Communications in Mathematics and Applications
Vol. 12, No. 3, pp. 603–617, 2021
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v12i3.1571

Research Article

Bifurcation of Function in Four Dimensions With Eight Parameters Based on Lyapunov-Schmidt Reduction

Zainab S. Madhi *[1] and Mudhir A. Abdul Hussain[2]

[1] *Department of Mathematics, College of Sciences, University of Basrah, Basrah, Iraq*
[2] *Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq*

Received: April 20, 2021 **Accepted:** June 28, 2021

Abstract. In this paper, we investigate the bifurcation-spreading of critical points for a certain smooth function with eight parameters which have codimension 80. In addition, we found five plots of caustic (bifurcation set) corresponding different cases of parameters. Finally, using the method of alternative problems (Lyapunov-Schmidt method) we obtained the bifurcation solution for the equation of sixth order with boundary conditions as applicable.

Keywords. Classification of critical points; Caustic set and Lyapunov-Schmidt method

Mathematics Subject Classification (2020). 34K10; 34K18

1. Introduction

A bifurcation is a qualitative change in a family of solutions for equation when the parameter value changes such that it causes a sudden qualitative change in its behavior. A bifurcation diagram It is a graphical representation of the location of the critical points versus the bifurcation parameters. The nonlinear problems, parameter that appears in many scientific research fields can describe as operator equations of the form

$$\Omega(u,t) = p, \quad u \in \mathcal{O} \subset \mathcal{C}, p \in \mathcal{W}, t \in \mathcal{R}^n, \tag{1.1}$$

*Corresponding author: zienab.madhi@uobasrah.edu.iq

where $\mathcal{C}$ and $\mathcal{W}$ are Banach spaces and $\mathcal{O}$ is an open set. The foundation to investigate the bifurcation solution of this equation is produced by Lyapunov [12], and Schmidt [15] in his method of reduction it is efficient for solving this problem by obtaining the following equivalent equation

$$\mathcal{Q}(r,t)=\epsilon, \quad r\in\hat{E}, \epsilon\in\hat{K}, \tag{1.2}$$

where $\hat{E}$ and $\hat{K}$ are smooth finite dimensional manifold.

So that equation (1.2) preserves all the analytical and topological features of the equation (1.1) (bifurcation diagram, multiplicity, etc.) see [2, 5, 6, 16].

Critical points of smooth functions play an important role in the study of bifurcation solutions of BVPs. There are many researches of them in more than variable (see [3]).

More than one study of classification of critical points, geometric description of caustic and bifurcation solutions of nonlinear differential equations having fredholm operator equation using lyapanov-schmidt methodintroduced by ([3, 4, 7–11, 14]).

In this work, we study to classify the critical points of functions with four variables and eight parameters which have codimension 80, and we found a caustic for the function. Finally, by Lyapunov-Schmidt method in the variational case the bifurcation solutions to the equation of sixth order with boundary conditions we took as an application of our work.

Definition 1.1 ([13])**.** Let h be a function from $\mathcal{R}^n$ into $\mathcal{R}$, then a function h has a critical point at u_0 if $\frac{\partial h}{\partial u}(u_0)=0$.

Definition 1.2 ([13])**.** Let Υ be an open subset of $\mathcal{R}^n$, then the Hessian matrix of the function h at u_0 is the symmetric $m\times m$ matrix of second partial derivative

$$\mathcal{H}_h(u_0)=(A_{ij}), \quad i,j=1,2,\dots,n,$$

where $A_{ij}=\frac{\partial^2 h}{\partial u_i \partial u_j}$.

Definition 1.3 ([13])**.** The critical point u_0 of h is nondegenerate if $\det(\mathcal{H}_h(u_0))\neq 0$.

Definition 1.4 ([13])**.** Suppose that $\Upsilon\subset\mathcal{R}^n$ and $\Gamma\subset\mathcal{R}^m$ are open sets. A function $h:\Upsilon\to\Gamma$ is said to be a diffeomorphism if

(I) h is differentiable and bijective function;

(II) $h^{-1}$is differentiable function.

Definition 1.5 ([13])**.** A function $h,k:(\mathcal{R}^n,0)\to(\mathcal{R}^m,0)$ are said to be germ equivalent at $m\in\mathcal{R}^n$ if m is in the domain of both and there is a neighborhood Υ of m such that for all $u\in\Upsilon$, $h(u)=k(u)$. A function germ at a point m is an equivalent class of germ equivalent function.

Definition 1.6 ([3])**.** A continues-linear operator of Banach spaces $\Omega:\mathcal{C}\to\mathcal{W}$ is called fredholm operator if and only if $\dim(\ker\Omega)\prec\infty$ and $\dim(\mathrm{Coker}\,\Omega)\prec\infty$.

A fredholm index ($\mathrm{ind}(\Omega)$) is given by $\dim(\ker\Omega)-\dim(\mathrm{Coker}\,\Omega)$.

Definition 1.7 ([3]). The nonlinear map $\rho : l \subset \mathcal{C} \to \mathcal{W}$, where l is an open set. Then ρ is called fredholm if $\frac{\partial\rho}{\partial u}(u)$ is a Fredholm operator $u \in l$ (i.e. $\operatorname{ind}(\rho) = \operatorname{ind}\left(\frac{\partial\rho}{\partial u}(u)\right)$).

Definition 1.8 (Local-algebra [1]). The local algebra $\sum_h^u$ of the function h the origin is the quotient of the algebra of function germs by the gradient ideal of the function h

$$\sum_h^u = \frac{\zeta_n}{\mathfrak{J}(h)},$$

where $\mathfrak{J}(h)$ is the Jacobin-ideal $= \left(\frac{\partial h}{\partial u_1}, \frac{\partial h}{\partial u_2}, \dots, \frac{\partial h}{\partial u_n}\right)$.

The multiplicity $\omega(h)$ of the critical point is the dimension of its local-algebra:

$$\omega(h) = \dim \sum_h^u.$$

If $\dim \sum_h^u < \infty$, then a critical point is called isolated point.

Theorem 1.9 ([13]). *The multiplicity $\omega(h)$ of the isolated critical point is equal to the number of Morse-critical points into which it decomposes under a generic deformation of the function.*

Definition 1.10 ([13]). If $h, k : (\mathcal{R}^n, 0) \to (\mathcal{R}^m, 0)$ two function germs are said to be contact equivalent (C-equivalent), if there exist:

(I) a diffeomorphism Ω of the source $(\mathcal{R}^n, 0)$,

(II) a matrix $\mathcal{M} \in GL_m(\zeta_n)$ when $GL_m(\zeta_n)$ is the set of invertible $m \times m$ matrices whose entries are in ζ_n, such that $h \circ \aleph(u) = \mathcal{M}(u)k(u)$, where $h(u)$ and $k(u)$ are written as column-vectors, and $h \circ \aleph(u)$ is the usual product of matrix times vector.

The idea of C-equivalent of two function above h, k is that the solution sets $h^{-1}(0)$ and $k^{-1}(0)$ should be diffeomorphic.

Lemma 1.11 ([13]). *If h and k are C-equivalent, then their o-sets are diffeomorphic.*

Theorem 1.12 ([13]). *Let h, k be two function germs. Then*

(I) *h and k are contact equivalent,*

(II) *the ideal P and S are induced isomorphic ideal in ζ_n, where P and S are generated by the components of h and k,*

(III) *$\frac{\zeta_n}{P}$ and $\frac{\zeta_n}{S}$ are induced isomorphic.*

2. Main Results

This part includes finding the critical points and geometric description of the caustic set of the function defined by

$$\begin{aligned}\mathcal{M}(v_j;\eta) = {}& \frac{1}{4}v_1^4 + \frac{1}{4}v_2^4 + \frac{1}{4}v_3^4 + \frac{1}{4}v_4^4 + v_1^2v_2^2 + v_3^2v_1^2 + v_4^2v_1^2 + v_3^2v_2^2 + v_2^2v_4^2 + v_3^2v_4^2 - v_1^3v_3 + v_1v_2^2v_3 - v_1^2v_2v_4 \\ & + v_2v_3^2v_4 + v_1v_2v_3v_4 + p_1v_1^2 + p_2v_2^2 + p_3v_3^2 + p_4v_4^2 - q_1v_1 - q_2v_2 - q_3v_3 - q_4v_4, \end{aligned} \tag{2.1}$$

where $v_j = v_{1,2,3,4}$, $\eta = \{p_{1,2,3,4}, q_{1,2,3,4}\}$ such that p,q are parameters.

