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Research Article

A Study of New Separation Axioms in Topology Induced by θ-sgp-Open Sets

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Abstract. This paper introduces separation axioms using θ-sgp-open sets in topological spaces. We also focus on some of their basic properties and also made the implication of these axioms among themselves and with other known axioms. Also, we investigated characterizations of these axioms. It also introduces the concept of sober θ-sgp-R_0 spaces.

Keywords. θ-sgp-open, θ-sgp-T_0, θ-sgp-T_1, θ-sgp-T_2, θ-sgp-R_0, sober θ-sgp-R_0

Mathematics Subject Classification (2020). 54D10

1. Introduction

Separation axioms in topology give a very influential role in evaluation. The initial step of generalized closed set was done by Levine [5] in general topology and also defined $T_{1/2}$-space which was accordingly fixed between T_0-space and T_1-space. Thereupon, several separation axioms have been defined and studied. Recently, Navalagi [7] introduced the concept of semi-generalized-T_i spaces using sg-open sets. The authors in [8] defined new set called θ-sgp-open set.

We continue the study on new set defined in [8] and we introduce new separation axioms called θ-sgp-T_0, θ-sgp-T_1 and θ-sgp-T_2 spaces using θ-sgp-open sets and also, we define weak

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separation axiom called sober θ-sgp-R_0-space using θ-sgp-closure and we characterize and made implications of these axioms mutually themselves and by all of other supported axioms. some examples are given to illustrate the results.

2. Preliminaries

In the entire paper no separation axioms are assumed on the notations (X,τ) also (Y,σ) (or simply X and Y) which stands for topological spaces except clearly stated. If $Q \subset X$, then $Cl(Q)$ and $Int(Q)$ stands for closure of Q and interior of Q in X, respectively.

Definition 2.1. Let $Q \subset X$. Then Q is called

(i) a semi-open set [4] when $Q \subseteq Cl(Int(Q))$.

(ii) a semi-closed set [2] when $Int(Cl(Q)) \subseteq Q$.

(iii) a pre-open set [6] when $Q \subseteq Int(Cl(Q))$.

Definition 2.2. A $Q \subset X$ is the intersection of pre-closed sets [6] that contain Q is called pre-closure [3] of Q and is symbolize as $pCl(Q)$.

Definition 2.3. Whenever $pCl(U) \cup Q \neq \emptyset$, for each pre-open set U having x, then a point $x \in X$ is known as a pre-θ-cluster point [1] of Q.

Definition 2.4. The set of all pre-θ-cluster points of Q is said to be pre-θ-closure of Q and is symbolize as $pCl_\theta(Q)$. A subset Q is called pre-θ-closed set [1] when $Q = pCl_\theta(Q)$. The complement of pre-θ-closed set is pre-θ-open set.

Definition 2.5. A space X is:

(i) T_0-space [13] (pre-θ-T_0-space [1]) when for each different points c and d in X, there exists an open set(pre-θ-open set) containing one of them but does not contain the other.

(ii) T_1-space [13] (pre-θ-T_1-space [1]) when for each different points c and d in X there exist open(pre-θ-open) sets P and Q so that $c \in P$, $d \notin P$ and $d \in Q$, $c \notin Q$.

(iii) T_2-space [13] (pre-θ-T_2-space [1]) when for each pair of different points $c, d \in X$, there exist different open(pre-θ-open) sets P and Q so that $c \in P$, $d \in Q$ and $P \cap Q = \phi$.

(iv) $T_\theta$sgp-space [12] if θ-sgp-closed set is semi-closed set.

Definition 2.6. If $pCl_\theta(C) \subset P$ when $C \subset P$ and P is semi-open in X, then a subset C of a space X is named as θ-semigeneralized pre-closed set [8] (in short, θ-sgp-closed).
The complement of θ-semigeneralized pre-closed set is termed as θ-semigeneralized pre-open (in short, θ-sgp-open).

Definition 2.7. For a subset C belongs to X, θ-semigeneralized pre-closure [9] of C, symbolize as θ-$sgpCl(C)$ and is illustrate as θ-$sgpCl(C) = \cup\{G : C \subseteq G, G$ is θ-sgp-closed in $X\}$.

