9) we have

$$irr_t(\infty(e-1, f, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_{r+1})) - irr_t(\infty(e, f, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r))$$
$$= 2e + 2f + 3r + 6 > 0, \tag{2.10}$$
$$irr_t(\infty(e, f-1, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_{r+1})) - irr_t(\infty(e, f, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r))$$
$$= 2e + 2f + 3r + 6 > 0. \tag{2.11}$$

The graph $\infty(e-1, f, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_{r+1})$ is obtained from $H_1 = \infty(e, f, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)$ by contracting an edge of the cycle C_e which is different from $w_1 w_2$ and adding a pendant edge to w_1 and the graph $\infty(e, f-1, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_{r+1})$ is obtained from $H_1 = \infty(e, f, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_r)$ by contracting an edge of the cycle C_f which is different from $w_1 w_3$ and adding a pendant edge to w_1. Let H_2 be the graph obtained from H_1 by repeating this kind of edge replacements until the length of the cycles C_e and C_f cannot be reduced. Then we know that $H_2 \cong \infty_n(3, 3, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_{n-6})$ and is *Totally Segregated ∞-Bicyclic Graph* for $n \geq 7$ (see Figure 7). By equations (2.10) and (2.11), we have $irr_t(H_1) < irr_t(H_2)$ and by simple calculation we get $irr_t(H_2) = n^2 + n - 28$. □

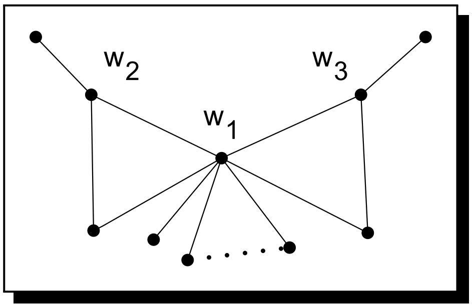

Figure 7. The graph $\infty_n(3, 3, 1, w_2 \circ R_2, w_3 \circ R_2, w_1 \circ R_{n-6})$

3. Conclusion

We have studied total irregularity of certain graphs and found maximum total irregularity of totally segregated ∞ Bicyclic graphs. More problems in this area still remain unsettled. More studies on different types of irregularities for different graph classes can be done. We focused our investigation to find greatest total irregularity of certain class of graphs. It would be interesting to compare the irregularity indices of various graphs. Another interesting problem is to characterise the set of graphs having identical irregularity indices. It is also interesting to find the relationship between number of edges and number of vertices in the case of extreme irregularities.

Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] H. Abdo, S. Brandt and D. Dimitrov, The total irregularity of a graph, *Discrete Mathematics and Theoretical Computer Science* **16**(1) (2014), 201 – 206, DOI: 10.46298/dmtcs.1263.

[2] F. K. Bell, A note on the irregularity of graphs, *Linear Algebra and its Applications* **161** (1992), 45 – 54, DOI: 10.1016/0024-3795(92)90004-T.

[3] G. Chartrand, P. Erdös and O. R. Oellermann, How to define an irregular graph, *The College Mathematics Journal* **19** (1988), 36 – 42, DOI: 10.1080/07468342.1988.11973088.

[4] G. H. Fath-Tabar, I. Gutman and R. Nasiri, Extremely irregular trees, *Bulletin* (*Académie serbe des sciences et des arts. Classe des sciences mathématiques et naturelles. Sciences mathématiques*) **38** (2013), 1 – 8, URL: `https://www.jstor.org/stable/44097193`.

[5] I. Gutman, P. Hansen and H. Mélot, Variable neighborhood search for extremal graphs. 10. Comparison of Irregularity Indices for Chemical Trees, *Journal of Chemical Information and Modeling* **45**(2) (2005), 222 – 230, DOI: 10.1021/ci0342775.

[6] D. E. Jackson and R. Entringer, Totally segregated graphs, *Congress. Numer.* **55** (1986), 159 – 165.

[7] L. H. You, J. Yang, Y. Zhu and Z. You, The maximal total irregularity of bicyclic graphs, *Journal of Applied Mathematics* **2014** (2014), Article ID 785084, 9 pages, DOI: 10.1155/2014/785084.