The Jacobian ideal of this function is

$$JI_{\mathcal{M}} = (\mathcal{M}_{v_1}, \mathcal{M}_{v_2}, \mathcal{M}_{v_3}, \mathcal{M}_{v_4}) = (v_1^3, v_2^3, v_3^3, v_4^3).$$

It has multiplicity 81 and codimension 80.

To study the bifurcation of function (2.1) using replace variables to complex, assume that

$$\zeta_1 = v_1 + iv_2, \quad \zeta_2 = v_3 + iv_4.$$

In the complex plane the function (2.1) has the form

$$\begin{aligned}\mathcal{M}(\zeta;\eta) = &\left(\frac{p_1}{4} - \frac{p_2}{4}\right)\bar{\zeta}_1^2 + |\zeta_1|^2|\zeta_2|^2 + \left(\frac{p_1}{4} - \frac{p_2}{4}\right)\zeta_1^2 + \left(\frac{p_3}{4} - \frac{p_4}{4}\right)\zeta_2^2 - \left(\frac{q_3}{2} + \frac{iq_4}{2}\right)\bar{\zeta}_1 \\ &- \left(\frac{q_1}{2} - \frac{iq_2}{2}\right)\zeta_1 - \left(\frac{q_1}{2} + \frac{iq_2}{2}\right)\bar{\zeta}_1 + \left(\frac{iq_4}{2} - \frac{q_3}{2}\right)\zeta_2 - \frac{\zeta_2^4}{32} - \frac{\bar{\zeta}_1^4}{32} - \frac{\zeta_1^4}{32} - \frac{\bar{\zeta}_2^4}{32} \\ &- \frac{3\bar{\zeta}_1^3\zeta_2}{16} - \frac{\zeta_1^2\zeta_2^2}{16} - \frac{3\zeta_2\zeta_1^3}{16} + \frac{\zeta_2^3\zeta_1}{16} - \frac{\zeta_1^3\zeta_2}{16} - \frac{\zeta_2\bar{\zeta}_1^3}{16} - \frac{\zeta_1\zeta_2^3}{16} - \frac{\zeta_1\bar{\zeta}_2^3}{16} + \frac{\bar{\zeta}_1^2\zeta_2^2}{16} + \frac{\bar{\zeta}_1\zeta_2^3}{16} \\ &- \frac{\bar{\zeta}_1^2\bar{\zeta}_2^2}{16} + \frac{\zeta_2^2\zeta_1^2}{16} + \left(\frac{p_1}{2} + \frac{p_2}{2}\right)|\zeta_1|^2 + \left(\frac{p_3}{2} + \frac{p_4}{2}\right)|\zeta_2|^2 + \left(\frac{p_3}{4} - \frac{p_4}{4}\right)\bar{\zeta}_2^2 + \frac{5|\zeta_1|^4}{16} \\ &+ \frac{5|\zeta_2|^4}{16} + \frac{\bar{\zeta}_2|\zeta_2|^2\zeta_1}{16} - \frac{\bar{\zeta}_1\bar{\zeta}_2|\zeta_1|^2}{16} - \frac{|\zeta_2|^2\zeta_1\zeta_2}{16} - \frac{|\zeta_1|^2\zeta_1\zeta_2}{16} - \frac{\bar{\zeta}_1\bar{\zeta}_2|\zeta_2|^2}{16} \\ &- \frac{3\bar{\zeta}_2|\zeta_1|^2\zeta_1}{16} - \frac{3\bar{\zeta}_1|\zeta_2|^2\zeta_2}{16} + \frac{\bar{\zeta}_1|\zeta_2|^2\zeta_2}{16}.\end{aligned}$$

Assume that $\left(\frac{q_1}{2} + \frac{iq_2}{2}\right) = \mu_1$, $\left(\frac{iq_4}{2} - \frac{q_3}{2}\right) = \mu_3$, $\left(\frac{q_1}{2} - \frac{iq_2}{2}\right) = \mu_2$, $\left(\frac{q_3}{2} + \frac{iq_4}{2}\right) = \mu_4$ and $p_1 - p_2 = \omega_1$, $p_1 + p_2 = \omega_2$, $p_3 - p_4 = \omega_3$, $p_3 + p_4 = \omega_4$.

Substituting the previous hypothesis into $\mathcal{M}(\zeta;\eta)$, we get

$$\begin{aligned}\mathcal{M}(\zeta;\eta) = &-\frac{\zeta_2^4}{32} - \frac{\bar{\zeta}_1^4}{32} - \frac{\zeta_1^4}{32} - \frac{\bar{\zeta}_2^4}{32} - \frac{3\bar{\zeta}_1^3\zeta_2}{16} - \frac{\zeta_1^2\zeta_2^2}{16} - \frac{3\bar{\zeta}_2\bar{\zeta}_1^3}{16} + \frac{\bar{\zeta}_2^3\zeta_1}{16} - \frac{\zeta_1^3\zeta_2}{16} - \frac{\bar{\zeta}_2\bar{\zeta}_1^3}{16} \\ &- \frac{\zeta_1\zeta_2^3}{16} - \frac{\bar{\zeta}_1\bar{\zeta}_2^3}{16} + \frac{\bar{\zeta}_1^2\zeta_2^2}{16} + \frac{\bar{\zeta}_1\zeta_2^3}{16} - \mu_2\zeta_1 + \frac{\bar{\zeta}_2^2\zeta_1^2}{16} - \frac{\bar{\zeta}_1^2\bar{\zeta}_2^2}{16} + \mu_3\zeta_2 - \mu_1\bar{\zeta}_1 \\ &+ |\zeta_1|^2|\zeta_2|^2 + \frac{q_1 z_1^2}{4} - \mu_4\bar{\zeta}_2 + \frac{q_3\zeta_2^2}{4} + \frac{\bar{\zeta}_2|\zeta_2|^2\zeta_1}{16} - \frac{\bar{\zeta}_1\bar{\zeta}_2|\zeta_1|^2}{16} + \frac{\bar{\zeta}_1|\zeta_2|^2\zeta_2}{16} \\ &- \frac{\bar{\zeta}_1\bar{\zeta}_2|\zeta_2|^2}{16} - \frac{3\bar{\zeta}_2|\zeta_1|^2\zeta_1}{16} - \frac{|\zeta_2|^2\zeta_1\zeta_2}{16} - \frac{|\zeta_1|^2\zeta_1\zeta_2}{16} - \frac{3\bar{\zeta}_1|\zeta_1|^2\zeta_2}{16} + \frac{\omega_1\bar{\zeta}_1^2}{4} \\ &+ \frac{\omega_3\bar{\zeta}_2^2}{4} + \frac{\omega_1|\zeta_1|^2}{2} + \frac{\omega_4|\zeta_2|^2}{2} + \frac{5|\zeta_2|^4}{16} + \frac{5|\zeta_1|^4}{16}\end{aligned} \tag{2.2}$$

where $|\zeta_1|^2 = v_1^2 + v_2^2$, $|\zeta_2|^2 = v_3^2 + v_4^2$ and $\bar{\zeta}_1$, $\bar{\zeta}_2$ conjugates of ζ_1, ζ_2, respectively.