Definition 2.8. If we have a θ-sgp-open set P so that $x \in P \subseteq C$, then a subset C of a space X is said to be θ-semigeneralized pre-neighbourhood [9] (in precis, θ-sgp-nbd) of a point x of X.

Definition 2.9. Let $C \subset X$. If each θ-sgp-nbd of x consist a point of C different from x, that is $[N - \{x\}] \cap C \neq \phi$, for all θ-sgp-nbd N of x, then a point $x \in X$ is said to be a θ-semi generalized pre-limit point [9] of C.
The collection of all θ-sgp-limit points of C is said to be θ-sgp-derived set [9] of C and is symbolize as $\theta\text{-}sgpd(C)$.

Definition 2.10. Let $C \subset X$. The θ-sgp-kernel of C [9], symbolize as θ-sgp-$ker(C)$ is defined as the set θ-sgp-$ker(C) = \cup\{P : C \subseteq P$ and P is θ-sgp-open in $X\}$.

Definition 2.11. Consider a point x of a space X. The θ-sgp-kernel of x [9], symbolize as θ-sgp-$ker(\{x\})$ is said to be a set θ-sgp-$ker(\{x\}) = \cup\{P : x \in P$ and P is θ-sgp-open in $X\}$.

Definition 2.12. A space X is termed as θ-semigeneralized pre-Ro (in short, θ-sgp-Ro) [9] space, whenever for each θ-sgp-open set K and $\alpha \in K$ implies $\theta\text{-}sgpCl(\{\alpha\}) \subseteq K$.

Definition 2.13. A function $f : X \to Y$ is named as θ-semigeneralized pre irresolute (in precis, θ-sgp-irresolute) [10] when $f^{-1}(F)$ is θ-sgp-closed in X for each θ-sgp-closed set F of Y.

Corollary 2.14 ([9])**.** *The subsequent properties are identical for a space X*

(a) *X is a θ-sgp-Ro-space.*

(b) $\theta\text{-}sgpCl(\{\alpha\}) = \theta\text{-}sgp\text{-}ker(\{\alpha\})$ *for all* $\alpha \in X$.

Remark 2.15 ([8])**.** The concept of θ-sgp-closed sets and closed sets are not dependent of each other.

3. Results and Discussion

Separation Axioms Associated with θ-sgp-open Sets

Definition 3.1. A spaces X is

(i) θ-semigeneralized pre-T_0 (in short, θ-sgp-T_0)-space, when for any pair $i, j \in X$ and $i \neq j$, there is a θ-sgp-open set having one point but not the other.

(ii) θ-semigeneralized pre-T_1 (in short, θ-sgp-T_1)-space, when for any pair of unequal points $i, j \in X$, there are different θ-sgp-open sets M and N so that $i \in M$, $j \notin M$ and $j \in N$, $i \notin N$.

(iii) θ-semigeneralized pre-T_2 (in short, θ-sgp-T_2)-space, when for any pair of unequal points $i, j \in X$, there are different θ-sgp-open sets M and N so that $i \in M$, $j \in N$ and $M \cap N = \emptyset$.

The subsequent examples illustrate the above definitions.

Example 3.2. Consider $X = \{m, n\}$ and $\tau = \{X, \emptyset, \{n\}\}$ as a topology on X. Then (X, τ) is θ-sgp-T_0-space, for different points m, n in X and $\{n\}$ is the θ-sgp-open set such that $m \notin \{n\}$, $n \in \{n\}$.

Example 3.3. Consider $X = \{i, j\}$ and $\tau = \{X, \emptyset, \{i\}, \{j\}\}$ as a topology on X. Then θ-sgp-open sets are X, $\emptyset$, $\{i\}$, $\{j\}$. Here $i, j \in X$ and $i \neq j$,

(a) then there are θ-sgp-open sets $\{i\}$, $\{j\}$ of X so that $i \in \{i\}$, $i \notin \{j\}$ and $i \notin \{j\}$, $j \in \{j\}$. Therefore, X is θ-sgp-T_1-space.

(b) then there are θ-sgp-open sets $\{i\}$, $\{j\}$ of X so that $i \in \{i\}$, $j \in \{j\}$ and $\{i\} \cap \{j\} = \emptyset$. Therefore, X is θ-sgp-T_2-space.

