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Research Article

Application of $G_k/G_d/1$ Queuing Model to Patient Flow at Hospital

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Abstract. The health systems should have an ability to deliver efficient and smooth and safe services to the patients. Now-a-day, in hospitals, to get timely appointments to doctors, is a very difficult task, for most of patients long wait for appointments, that means demand and supply are imbalanced in a queue. Queuing theory is the branch of operations research in applied mathematics and deals with the phenomenon of waiting lines. Therefore, the present paper deals with the application of $G_k/G_d/1$ queuing model to patient flow at hospital namely Raipur, India. The arrival process is measured by exponential distribution and the service process is measured by Poisson distribution. Finally, appointment probabilities of waiting time of patients have been derived, and also expected queue length, waiting time for the patients in the model have been shown. It has also been observed that waiting time for patients can be reduced by using multiple servers instead of a single server queued model. Lastly, a numerical illustration of the model has been provided. The proposed result would be useful for academic literature, queuing scientists, and practitioners.

Keywords. Queuing theory; $G_k/G_D/1$ model; Appointment probability

Mathematics Subject Classification (2020). 68M20; 90B22

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1. Introduction

Saima and Nisha [13] remarked that in 1930's operations research is taken as a discipline of science and branch of applied mathematics. The past of queuing theory was nearly 100 years. A Danish telephonic engineer A. K. Erlang developed queuing theory in early 1920's. During study in applications of automatic telephone switching, Erlang was concerned with

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the capacities and utilization of the equipment and lines. Queuing theory was continued in applications to furnish for a large number of situations, at the end of World War-II. Queuing theory is actually a study of waiting lines. The theory allows the calculation and derivation of a number of representative measures which includes the estimated number of receiving service. The probability of encounters the system in certain cases, e.g. having to wait for certain time to be served or an available server, empty or full and average waiting time in the queue.

As a result of the rapid growth of the population, the people of Indian have started to demand more efficient health care at resonable cost. Hospitals are one of the most important place in the healthcare service. Quality and their services directly affect human life. Simulation is one of the oldest analysis tools; simulation is the process of making a model of a system and conducting experiments with model for the purpose of knowing the behavior of the operation of the system. Simulation is a problem solving technique that helps to study problems that cannot be analyzed using direct and formal analytical methods.

Use of simulation in hospitals there are many examples as cited in the several reviews of the academic literature, such as those by Vassilacopoulos [16] determined the bed complement in hospital inpatient departments to meet a pre determined demand for service and considered the models: (i) emergency patients should be admitted without delay, (ii) occupancy should not fall below a pre specified level, and (iii) the waiting list length should not exceed a predetermined number. Lane *et al*. [9] described the collaborative process of building a simulation model in order to understand patient waiting times in an accident and emergency department and they focused about on some general themes that can be discovered running through the process. Brailsford *et al*. [3] analyzed the relative frequency to use of a range in operational research with modeling approaches in health care along the specific domains of application. The level of implementation and provides new insights into the level of activity across many areas of application, highlighting important relationships and pointing to key areas of omission and neglect in the literature. Gunal and Pidd [5] reviewed the literature about discrete event simulation for performance modeling in health care.