To study the function's behaviour (2.1) near the critical point, it is convenient to consider this function in polar coordination,

$$\zeta_1 = r_1 e^{i\theta_1}, \quad \zeta_2 = r_2 e^{i\theta_2}. \tag{*}$$

Since we want to find the set of all parameter values of the equation, which makes the equation have a real solution. So the real part of the equation (*) is

$$\Gamma(r_{1,2};\vartheta_1^9)=\vartheta_1 r_1^4+\vartheta_2 r_2^4+\vartheta_3 r_2 r_2^3+\vartheta_4 r_2 r_1^3+\vartheta_5 r_1^2 r_2^2+\vartheta_6 r_2^2+\vartheta_7 r_1^2-\vartheta_8 r_1-\vartheta_9 r_2\,, \tag{2.3}$$

where

$$\vartheta_1=\left(\frac{5}{16}-\frac{\cos(4\theta_1)}{16}\right),\qquad \vartheta_2=\left(\frac{5}{16}-\frac{\cos(4\theta_2)}{16}\right),$$

$$\vartheta_3=\left(\frac{\cos(\theta_1-\theta_2)}{8}+\frac{\cos(\theta_1-3\theta_2)}{8}-\frac{\cos(\theta_1+\theta_2)}{8}-\frac{\cos(\theta_1+3\theta_2)}{8}\right),$$

$$\vartheta_4=\left(-\frac{3\cos(\theta_1-\theta_2)}{8}-\frac{\cos(\theta_1+\theta_2)}{8}-\frac{3\cos(3\theta_1-\theta_2)}{8}-\frac{\cos(3\theta_1+\theta_2)}{8}\right),$$

$$\vartheta_5=\left(1+\frac{\cos(2\theta_1-2\theta_2)}{8}-\frac{\cos(2\theta_1+2\theta_2)}{8}\right),$$

$$\vartheta_6=\left(\frac{\omega_3\cos(2\theta_2)}{2}+\frac{\omega_4}{2}\right),\qquad \vartheta_7=\left(\frac{q_1\cos(2\theta_1)}{2}+\frac{\omega_2}{2}\right),$$

$$\vartheta_8=(-\mu_1\cos(\theta_1)-\mu_2\cos(\theta_1)),\qquad \vartheta_9=(-\mu_3\cos(\theta_2)-\mu_4\cos(\theta_2)).$$

Now, suppose that $r_1=\frac{1}{\sqrt[4]{\vartheta_1}}\cdot\varpi_1$, $r_2=\frac{1}{\sqrt[4]{\vartheta_2}}\cdot\varpi_2$ we obtain the equation below, which is equivalent to the equation (2.3)

$$\wp=\sigma_2\varpi_1^2\varpi_2^2+\sigma_3\varpi_1\varpi_2^3+\sigma_4\varpi_1^3\varpi_2+\varpi_1^4+\varpi_2^4+\sigma_1\varpi_1^2+\sigma_5\varpi_2^2-\sigma_6\varpi_1-\sigma_7\varpi_2, \tag{2.4}$$

where $\sigma_2=\frac{\vartheta_5}{\sqrt{\vartheta_2}\sqrt{\vartheta_1}}$, $\sigma_3=\frac{\vartheta_3}{\vartheta_1^{\frac14}\vartheta_2^{\frac34}}$, $\sigma_4=\frac{\vartheta_4}{\vartheta_2^{\frac14}\vartheta_1^{\frac34}}$, $\sigma_1=\frac{\vartheta_7}{\sqrt{\vartheta_1}}$, $\sigma_5=\frac{\vartheta_6}{\sqrt{\vartheta_2}}$, $\sigma_6=\frac{\vartheta_8}{\vartheta_1^{\frac14}}$, $\sigma_7=\frac{\vartheta_9}{\vartheta_2^{\frac14}}$.

The elements $\varpi_1^3\varpi_2$, $\varpi_1\varpi_2^3$, belongs to the tangent space generated by the first derivatives $\frac{\partial\wp}{\partial\varpi_1}$, $\frac{\partial\wp}{\partial\varpi_2}$ then from the theory of germs we have that the function is equivalent to the following function

$$\wp=\sigma_2\varpi_1^2\varpi_2^2+\varpi_1^4+\varpi_2^4+\sigma_1\varpi_1^2+\sigma_5\varpi_2^2-\sigma_6\varpi_1-\sigma_7\varpi_2\,. \tag{2.5}$$

The aim of the function ($\wp$) is to find the geometric description of the Caustic, then classify the critical points of this function by determining the types of the critical points. The critical points of the function ($\wp$) are the solutions of the following system of nonlinear-algebraic equations

$$2\sigma_2\varpi_1\varpi_2^2+4\varpi_1^3+2\sigma_1\varpi_1-\sigma_6=0,$$

$$2\sigma_2\varpi_1^2\varpi_2+4\varpi_2^3+2\sigma_5\varpi_2-\sigma_7=0.$$

All the critical points of function $\wp$ are degenerate on the surface given by the equation

$$-12\sigma_2^2\varpi_1^2\varpi_2^2+24\sigma_2\varpi_1^4+24\sigma_2\varpi_2^4+4\sigma_1\sigma_2 u\varpi_1^2+4\sigma_2\sigma_5\varpi_2^2+144\varpi_1^2\varpi_2^2+24\sigma_1\varpi_2^2+24\sigma_5\varpi_1^2+4\sigma_1\sigma_5.$$

To get the caustic of the function $\wp$ we make the following parameterization

$$2\sigma_2\varpi_1\varpi_2^2+4\varpi_1^3+2\sigma_1\varpi_1=\sigma_6,$$

$$2\sigma_2\varpi_1^2\varpi_2+4\varpi_2^3+2\sigma_5\varpi_2=\sigma_7.$$

By change values ϖ_1, ϖ_2 and ϖ_5, we have the following geometric descriptions of the caustic of function $\wp$ in the $\sigma_6\sigma_7$-plane, the follow figures have been using MAPLE 17.

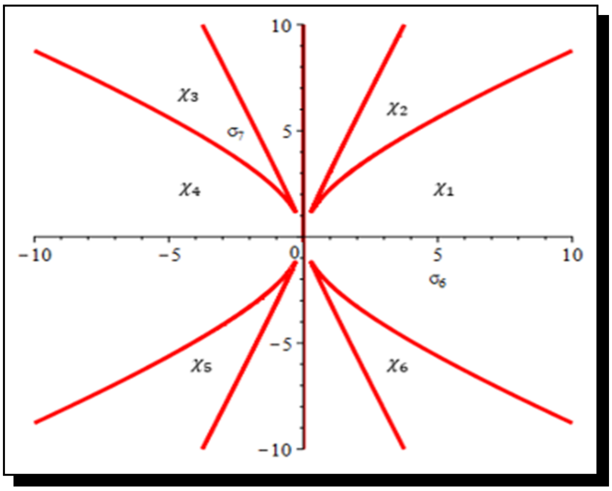


Figure 1. Describe caustic when $\sigma_1 = -0.02$, $\sigma_2 = 45$, $\sigma_5 = 3$

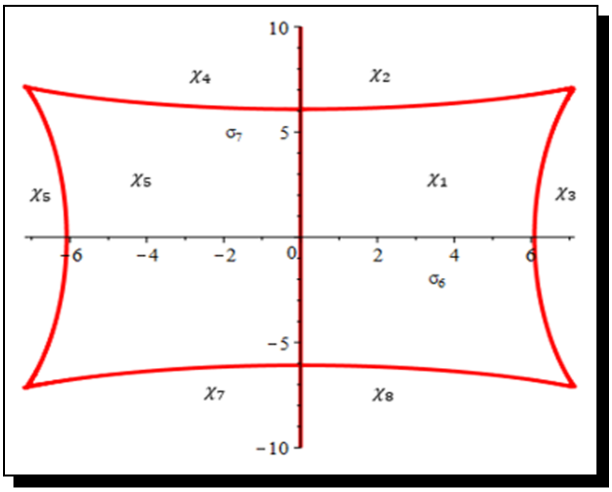


Figure 2. Describe caustic when $\sigma_1 = -5$, $\sigma_2 = -25$, $\sigma_5 = -5$