Proposition 3.4. *A space X is θ-sgp-T_0-space iff θ-sgp-closures of different points are disjoint.*

Proof. Consider $p, q \in X$ with $p \neq q$ and X is θ-sgp-T_0-space. We shall show that θ-$sgpCl(\{p\}) \neq \theta$-$sgpCl(\{q\})$. As X is θ-sgp-T_0, we have a θ-sgp-open set M so that $p \in M$ but $q \notin M$. Also, $p \notin X - M$ and $q \in X - M$ where $X - M$ is θ-sgp-closed set in X. Now by definition $\{q\}$ is the intersection of all θ-sgp-closed sets which have q. Hence $q \in \theta$-$sgpCl(\{q\})$ but $p \notin \theta$-$sgpCl(\{q\})$ as $p \notin X - M$. Therefore, θ-$sgpCl(\{p\}) \neq \theta$-$sgpCl(\{q\})$.

Conversely, for any pair of different points $p, q \in X$ and θ-$sgpCl(\{p\}) \neq \theta$-$sgpCl(\{q\})$. Then there is at least one point $r \in X$ so as $r \in \theta$-$sgpCl(\{p\})$ but $r \notin \theta$-$sgpCl(\{q\})$. We claim that $p \notin \theta$-$sgpCl(\{q\})$. If $p \in \theta$-$sgpCl(\{q\})$ then $p \subseteq \theta$-$sgpCl(\{q\})$ implies θ-$sgpCl(\{p\}) \subseteq \theta$-$sgpCl(\{q\})$. So, $r \in \theta$-$sgpCl(\{q\})$, which is not true. Hence, $p \notin \theta$-$sgpCl(\{q\})$. Now, $p \notin \theta$-$sgpCl(\{q\})$ implies $p \in X$-θ-$sgpCl(\{q\})$, which is a θ-sgp-open set in X having p but not q. Thus X is a θ-sgp-T_0-space. □

Proposition 3.5. *Whenever X is a θ-sgp-T_0-space, then each subspace of X is θ-sgp-T_0-space.*

Proof. Consider X as a θ-sgp-T_0-space and $Y \subset X$. Take α and β as unequal points of Y. As $Y \subset X$, α and β are also unequal points of X. As per given, X is θ-sgp-T_0-space, we have a θ-sgp-open set K so that $\alpha \in K$, $\beta \notin K$. Then we have $Y \cap K$ is θ-sgp-open in Y having α but not β. Thus Y is also θ-sgp-T_0-space. □

Proposition 3.6. *Whenever X is θ-sgp-T_0-space, $T_\theta$sgp-space and Y is θ-sgp-closed subspace of X, then Y is θ-sgp-T_0-space.*

Proof. Consider X as θ-sgp-T_0-space, $T_\theta$sgp-space and Y as θ-sgp-closed subspace of X. Taking $c, d \in Y$ and $c \neq d$. As $Y \subset X$, c and d are different points of X. As per given X is θ-sgp-T_0-space, we have a θ-sgp-open set G then $c \in G$ and $d \notin G$. Also as X is $T_\theta$sgp-space, G is semi-open in X. Then we have $Y \cap G$ is semi-open in Y. Therefore $c \in Y \cap G$ and $d \notin Y \cap G$. Thus Y is also θ-sgp-T_0-space. □

Definition 3.7. A function $g : X \to Y$ is a point θ-sgp-closure one-to-one if for each $p, q \in X$ such that θ-$sgpCl(\{p\}) \neq \theta$-$sgpCl(\{q\})$, then θ-$sgpCl(\{f(p)\}) \neq \theta$-$sgpCl(\{f(q)\})$.

Proposition 3.8. *g is one-to-one, whenever $g : X \to Y$ is a point θ-sgp-closure one-to-one function and X is θ-sgp-T_0-space.*

Proof. Consider $p, q \in X$ with $p \neq q$. As X is θ-sgp-T_0, by Proposition 3.4, θ-$sgpCl(\{p\}) \neq \theta$-$sgpCl(\{q\})$. But g is point θ-sgp-closure one-to-one, so θ-$sgpCl(\{(p)\}) \neq \theta$-$sgpCl(\{g(q)\})$. Hence $g(p) \neq g(q)$. Thus, g is one-to-one. □

Proposition 3.9. *Consider $g : X \to X$ be a function from a θ-sgp-T_0-space X into a θ-sgp-T_0-space Y. Then g is point θ-sgp-closure one-to-one iff g is one-to-one.*