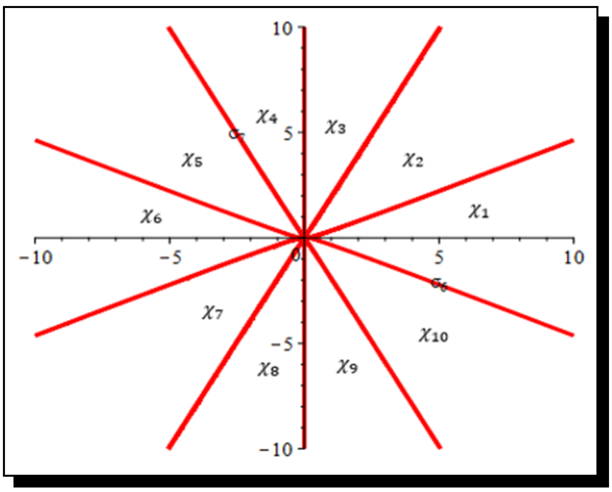


Figure 3. Describe caustic when $\sigma_1 = 0.02$, $\sigma_2 = 23$, $\sigma_5 = -0.2$

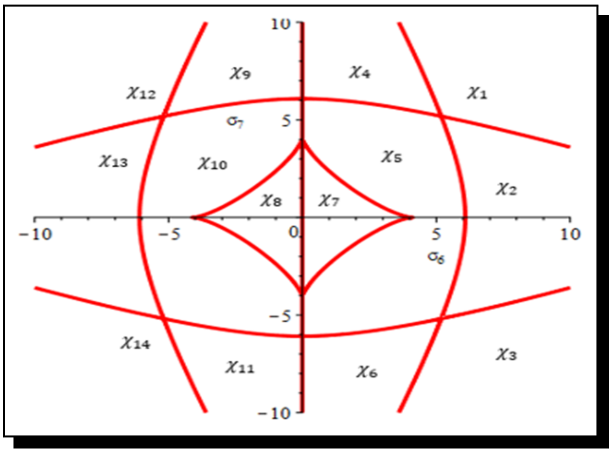


Figure 4. Describe caustic when $\sigma_1 = -5$, $\sigma_2 = 25$, $\sigma_5 = -5$

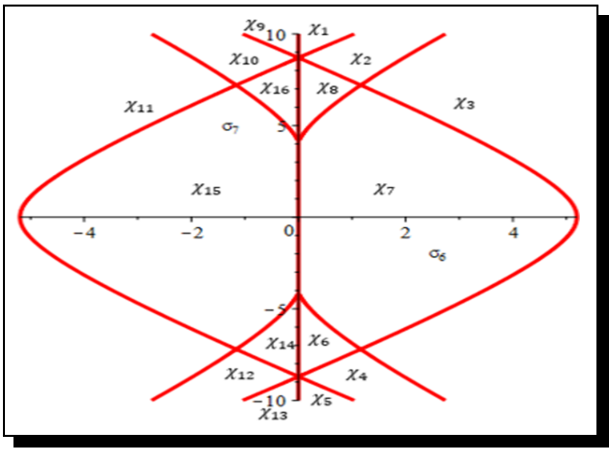


Figure 5. Describe caustic when $\sigma_1 = -4.5$, $\sigma_2 = 4.5$, $\sigma_5 = 0$

The bifurcation spread for the critical points of function (2.5) is given in five cases depending on parameters, we can determined type of them points in every region depending on the parameters (σ_6, σ_7), this lead into the following cases:

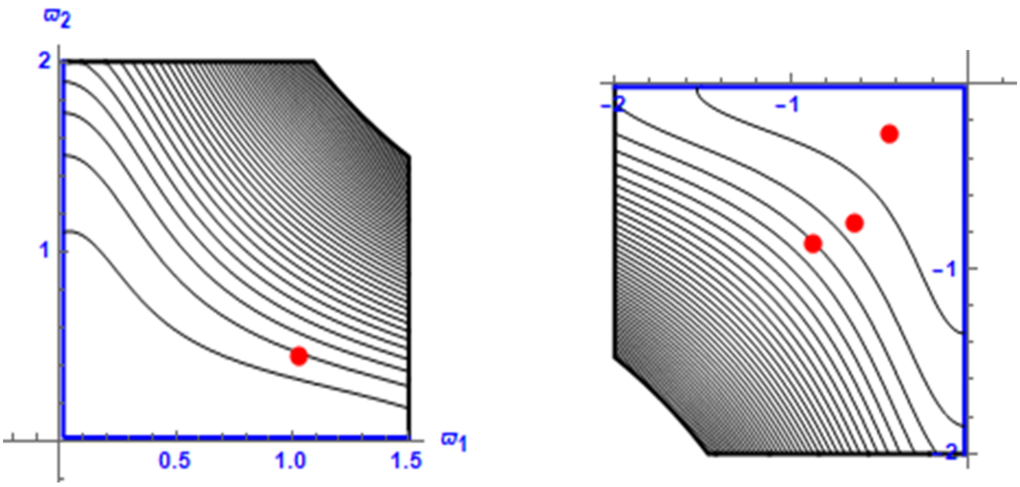
Case 1: If $\sigma_1 < 0 < \sigma_5 < \sigma_2$ (as in Figure 6). In this case the Caustic divided the plane of parameters into (6) regions:

(1) if (σ_6, σ_7) belongs to $(\mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_5 \text{or} \mathcal{X}_6)$, then this region contains three non-degenerate real critical point (2 minimum, 1 saddle) point, and

(2) if (σ_6, σ_7) belong to region $(\mathcal{X}_1$ or $\mathcal{X}_4)$, then this region contains one non-degenerate real critical point (minimum).

From the above dissociation we can represent the bifurcation-spreading extremals bythe rows $(2,1,0)$, $(1,0,0)$.

In addition, Figures 6a and 6b shown the locations of contour lines with respect to the domain of function (2.5), the number and type of critical points corresponding for the region $(\mathcal{X}_1)$ and region $(\mathcal{X}_5)$ in the caustic of function (2.5).



(a) Contours and crtical points in $\mathcal{X}_1$ **(b)** Contours and crtical points in $\mathcal{X}_2$

Figure 6

Case 2: If $\sigma_2 < \sigma_1 < 0$ and $\sigma_1 = \sigma_5$ (as in Figure 7).

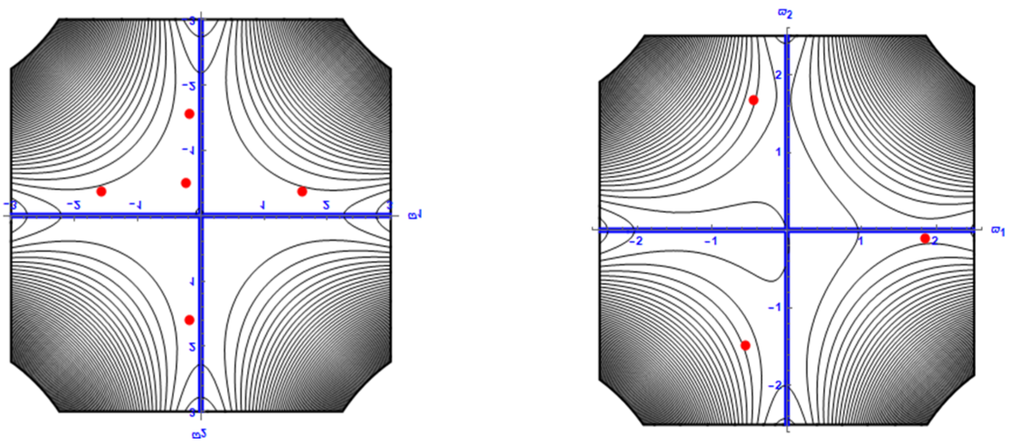
In this case the Caustic divided the plane of parameters into (8) regions:

(1) if (σ_6, σ_7) belong to region ($\mathcal{X}_1$ or $\mathcal{X}_5$), then this region contains five non-degenerate real critical point (3 minimum, 2 saddle).

(2) if (σ_6, σ_7) belongs to ($\mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4, \mathcal{X}_6, \mathcal{X}_7$ or $\mathcal{X}_8$), then this region contains three non-degenerate real critical point (2 minimum, 1 saddle).

From the above dissociation, we can represent the bifurcation-spreading extremals bythe rows (2, 1, 0), (3, 2, 0).

In addition, Figures 7a and 7b below: show the locations of contour lines with respect to the domain of function (2.5), the number and type of critical points corresponding for the region ($\mathcal{X}_1$) and region ($\mathcal{X}_3$) in the caustic of function (2.5).



(a) Contours and crtical points in $\mathcal{X}_1$ **(b)** Contours and crtical points in $\mathcal{X}_3$

Figure 7

Case 3: If $\sigma_5 < 0 < \sigma_1 < \sigma_2$ (as in Figure 8).

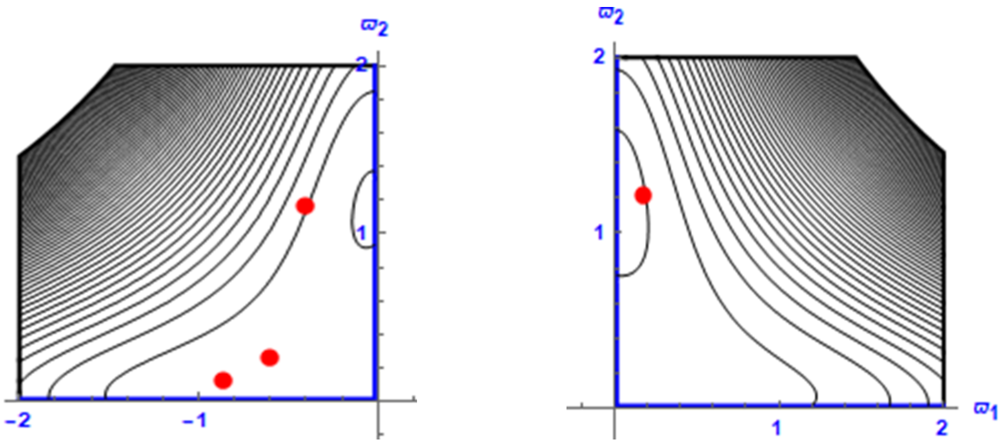
In this case the Caustic divided the plane of parameters into (10) regions:

(1) if (σ_6, σ_7) belongs to $(\mathcal{X}_2, \mathcal{X}_5, \mathcal{X}_7$ or $\mathcal{X}_{10})$, then this region contains three non-degenerate real critical point (2 minimum, 1 saddle) point, and

(2) if (σ_6, σ_7) belong to region $(\mathcal{X}_1, \mathcal{X}_3, \mathcal{X}_4, \mathcal{X}_6, \mathcal{X}_8$ or $\mathcal{X}_9)$, then this region contains one non-degenerate real critical point (minimum).

From the above dissociation we can represent the bifurcation-spreading extremals bythe rows $(2,1,0)$, $(1,0,0)$.

In addition, Figures 8a and 8b below: show the locations of contour lines with respect to the domain of function (2.5), the number and type of critical points corresponding for the region $(\mathcal{X}_5)$ and region $(\mathcal{X}_3)$ in the caustic of function (2.5).



(a) Contours and crtical points in $\mathcal{X}_5$ **(b)** Contours and crtical points in $\mathcal{X}_3$

Figure 8

Case 4: If $\sigma_1 < 0 < \sigma_2$ and $\sigma_1 = \sigma_5$ (as in Figure 9).

In this case the Caustic divided the plane of parameters into (14) regions:

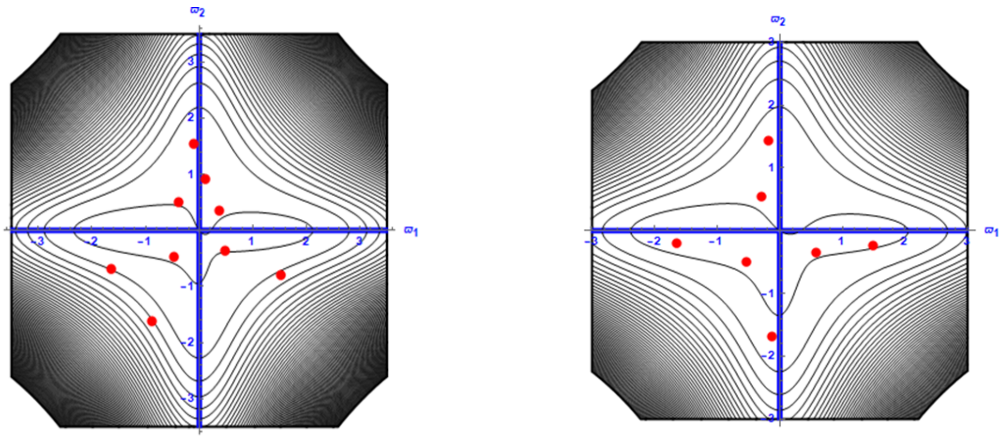
(1) if (σ_6, σ_7) belong to region $(\mathcal{X}_7$ or $\mathcal{X}_8)$, then this region containsnine non-degenerate real critical point (4 minimum, 4 saddles, 1 maximum).

(2) if (σ_6, σ_7) belong to region $(\mathcal{X}_2, \mathcal{X}_4, \mathcal{X}_6, \mathcal{X}_9, \mathcal{X}_{11}$ or $\mathcal{X}_{13})$, then this region contains five non-degenerate real critical point (3 minimum, 2 saddle).

(3) if (σ_6, σ_7) belongs to $(\mathcal{X}_1, \mathcal{X}_{12}$ or $\mathcal{X}_{14})$, then this region contains three non-degenerate real critical point (2 minimum, 1 saddle).

From the above dissociation we can represent the bifurcation-spreading extremals bythe rows $(4,4,1)$, $(3,2,0)$, $(2,1,0)$.

In addition, Figures 9a and 9b shows the locations of contour lines with respect to the domain of function (2.5),the number and type of critical points corresponding for the region $(\mathcal{X}_7)$ and region $(\mathcal{X}_2)$ in the caustic of function (2.5).



(a) Contours and crtical points in $\mathcal{X}_8$ **(b)** Contours and crtical points in $\mathcal{X}_{10}$

Figure 9

Case 5: If $\sigma_1 < \sigma_5 < \sigma_2$ and $\sigma_5 = 0$ (as in Figure 10).

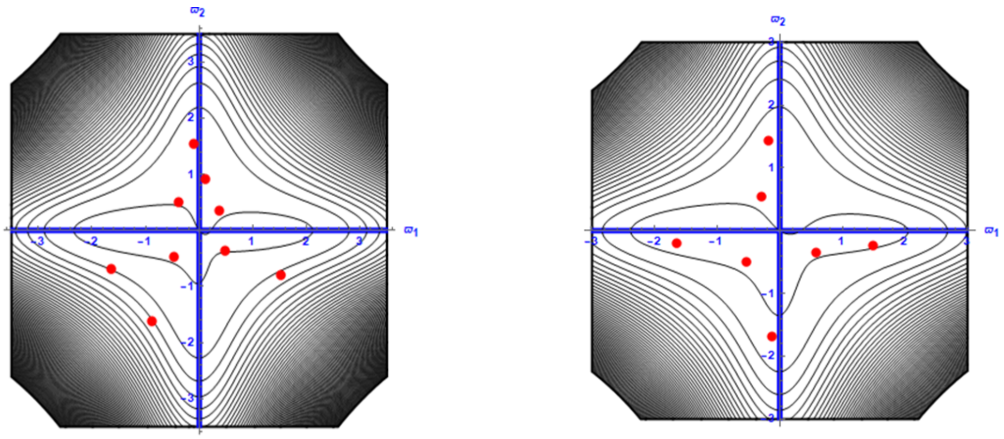
In this case the Caustic divided the plane of parameters into (16) regions:

(1) if (σ_6, σ_7) belong to region $\mathcal{X}_6, \mathcal{X}_8, \mathcal{X}_{14}$ or $\mathcal{X}_{16}$), then this region containsfive non-degenerate real critical point (3 minimum, 2 saddle).