Proof. It follows from Proposition 3.8 and the definitions. □

Proposition 3.10. *Consider $g : X \to Y$ as an one-to-one, θ-sgp-irresolute function. If Y is θ-sgp-T_0, then X is also θ-sgp-T_0.*

Proof. Consider $x, y \in X$ where $x \neq y$. As g is one-to-one, $g(x) \neq g(y)$, but Y is θ-sgp-T_0, so there prevails a θ-sgp-open set V_x in Y so that $g(x) \in Vx$ and $g(y) \notin Vx$ or there prevails a θ-sgp-open set V_y in Y then $g(y) \in Vy$ and $g(x) \notin Vy$. By θ-sgp-irresoluteness of g, $g^{-1}(V_x)$ θ-sgp-open in X so that $x \in g^{-1}(V_x)$ and $y \notin g^{-1}(V_x)$ or $g^{-1}(V_y)$ is θ-sgp-open in X so that $y \in g^{-1}(V_y)$ and $x \notin g^{-1}(V_y)$. This gives X is θ-sgp-T_0. □

Proposition 3.11. *Y is θ-sgp-T_1-space, whenever X is a θ-sgp-T_1-space, $T_\theta$sgp-space and $Y \subset X$.*

Proof. Considering X as a θ-sgp-T_1-space and also $Y \subset X$. Taking c and d as different points of Y. As per given X is a θ-sgp-T_1-space, we have θ-sgp-open sets M and N then $c \in M$, $d \notin M$ and $c \notin N$, $d \in N$. Again, as X is a $T_\theta$sgp-space, M and N are semi-open sets in X. Then $Y \cap M$ and $Y \cap N$ are semi-open sets so θ-sgp-open sets of Y so as $c \in Y \cap M$, $d \notin Y \cap M$ and $c \notin Y \cap N$, $d \in Y \cap N$. Thus, Y is also θ-sgp-T_1-space. □

Proposition 3.12. *The subsequent statements are identical for a space X:*

(a) *X is θ-sgp-T_1*

(b) *Each singleton subset of X is θ-sgp-closed in X.*

(c) *For each $A \subset X$, $A = \theta$-sgp-$ker(A)$, or equivalently, every subset of X has been the intersection of θ-sgp-open sets.*

(d) *For every $x \in X$, $\{x\} = \theta$-sgp-$ker(\{x\})$, or equivalently, every singleton subset of X has been the intersection of θ-sgp-open sets.*

Proof. (i)⇒(ii): Taking $x \in X$. Now by (i), for any $y \in X$, $y \neq x$, we have a θ-sgp-open set V_y having y but not x. Hence $y \in V_y \subset \{x\}^c$. Now varying Y over $\{x\}^c$, we get $\{x\}^c = \cup\{V_y : y \in \{x\}^c$. So $\{x\}^c$ is the union of θ-sgp-open sets. Thus $\{x\}$ is θ-sgp-closed.

(ii)⇒(iii): If $A \subset X$ then for all point $y \notin A$, $\{y\}^c$ is θ-sgp-open by (ii). Hence $A = \cap\{\{y\}^c : y \in A^c$ is the intersection of θ-sgp-open sets.

(iii)⇒(iv): Trivial.

(iv)⇒(i): Taking $x, y \in X$ and $x \neq y$. Then by (iv), we have a θ-sgp-open set U_x so that $x \in U_x$ and $y \notin U_x$. Similarly, we have a θ-sgp-open set U_y so that $y \in U_y$ and $x \notin U_y$Hence X is θ-sgp-T_1. □

Proposition 3.13. *Every θ-sgp-nbd of x having infinitely many points of C, whenever X is θ-sgp-T_1 and $x \in \theta$-$sgpd(C)$ for some $C \subset X$.*

Proof. Assume that U is a θ-sgp-nbd of x, then $U \cap C$ is finite. Taking $U \cap C = \{x_1, x_2, \ldots, x_n\} = C$.