(2) if (σ_6, σ_7) belong to region $(\mathcal{X}_2, \mathcal{X}_4, \mathcal{X}_7, \mathcal{X}_{10}, \mathcal{X}_{12}$ or $\mathcal{X}_{15})$, then this region contains three non-degenerate real critical point (2 minimum, 1 saddle).

(3) if (σ_6, σ_7) belongs to $(\mathcal{X}_1, \mathcal{X}_3, \mathcal{X}_5, \mathcal{X}_9, \mathcal{X}_{11}$ or $\mathcal{X}_{13})$, then this region contains one non-degenerate real critical point (minimum).

From the above dissociation we can represent the bifurcation-spreading extremals bythe rows $(3,2,0)$, $(2,1,0)$, $(1,0,0)$.



(a) Contours and crtical points in $\mathcal{X}_6$ **(b)** Contours and crtical points in $\mathcal{X}_{15}$

Figure 10

In addition, Figures 10a and 10b shows the locations of contour lines with respect to the domain of function (2.5), the number and type of critical points corresponding for the region $(\mathcal{X}_6)$ and region $(\mathcal{X}_7)$ in the caustic of function (2.5).

3. Application

As an application of our results obtained in Section 2, we take the following nonlinear differential equation

$$\upsilon\left(\frac{d^6\varrho}{dg^6}\right)+\tau\left(\frac{d^4\varrho}{dg^4}\right)+\sigma\left(\frac{d^2\varrho}{dg^2}\right)+\Phi\varrho+\varrho^3=\mathcal{F}, \tag{3.1}$$
$$\frac{d^4\varrho}{dg^4}(0)=\frac{d^2\varrho}{dg^2}(0)=\varrho(0)=\frac{d^4\varrho}{dg^4}(\pi)=\frac{d^2\varrho}{dg^2}(\pi)=\varrho(\pi)=0,$$

where υ,τ,σ and Φ are the parameters of the problem, $\varrho=\varrho(g)$, $g\in[0,\pi]$.

Suppose that $\widetilde{\mathcal{B}}:\mathcal{G}\to\mathcal{H}$is Fredholm operator of index zero, where $\mathcal{G}=C^6([0,\pi],\mathcal{R})$ and $\mathcal{H}=C^0([0,1],\mathcal{R})$ is defined by the operator equation,

$$\widetilde{\mathcal{B}}(\varrho,l)=\upsilon\left(\frac{d^6\varrho}{dg^6}\right)+\tau\left(\frac{d^4\varrho}{dg^4}\right)+\sigma\left(\frac{d^2\varrho}{dg^2}\right)+\Phi\varrho+\varrho^3,\quad l=(\upsilon,\tau,\sigma,\Phi).$$

Every solution of the equation in (3.1) is a solution of the operator equations,

$$\widetilde{\mathcal{B}}(\varrho,l)=\mathcal{F},\quad \mathcal{F}\in\mathcal{H}. \tag{3.2}$$

Since, the operator $\widetilde{\mathcal{B}}$ have variational property, then there exists functional $\widetilde{\mathcal{A}}$ such that $\widetilde{\mathcal{B}}(\varrho,l)=\nabla_H\widetilde{\mathcal{A}}(\varrho,l,0)$ or equivalently,

$$\frac{\partial\widetilde{\mathcal{A}}}{\partial\varrho}(\varrho,l)k=\langle\mathcal{H}(\varrho,l),k\rangle,\quad \forall\ \varrho\in\mathcal{C},\ k\in\mathcal{G},$$

and then every solution of equation (3.2) is a critical point of the functional $\widetilde{\mathcal{A}}$

$$\widetilde{\mathcal{A}}(\varrho,l,\mathcal{F})=\int_0^\pi(\upsilon(\frac{(\varrho'''(g))^2}{2})+\tau\left(\frac{(\varrho''(g))^2}{2}\right)+\sigma\left(\frac{(\varrho'(g))^2}{2}\right)+\frac{1}{2}\Phi\varrho^2+\frac{1}{4}\varrho^4-\mathcal{F}\varrho)dg.$$

Thus, the study of the equation (3.2) is equivalent to the study extremely problem

$$\widetilde{\mathcal{A}}(\varrho,l,\mathcal{F})\to extr,\quad \varrho\epsilon\mathcal{G}.$$

The analysis of bifurcation can be found using the local method of Lyapunov-Schmidt to reduce it into finite dimensional space and by localized parameters $\upsilon=\tilde{\upsilon}+o_1$, $\tau=\tilde{\tau}+o_2$, $\sigma=\tilde{\sigma}+o_3$ and $\Phi=\tilde{\Phi}+o_4$, o_1, o_2, o_3 and o_4 are small parameters.

The reduction leads to the function in four variables,

$$\Xi(\eta,o)=\inf_{g^{\pm}(\varrho,r_j)=\eta_j\ \forall\ j}\widetilde{\mathcal{A}}(\varrho,o),\quad \eta=(\eta_1,\eta_2,\eta_3,\eta_4),\ o=(o_1,o_2,o_3,o_4).$$

It is well known that in the reduction of Lyapunov-Schmidt the function, $\Xi(\eta,o)$ is smooth. This function has all the topological and analytical properties of functional $\widetilde{\mathcal{A}}$ [3]. In particular, for small o there is 1-1 corresponding between the critical points of functional $\widetilde{\mathcal{A}}$ and smooth function Ξ, preserving the type of critical points (multiplicity, index Morse, etc.) [3]. By the scheme of Lyapunov-Schmidt, the linearized equation corresponding to the equation (3.2) has the form:

$$\upsilon\tilde{y}''''''+\tau\tilde{y}''''+\sigma\tilde{y}''+\Phi\tilde{y}=0,\quad \tilde{y}\in\mathcal{H},$$
$$\frac{d^4\tilde{y}}{dg^4}(0)=\frac{d^2\tilde{y}}{dg^2}(0)=\tilde{y}(0)=\frac{d^4\tilde{y}}{dg^4}(\pi)=\frac{d^2\tilde{y}}{dg^2}(\pi)=\tilde{y}(\pi)=0.$$

The point $(v,\tau,\sigma,\Phi) = (0,0,0,0)$ is a bifurcation point [7]. Localized parameters v,τ,σ,Φ as follow $v = 0 + o_1$, $\tau = 0 + o_2$, $\sigma = 0 + o_3$ and $\Phi = 0 + o_4$, lead to bifurcation along the modes, $r_j(g) = c_j \sin(jg)$, $j = 1,2,3,4$, where $\|r_j\| = 1$ and $c_j = \sqrt{\frac{2}{\pi}}$.

Let $S_d = \ker(N^*) = span\{r_1,r_2,r_3,r_4\}$, where $N^* = d\widetilde{\mathcal{B}}(0,l) = v\left(\frac{d^6\varrho}{dg^6}\right) + \tau\left(\frac{d^4\varrho}{dg^4}\right) + \sigma\left(\frac{d^2\varrho}{dg^2}\right) + \Phi$.