Clearly, D is a θ-sgp-closed set. Therefore $V = U-(D-\{x\})$ is a θ-sgp-nbd of x and $V\cap(C-\{x\}) = \emptyset$, which gives that $x \notin \theta\text{-}sgpd(C)$, which is false. □

Proposition 3.14. *Consider $g : X \to Y$ as an one-to-one, θ-sgp-irresolute function. X is θ-sgp-T_1, whenever Y is θ-sgp-T_1.*

Proof. Considering p and q as two distant points in X and Y is θ-sgp-T_1, then there prevail a pair of θ-sgp-open sets C, D of Y so that $p \in C$, $q \in D$ and $p \notin D$, $q \notin C$. Since g is one-to-one, θ-sgp-irresolute, to each pair of distinct points $g^{-1}(p), g^{-1}(q)$ in X, there prevail a pair of θ-sgp-open sets $g^{-1}(C), g^{-1}(D)$ of X so that $g^{-1}(p) \in g^{-1}(C)$, $g^{-1}(q) \in g^{-1}(D)$ and $g^{-1}(p) \notin g^{-1}(D)$, $g^{-1}(q) \notin g^{-1}(C)$. Hence X is θ-sgp-T_1. □

Proposition 3.15. *A space X is θ-sgp-T_1 iff it is θ-sgp-T_0 and θ-sgp-R_0.*

Proof. Consider X as a θ-sgp-T_1-space. Then by definitions, X is θ-sgp-T_0. It follows also by Corollary 2.14 that X is θ-sgp-R_0.

Contrarily, suppose that X is both θ-sgp-T_0 and θ-sgp-R_0. We want to prove that X is θ-sgp-T_1. Taking $c, d \in X$ and $c \neq d$. Since X is θ-sgp-T_0, we have a θ-sgp-open set M so that $c \in M$ and $d \notin M$ or we have a θ-sgp-open set N so that $d \in N$ and $c \notin N$. Without loss of generality, we may consider that there is a θ-sgp-open set M so that $c \in M$ and $d \notin M$. Since X is θ-sgp-R_0, $\theta\text{-}sgpCl(\{c\}) \subset M$. As $d \notin M$, $d \notin \theta\text{-}sgpCl(\{c\})$. Hence $d \in N = X - \theta\text{-}sgpCl(\{c\})$ and it is very clear that $c \notin N$. Thus it gives that there are θ-sgp-open sets M and N having c and d, respectively, so that $d \notin M$ and $c \notin N$. Hence X is θ-sgp-T_1. □

Proposition 3.16. *If X is θ-sgp-T_2-space, $T_\theta$sgp-space and $Y \subset X$, then Y is also θ-sgp-T_2-space.*

Proof. Consider X as a θ-sgp-T_2-space and $Y \subset X$. Taking a, b as two different points of Y. As given $Y \subseteq X$, a, b are also different points of X. Again as X is θ-sgp-T_2-space, we have different θ-sgp-open sets U and V of a and b, respectively. As per given X is $T_\theta$sgp-space, we have separate θ-sgp-open sets U and V are semi-open sets. So $U \cap Y$ and $V \cap Y$ are semi-open sets and so θ-sgp-open sets in Y, and also, $a \in U$, $a \in Y$ implies $a \in U \cap Y$ and $b \in V$ and $b \in Y$ which gives $b \in Y \cap V$. As $U \cap V = \emptyset$ we have $(Y \cap U) \cap (Y \cap V) = \emptyset$. Therefore $U \cap Y$ and $V \cap Y$ are different θ-sgp-open sets of a and b, respectively. Thus, we get Y is θ-sgp-T_2-space. □

Proposition 3.17. *The successive statements are identical for X:*

(i) *X is θ-sgp-T_2*

(ii) *For each $i \in X$, $\cap\{\theta\text{-}sgpCl(U_i) : U_i$ is a θ-sgp-nbd of $i\} = \{i\}$ or equivalently, each singleton subset of X is the intersection of θ-sgp-closed nbd of i.*

Proof. (i)⇒(ii): Consider X as a θ-sgp-T_2-space and $i \in X$. Then to each $j \in X$, $j \neq i$, there are θ-sgp-open sets P and Q so that $i \in P$, $j \in Q$ and $P \cap Q = \emptyset$. As $i \in P \subset X - Q$, $X - Q$ is a θ-sgp-closed θ-sgp-nbd of i to which j does not belong. Therefore, the intersection of all θ-sgp-closed θ-sgp-nbd of i is reduced to $\{i\}$.

(ii)⇒(i): Suppose that $i,j \in\in X$ and $i \neq j$. Then by hypothesis we have a θ-sgp-closed θ-sgp-nbd U of i so that $j \notin U$. Now, we have a θ-sgp-open set P so that $i \in P \subset U$. Thus, P and $X-U$ are different θ-sgp-open sets having i and j, respectively. Thus, X is θ-sgp-T_2. □

Proposition 3.18. *Any space X is θ-sgp-T_2 iff for each $m,n \in X$ so that $m \neq n$, there prevail θ-sgp-closed sets G_1 and G_2 so that $m \in G_1$, $n \notin G_1$, $n \in G_2$, $m \notin G_2$ and $X = G_1 \cup G_2$.*

Proof. Trivial. □

Proposition 3.19. *X is θ-sgp-T_2, whenever $g : X \to Y$ is an one-to-one θ-sgp-irresolute function and Y is θ-sgp-T_2.*

Proof. Suppose that $r,s \in X$ and $r \neq s$. As per given g is one-to-one, $g(r) \neq g(s)$, but Y is θ-sgp-T_2, so there prevail θ-sgp-open sets G, H in Y so that $g(r) \in G$, $g(s) \in H$ and $G \cap H = \emptyset$. Taking $P = g^{-1}(G)$ and $Q = g^{-1}(H)$. Then by hypothesis, P and Q are θ-sgp-open sets in X. Also, $r \in g^{-1}(G) = P$, $s \in g^{-1}(H) = Q$ and $P \cap Q = \emptyset$. Thus, we get X is θ-sgp-T_2. □

Proposition 3.20. *T_0-space (resp. T_1-space, T_2-space) and θ-sgp-T_0(resp. θ-sgp-T_1, θ-sgp-T_2)-spaces are independent of each other.*

Proof. Follows from Remark 2.15. □

Proposition 3.21. (a) *Every θ-sgp-T_1-space is θ-sgp-T_0-space. (ii) Every θ-sgp-T_2-space is θ-sgp-T_1-space.*

(b) *Every pre-θ-T_0-space is θ-sgp-T_0-space. (iv) Every pre-θ-T_1-space is θ-sgp-T_1-space.*

(c) *Every pre-θ-T_2-space is θ-sgp-T_2-space.*

Proof. Straight forward by their definitions and also each pre-θ-closed set is θ-sgp-closed set. □

Remark 3.22. The reverse implication is not possible for the above proposition as exhibited in the examples below.

Example 3.23. Taking $X = \{p,q\}$, $\tau = \{X,\emptyset,\{q\}\}$ and θ-sgp-open sets $= \{X,\emptyset,\{q\}\}$. Now, X is θ-sgp-T_0-space however not θ-sgp-T_1-space. For different points p and q of X and $\{q\}$ is θ-sgp-open set so that $p \notin \{q\}$, $q \in \{q\}$, however there is no θ-sgp-open set G_1 with $p \in G_1$, $q \notin G_1$ for $p \neq q$.

Example 3.24. Let $X = \{1,2,3\}$, $\tau = \{X,\emptyset,\{1,2\}\}$ and θ-sgp-open sets $= \{X,\emptyset,\{1\},\{2\},\{1,2\},\{1,3\},\{2,3\}\}$. Now X is θ-sgp-T_1-space however not a θ-sgp-T_2-space. For different points 1, 2 of X and $\{1,3\}$, $\{2,3\}$ are θ-sgp-open sets such that $1 \in \{1,3\}$, $2 \notin \{1,3\}$ and $1 \notin \{2,3\}$, $2 \in \{2,3\}$, but $\{1,3\} \cap \{2,3\} \neq \emptyset$.

Example 3.25. Taking $X = \{i,j,k\}$, $\tau = \{X,\emptyset,\{i\},\{j\},\{i,j\},\{i,k\}\}$ and θ-sgp-open sets $= \{X,\emptyset,\{i,k\},\{i,j\},\{i\},\{j\}\}$. Now, X is θ-sgp-T_0-space however not pre-θ-T_0-space. For two separate points j, k of X and $\{k\}$ is θ-sgp-open set so as $j \notin \{k\}$, $k \in \{k\}$, but $\{k\}$ is not pre-θ-open set of X.