Then the space $\mathcal{G}$ can be decomposed in direct sum of two subspaces, $\ker(N^*)$ and the orthogonal complement to $\ker(N^*)$,

$$\mathcal{G} = \ker(N^*) \oplus \mathcal{G}^{\infty-4},\ \mathcal{G}^{\infty-4} = \ker(N^*)^{\perp} \cap \mathcal{G} = \{s \in \mathcal{G} : s \perp \ker(N^*)\}.$$

Similarly, the space $\mathcal{H}$ can be decomposed in direct sum of two subspaces, $\ker(N^*)$ and the orthogonal complement to $\ker(N^*)$,

$$\mathcal{H} = \ker(N^*) \oplus \mathcal{H}^{\infty-4},\ \mathcal{H}^{\infty-4} = \ker(N^*)^{\perp} \cap \mathcal{H} = \{a \in \mathcal{H} : a \perp \ker(N^*)\}.$$

Hence every vector $\varrho \in \mathcal{G}$, can be written in the form,

$$\varrho = t + s,\ t = \sum_{j=1}^{4} r_j\eta_j \in \ker(N^*),\ \ \ker(N^*) \perp s \in \mathcal{G}^{\infty-4},\ \ \eta_j = \langle v, r_j\rangle$$

Similarly,

$$\widetilde{\mathcal{B}}(\varrho,l) = \widetilde{\mathcal{B}}^{(4)}(\varrho,l) + \widetilde{\mathcal{B}}^{(\infty-4)}(\varrho,l);$$

$$\widetilde{\mathcal{B}}^{(4)}(\varrho,l) = \sum_{j=1}^{4} s_j(\varrho,l) r_j \in \ker(N^*),\ \ \widetilde{\mathcal{B}}^{(\infty-4)}(\varrho,l) \in \mathcal{H}^{\infty-4},$$

$$s_j(\varrho,l) = \langle \widetilde{\mathcal{B}}(\varrho,l), r_j\rangle,$$

where $\widetilde{\mathcal{B}}^{(4)}(\varrho,l)$ is the projection of the space $\mathcal{H}$ on $\ker(N^*)$ and $\widetilde{\mathcal{B}}^{(\infty-4)}(\varrho,l)$ is the projection of the space $\mathcal{H}$ on $\mathcal{H}^{\infty-4}$.

Since $\mathcal{F} \in \mathcal{H}$ implies that

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2,\ \mathcal{F}_1 = \sum_{j=1}^{4} r_j\eta_j \in N^*,\ \mathcal{F}_2 \in \mathcal{H}^{\infty-4}. \tag{3.3}$$

According to equation (3.2), can be written in the form:

$$\widetilde{\mathcal{B}}^{(4)}(\varrho,l) = \mathcal{F}_1,$$
$$\widetilde{\mathcal{B}}^{(\infty-4)}(\varrho,l) = \mathcal{F}_2.$$

By the implicit function theorem, there exist smoothfunction $\mathcal{C}^* : \ker(N^*) \to \mathcal{G}^{\infty-4}$, such that $\Xi(\eta,o,\mathcal{F}) = \widetilde{\mathcal{A}}(\mathcal{C}^*(\eta,o,\mathcal{F}),o,\mathcal{F})$ and then, the key-function can be written in the form,

$$\begin{aligned}\Xi(\eta,o) &= \widetilde{\mathcal{A}}(r_1\eta_1 + r_2\eta_2 + r_3\eta_3 + r_4\eta_4 + \mathcal{C}^*(r_1\eta_1 + r_2\eta_2 + r_3\eta_3 + r_4\eta_4, o), o)\\ &= P^*(\eta,o) + o(|\eta|^4) + O(|\eta|^4)O(o).\end{aligned}$$

The function $P^*(\eta,o)$ can be found as follows, substitute the value of ϱ in the above Functional, we obtain

$$v\int_0^{\pi} \frac{(\varrho'''(g))^2}{2} dg = v \cdot \left(32\eta_2^2 + 2048\eta_4^2 + \frac{729\eta_3^2}{2} + \frac{\eta_1^2}{2}\right),$$

$$\tau\int_0^{\pi}\frac{(\varrho''(g))^2}{2}dg=\tau\cdot\left(\frac{81\eta_3^2}{2}+128\eta_4^2+8\eta_2^2+\frac{\eta_1^2}{2}\right),$$
$$\sigma\int_0^{\pi}\frac{(\varrho'(g))^2}{2}dg=\sigma\cdot\left(\frac{9\eta_3^2}{2}+8\eta_4^2+2\eta_2^2+\frac{\eta_1^2}{2}\right),$$
$$\Phi\int_0^{\pi}\frac{1}{2}\varrho^2dg=\Phi\cdot\left(\frac{\eta_3^2}{2}+\frac{1}{2}\eta_4^2+\frac{1}{2}\eta_2^2+\frac{\eta_1^2}{2}\right),$$
$$\int_0^{\pi}\frac{1}{4}\varrho^4dg=\sqrt{2}\left(\frac{1}{\pi}\right)^{3/2}\left(\frac{4}{9}\eta_3^3+\frac{4}{3}\eta_1^3+\frac{16}{5}\eta_1\eta_2^2+\frac{108}{35}\eta_1\eta_3^2+\frac{64}{21}\eta_1\eta_4^2-\frac{4}{5}\eta_1^2\eta_3+\frac{16}{7}\eta_2^2\eta_3\right.$$
$$\left.+\frac{64}{55}\eta_3\eta_4^2-\frac{64}{35}\eta_1\eta_2\eta_4+\frac{64}{15}\eta_2\eta_3\eta_4\right),$$
$$\int_0^{\pi}\mathcal{F}_1\varrho dg=\tilde{q}_1\cdot\eta_1+\tilde{q}_2\cdot\eta_2+\tilde{q}_3\cdot\eta_3+\tilde{q}_4\cdot\eta_4.$$

Since the function $\mathcal{F}$ is not symmetric with respect to the involution $L:\mathcal{F}(\eta)\mapsto\mathcal{F}(\pi-\eta)$, then we get

$$P^*(\eta,o)=\frac{3\eta_1^4}{8\pi}+\frac{3\eta_3^4}{8\pi}+\frac{3\eta_4^4}{8\pi}+\frac{3\eta_2^4}{8\pi}+\frac{3\eta_1\eta_2^2\eta_3}{2\pi}+\frac{3\eta_2\eta_3^2\eta_4}{2\pi}-\frac{3\eta_1^2\eta_2\eta_4}{2\pi}+\frac{3\eta_1\eta_2\eta_3\eta_4}{\pi}$$
$$-\tilde{q}_1\cdot\eta_1-\tilde{q}_2\cdot\eta_2-\tilde{q}_3\cdot\eta_3-\tilde{q}_4\cdot\eta_4+\frac{\Phi\eta_1^2}{4}+\frac{\Phi\eta_2^2}{4}+\frac{\Phi\eta_3^2}{4}+\frac{\Phi\eta_4^2}{4}+\frac{\upsilon\eta_1^2}{2}$$
$$+\frac{729\upsilon\eta_3^2}{2}+\frac{\tau\eta_1^2}{2}+\frac{81\tau\eta_3^2}{2}+\frac{\sigma\eta_1^2}{2}+\frac{9\sigma\eta_3^2}{2}-\frac{\eta_1^3\eta_3}{2\pi}+\frac{3\eta_1^2\eta_2^2}{2\pi}+\frac{3\eta_1^2\eta_3^2}{2\pi}+\frac{3\eta_1^2\eta_4^2}{2\pi}$$
$$+\frac{3\eta_2^2\eta_3^2}{2\pi}+\frac{3\eta_2^2\eta_4^2}{2\pi}+\frac{3\eta_3^2\eta_4^2}{2\pi}+32\upsilon\eta_2^2+2048\upsilon\eta_4^2+8\tau\eta_2^2+128\tau\eta_4^2+2\sigma\eta_2^2+8\sigma\eta_4^2.$$

Hence

$$P^*(\eta,o)=-\frac{\eta_1^3\eta_3}{2\pi}+\frac{3\eta_1^2\eta_2^2}{2\pi}+\frac{3\eta_1^2\eta_3^2}{2\pi}+\frac{3\eta_1^2\eta_4^2}{2\pi}+\frac{3\eta_2^2\eta_3^2}{2\pi}+\frac{3\eta_2^2\eta_4^2}{2\pi}+\frac{3\eta_3^2\eta_4^2}{2\pi}+\frac{3\pi\eta_1\eta_2^2\eta_3}{2\pi}$$
$$+\frac{3\eta_2\eta_3^2\eta_4}{2\pi}-\frac{3\eta_1^2\eta_2\eta_4}{2\pi}+\frac{3\eta_4^4}{8\pi}+\frac{3\eta_3^4}{8\pi}+\frac{3\eta_2^4}{8\pi}+\frac{3\eta_1^4}{8\pi}$$
$$-\tilde{q}_4\eta_4+\left(2048\upsilon+128\tau+8\sigma+\frac{\Phi}{4}\right)\eta_4^2-\tilde{q}_3\eta_3+\left(\frac{729\upsilon}{2}+\frac{81\tau}{2}+\frac{9\sigma}{2}+\frac{\Phi}{4}\right)\eta_3^2$$
$$-\tilde{q}_2\eta_2+\left(32\upsilon+8\tau+2\sigma+\frac{\Phi}{4}\right)\eta_2^2-\tilde{q}_1\eta_1+\left(\frac{\upsilon}{2}+\frac{\tau}{2}+\frac{\sigma}{2}+\frac{\Phi}{2}\right)\eta_1^2+\frac{3\eta_1\eta_2\eta_3\eta_4}{\pi}.$$

Now, assume that $\eta_j:=\sqrt[4]{\frac{2\pi}{3}}\cdot v_j$, $j=1,2,3,4$, we obtain

$$-\frac{v_1^3v_3}{3}+v_1^2v_2^2+v_1^2v_3^2+v_1^2v_4^2+v_2^2v_3^2+v_2^2v_4^2+v_3^2v_4^2+v_1v_2^2v_3+v_2v_3^2v_4-v_1^2v_2v_4$$
$$+\frac{v_4^4}{4}+\frac{v_3^4}{4}+\frac{v_2^4}{4}+\frac{v_1^4}{4}+\frac{\left(2048\upsilon+128\tau+8\sigma+\frac{\Phi}{4}\right)\sqrt{2}\sqrt{3}\sqrt{\pi}v_4^2}{3}+\frac{\left(\frac{729\upsilon}{2}+\frac{81\tau}{2}+\frac{9\sigma}{2}+\frac{\Phi}{4}\right)\sqrt{2}\sqrt{3}\sqrt{\pi}v_3^2}{3}$$
$$+\frac{\left(32\upsilon+8\tau+2\sigma+\frac{\Phi}{4}\right)\sqrt{2}\sqrt{3}\sqrt{\pi}v_2^2}{3}+\frac{\left(32\upsilon+8\tau+2\sigma+\frac{\Phi}{4}\right)\sqrt{2}\sqrt{3}\sqrt{\pi}v_2^2}{3}+\frac{\left(\frac{\upsilon}{2}+\frac{\tau}{2}+\frac{\sigma}{2}+\frac{\Phi}{2}\right)\sqrt{2}\sqrt{3}\sqrt{\pi}v_1^2}{3}$$
$$+2v_1v_2v_3v_4-\frac{\sqrt[4]{2}\sqrt[4]{3}\sqrt[4]{\pi}}{3}\left(\tilde{q}_1\eta_1+\tilde{q}_2\eta_2+\tilde{q}_3\eta_3+\tilde{q}_4\eta_4\right).$$

The geometrical form of bifurcations of critical points and the first asymptotic of branches of bifurcating for the functions Ξ is completely determined by its principal part P^*. By changing variables in the function P^* as follows

$$\eta_j = v_j,$$

we have the function P^* is equivalent to the function $\mathcal{M}(v_j;\tau)$.

Hence the caustic of the function P^* coincides with the caustic of the function $\mathcal{M}(v_j;\tau)$.

The functions $\mathcal{M}(v_i;\tau)$ have all the topological and analytical properties of functional $\widetilde{\mathcal{A}}$, so the study of bifurcation analysis of the equation (3.2) is equivalent to the study of bifurcation analysis of the function $\mathcal{M}(v_j;\tau)$. This shows that the study of bifurcation of extremals of the functional $\widetilde{\mathcal{A}}$ is reduced to the study of bifurcation of extremals of the function $\mathcal{M}(v_j;\tau)$.

4. Conclusion

In this work, we found the caustic set of the function (2.1) with four variables and eight parameters, and we observed that the caustic set decomposed the space of parameters to different regions according to the change of parameter values. Moreover, we discuss the classification of critical points in each region. Finally, we studied the branching solutions for the equation (3.1) is an application of our work.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] D. P. L. Castrigiano and S. A. Hayes, *Catastrophe Theory*, 2nd edition, CRC Press (2018).

[2] O. Y. Danilova, Bifurcations of extremals under symmetric and boundary singularities, *Trudy Molodykh Uchenykh Mat. Fak. VSU* **6** (2001), 44 – 53 (in Russian).

[3] B. M. Darinskii, Y. I. Sapronov and S. L. Tsarev, Bifurcations of extremals of Fredholm functionals, *Journal of Mathematical Sciences* **145**(6) (2007), 5311 – 5453, DOI: 10.1007/s10958-007-0356-2.

[4] E. V. Derunova and Y. I. Sapronov, Application of normalized key functions in a problem of branching of periodic extremals, *Russian Mathematics* **59**(8) (2015), 9 – 18, DOI: 10.3103/S1066369X15080022.

[5] A. V. Gnezdilov, Corner singularities of Fredholm functional, *Scientific Journal Proceedings of Vor. State University Series: Physics - Mathematics* **1** (2003), 99 – 114.

[6] M. A. A. Hussain, *Corner Singularities of Smooth Functions in the Analysis of Bifurcations Balance of the Elastic Beams and Periodic Waves*, PhD Thesis, Voronezh University (2005) (in Russian).

[7] M. A. A. Hussain and M. J. Mohammed, Classification of critical points and caustic of the functions of codimensions eight and fifteen, *Journal of Karbala University* **10**(4) (2014), 64 – 77, URL: `https://kj.uokerbala.edu.iq/article_100456.html#downloadTab`.

[8] H. K. Kadhim, Three-mode bifurcation of extremals in the analysis of bifurcation solutions of sixth order differential equation, *Journal of College of Education for Pure Sciences* **4** (2014), 149 – 162, URL: `https://www.iasj.net/iasj/download/5c0c10a21cf3797d`.

[9] H. K. Kadhim and M. A. Abdul Hussain, The analysis of bifurcation solutions of the Camassa-Holm equation by angular singularities, *Problemy Analiza – Issues of Analysis* **9**(27)(1) (2020), 66 – 82, DOI: 10.15393/j3.art.2020.6770.

[10] D. V. Kostin, Initial-boundary value problems for Fuss-Winkler-Zimmermann and Swift-Hohenberg nonlinear equations of 4th order, *Matematicki Vesnik* **70**(1) (2018), 26 – 39, URL: `http://www.vesnik.math.rs/vol/mv18103.pdf`.

[11] D. V. Kostin, T. I. Kostina and L. V. Stenyuhin, On the existence of extremals of some nonlinear Fredholm operators, *Journal of Physics: Conference Series* **1203** (2019), 012066, DOI: 10.1088/1742-6596/1203/1/012066.

[12] A. Liapounoff, Sur la stabilité des figures ellipsoïdales d'équilibre d'un liquide animé d'un mouvement de rotation, *Annales De La Faculté Des Sciences De Toulouse* **6**(1) (1904), pp. 5 – 116, URL: `http://www.numdam.org/item/AFST_1904_2_6_1_5_0.pdf`.

[13] J. Montaldi, *Singularities Bifurcations and Catastrophes*, University of Manchester (2020), URL: `https://personalpages.manchester.ac.uk/staff/j.montaldi/SBC-LookInside.pdf`.

[14] Yu. I. Sapronov and E. V. Derunova, Bifurcations of critical orbits of SO(2)-invariant Fredholm functionals at critical points with double resonances, *Global and Stochastic Analysis* **2**(1) (2015), URL: `http://www.mukpublications.com/resources/gsa4no2.pdf`.

[15] E. Schmidt, Zur Theorie der linearen und nichtlinearen Integralgleichungen. III. Teil, Über die Auflösung der nichtlinearen Integralgleichung und die Verzweigung ihrer Lösungen, *Mathematische Annalen* **65** (1908), 370 – 399, DOI: 10.1007/BF01456418.

[16] N. Sidorov, B. Loginov, A. Sinitsyn and M Falaleev, *Lyapunov-Schmidt Methods in Nonlinear Analysis and Applications*, Springer, Vol. **550** (2002), DOI: 10.1007/978-94-017-2122-6.