Example 3.26. Taking $X = \{1,2,3\}$, $\tau = \{X, \emptyset, \{1,2\}\}$ and θ-sgp-open sets $= \{X, \emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}\}$. Now X is θ-sgp-T_1-space however not a pre-θ-T_1-space. For two different points 1, 2 of X and $\{1,3\}$, $\{2,3\}$ are θ-sgp-open sets such that $1 \in \{1,3\}$, $2 \notin \{1,3\}$ and $1 \notin \{2,3\}$, $2 \in \{2,3\}$, but $\{1,3\}$, $\{2,3\}$ are not pre-θ-open sets of X.

Example 3.27. Taking $X = \{i, j, k\}$ and also τ is indiscrete topology on X, so X is θ-sgp-T_2-space but not pre-θ-T_2-space.

Remark 3.28. The implication diagram:

$$\begin{array}{ccccc} \text{pre-}\theta\text{-}T_0 & \rightarrow & \theta\text{-sgp-}T_0 & \nleftrightarrow & T_0 \\ \uparrow & & \uparrow & & \uparrow \\ \text{pre-}\theta\text{-}T_1 & \rightarrow & \theta\text{-sgp-}T_1 & \nleftrightarrow & T_1 \\ \uparrow & & \uparrow & & \uparrow \\ \text{pre-}\theta\text{-}T_2 & \rightarrow & \theta\text{-sgp-}T_2 & \nleftrightarrow & T_2 \end{array}$$

where $A \rightarrow B$ (resp. $A \nleftrightarrow B$) represents A implies B but not conversely (resp. A and B are independent).

4. Weaker Separation Axiom

Definition 4.1. A topological space X is termed as sober θ-sgp-R_0 if $\cap\{\theta\text{-}sgpCl(\{x\}) : x \in X\}$.

Proposition 4.2. *A space X is sober θ-sgp-R_0 iff θ-sgp-$ker(\{x\}) = X$ for any $x \in X$.*

Proof. Necessity: Consider the space X as sober θ-sgp-R_0. Assuming we have a point y in X so that θ-sgp-$ker(\{y\}) = X$. Then $y \notin U$ where U is some proper θ-sgp-open subset of X. This gives $y \in \cap\{\theta\text{-}sgpCl(\{x\}) : x \in X\}$. However, this a contradiction.

Sufficiency: Now assuming θ-sgp-$ker(\{x\}) \neq X$ for any $x \in X$. Whenever there is a point y in X so that $y \in \cap\{\theta\text{-}sgpCl(\{x\}) : x \in X\}$, then each θ-sgp-open set having y must have every point of X. This gives that the space X is the exceptional θ-sgp-open set having y. Therefore θ-sgp-$ker(\{y\}) = X$ which is false. Therefore, the space X is sober θ-sgp-R_0. □

Proposition 4.3. *$X \times Y$ is sober θ-sgp-R_0, whenever the space X is sober θ-sgp-R_0 and Y is any topological space.*

Proof. By exhibiting $\cap\{\theta\text{-}sgpCl(\{\alpha,\beta\}) : (\alpha,\beta) \in X \times Y\} = \emptyset$ we are done. Now, we have $\cap\{\theta\text{-}sgpCl(\{\alpha,\beta\}) : (\alpha,\beta) \in X \times Y\} \subseteq \cap\{\theta\text{-}sgpCl(\{\alpha\}) \times \theta\text{-}sgpCl(\{\beta\}) : \alpha,\beta) \in X \times Y\} = \cap\{\theta\text{-}sgpCl(\{\alpha\}) : \alpha \in X\} \times \cap\{\theta\text{-}sgpCl(\{\beta\}) : \beta \in Y\} \subseteq \emptyset \times Y = \emptyset$. □

Definition 4.4. Let $g : X \rightarrow Y$. Then the graph $G(g) = \{(x_1, g(x_1)) : x_1 \in X\}$ of g is known as strongly θ-sgp-closed when for any $(x_1, y_1) \in (X \times Y) - G(g)$, there is a θ-sgp-open subset P of X and also a semi-open subset Q of Y having x_1 and y_1, respectively, so that $(P \times Q) \cap G(g) = \emptyset$.

Proposition 4.5. *X is θ-sgp-T_1, whenever $g : X \rightarrow Y$ is an one-to-one function with a strongly θ-sgp-closed graph.*

Proof. Suppose $m,n \in X$ and $m \neq n$. As per given g is one-to-one, $g(m) \neq g(n)$. Thus $(m,g(n)) \in (X \times Y) - G(g)$, but $G(g)$ is strongly θ-sgp-closed, so there is a θ-sgp-open subset U and a semi-open subset V having m and $g(n)$, respectively, so that $g(U) \cap V = \emptyset$. Hence $n \notin U$. Similarly, there is a θ-sgp-open set K_1 and a semi-open set K_2 having n and $g(m)$, respectively, so that $g(K_1) \cap K_2 = \emptyset$. Hence $m \notin K_1$. Accordingly, it follows that X is θ-sgp-T_1. □

Proposition 4.6. *Y is T_1, whenever $g : X \to Y$ is an onto with a strongly θ-sgp-closed graph.*

Proof. Considering y_1 and y_2 as distinct points of Y. As per given g is onto, there is $x \in X$ so that $g(x) = y_2$. Hence $(x, y_1) \notin G(g)$ and thus by Proposition 4.5 we have a θ-sgp-open set K and a semi-open set H having x and y_1, respectively, so that $g(K) \cap H = \emptyset$. Hence $y_2 \notin H$. Similarly, there is $x_0 \in X$ so that $g(x_0) = y_1$. Therefore $(x_0, y_2) \notin G(g)$ and thus there is a θ-sgp-open set D_1 and a semi-open set D_2 having x_0 and y_2, respectively, so as $g(D_1) \cap D_2 = \emptyset$. Then $y_1 \notin N$. So Y is T_1. □

5. Conclusions

By researching generalization of closed sets, some separation axioms have been found and which interns are helpful in the study of digital topology. In this paper using θ-sgp-open sets new separation axioms have been composed and their implication with other separation axioms have also been examined and emphasized which expand the future scope of normal and regular topological spaces.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. El-Atik, Some more results on pre-θ-open sets, *Antarctica Journal of Mathematics* **2**(1) (2005), 111 – 121.

[2] S. G. Crossley and S. K. Hildebrand, Semi-closure, *Texas Journal of Science* **22** (1971), 99 – 112.

[3] N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, On p-regular spaces, *Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie* **22**(75) (1983), 311 – 315.

[4] N. Levine, Semi-open sets and semi-continuity in topological spaces, *The American Mathematical Monthly* **70** (1963), 36 – 41, DOI: 10.1080/00029890.1963.11990039.

[5] N. Levine, Generalized closed sets in topology, *Rendiconti del Circolo Matematico di Palermo* **19**(2) (1970), 89 – 96, DOI: 10.1007/BF02843888.

[6] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On pre-continuous and weak pre-continuous mapping, *Proceedings of the Mathematical & Physical Society of Egypt* **53** (1982), 47 – 53.

[7] G. B. Navalagi, On semi-generalized separation axioms in topology, *International Journal of Mathematics and Computing Applications* **3**(1-2) (2011), 17 – 25.

[8] M. H. Page and V. T. Hosamath, On θ-semigeneralized pre closed sets in topological spaces, *International Journal of Scientific and Innovative Mathematical Research* **3**(3) (2015), 74 – 80, URL: https://www.arcjournals.org/pdfs/ijsimr/v3-i3/13.pdf.

[9] M. H. Page and V. T. Hosamath, Some properties of θ-sgp-neighbourhoods, *International Journal of Mathematics Trends and Technology* **20**(1) (2015), 47 – 54, DOI: 10.14445/22315373/IJMTT-V20P506.

[10] M. H. Page and V. T. Hosamath, Some properties of θ-semigeneralized pre continuous functions, *Global Journal of Pure and Applied Mathematics* **13**(7) (2017), 3627 – 3639, DOI: 10.37622/GJPAM/13.7.2017.3627-3639.

[11] M. H. Page and V. T. Hosamath, Characterizations of new open and closed Mappings in Topological Spaces, *Italian Journal of Pure and Applied Mathematics* **44** (2020), 764 – 774, URL: https://ijpam.uniud.it/online_issue/202044/67%20Page-Hosamath.pdf.

[12] M. H. Page and P. G Patil, Some new contra continuous functions in topology, *Communication in Mathematics and Application* **7**(2) (2016), 81 – 91, DOI: 10.26713%2Fcma.v7i2.377.

[13] S. Willard, *General Topology*, Addlson-Wesley Publishing Company (1970).