Shi *et al*. [15] analyzed a representative telephone response system of *Veterans Affairs* (VA) hospitals, address the existing inefficiency issues such as long call waiting time, and improve system resilience to changes use the methodology of discrete event simulation is adopted to model the current system and the resource sharing schemes and find out the resource sharing schemes dramatically improve system performance reflected by the decrease of call waiting time and queue, as well as the extreme high utilization of agents in a key unit. Pinto *et al*. [11] addressed the question of generic simulation models and their role in improving emergency care around the world and report the construction of a reusable model for ambulance systems and also described about the associated parameters, data sources, performance measures and report on the collection of information, as well as the use of optimization to configure the service to best effect. Jiang *et al*. [7] used the statistical method for statistical analysis of MRI inspection time and calculated the waiting queue length, average queue length, and waiting time then finally conclude the problem of health allocations of resources could be solved by establishing regional MRI examination center radiation regional around. Subsequently, queuing theory for the simulation in health care have been studied by several authors, Bhattacharjee and Ray [2] told about appointment systems for scheduling patients to a hospital facility play an important role in controlling and synchronizing the arrival of patients with resource availability thereby reducing the waiting time of patients and increasing the utilization of resources. Kuo *et al*. [8] presented a case study which uses simulation to analyze patient flows in a hospital emergency department in Hong Kong and analyzed the impact of the enhancements made to the system after the relocation of the emergency department after that, they developed a simulation model to capture all the key relevant processes of the department. Ben-Tovim *et al*. [1] outlined the design, development and application of a hospital patient flow management support tool – Hospital Event Simulation Model: Arrivals to Discharge (HEAD). Rodrigues *et al*. [12] developed a discrete event simulation model that estimates Level 2 bed needs for a large university hospital and they innovates by simulating the entirety of the hospital's inpatient flow and most importantly, the ICU's daily stochastic flows based on a nursing workload scoring metrics. Hu *et al*. [6] examined the contributions of queuing theory in modeling EDs and assess the strengths and limitations of this application. Cocchi *et al*. [4] developed a methodology able to improve non-clinical front office operation for the patients keeping the costs under control and find out that DES is a valuable tool that can be used to save money and improve clinical processes. Recently, Pandey and Gangeshwer [10] studied a queuing model with heterogeneous servers for specific service in health on health sector.

This paper solves the imbalance problem of demand and supply for queue in the field of hospitals which are managed by central government or state govt. In Section 2, we analyze about materials and methods in which three main sector of hospital Raipur is focused which are patient registration department in which we seen how to patient will register and take the service then next service for patients is out patients department i.e. OPD where some special area for patients treatment, after treatment of patient analysis of the pharmacy department. In Section 3, we analyze our model $G_k/G_d/1$ about probability and departure. Finally, in Section 4 result and discussion about appointment system and new appointment probabilities of waiting time of patients have been derived; also found the expected queue length and waiting time for the patients in the model. This model provides safe, efficient and smooth services to the patients.

2. Materials and Methods

The material and methods of the proposed study are as below:

(i) Data are collected from a district hospital of Raipur India, for two days of a week. Three main sector of district hospital Raipur is focused which are (a) Patient's registration, (b) Out patients department (OPD) and (c) Pharmacy Section. 'Questionnaire', 'direct observation', and 'interviews' methods are used for data collection. Some assumptions are used for data of queuing models.

(ii) Suppose that the patient's obey a Poisson probability distribution.

(iii) Patients are independent and exponentially distributed in the time of inter-arrival.

(iv) Service time is also exponentially distributed.

(v) Suppose that patients are following the rule *first-come first-served* (FIFO) basis and also without getting service no patient will leave the queue.

(vi) The queue is endless for OPD, doctors were only servers.

(vii) Rate of serving was not dependent on the queue length.

3. Analysis of $G_k/G_d/1$ Queuing Model

In this model, single doctor is available but patients are N as single server queuing system and patients required for service is a Poisson process with rate is λN. This is free from bulk patient in service and appointment backlog. Let finite queue length is K with finite waiting room, therefore excessive patients will not make an appointment and seek treatment. Here, service rule is FCFS order and doctor service times are with length T, because service times of doctor are fixed.

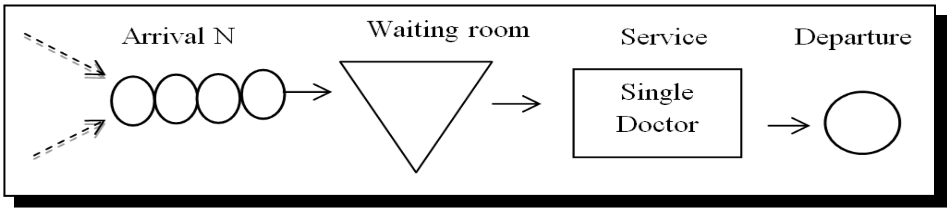

Figure 1

The results of this study stand up on the three characteristics of the rate of invalidations:

(a) There exists a no availability rate even for same day appointments.

(b) The rate of no-availability monotonically increase with the excessive patients until it reaches a maximum.

(c) The rate of no-availability fixed when it reaches the high value.

Now, no-availability function is as Sharma [14]

$$\begin{aligned}
\gamma(k) &= y_{\max} - (y_{\max} - \gamma_0)e^{-k/c} \\
&= y_{\max} - (y_{\max} - \gamma_0)\left\{1 - \left(\frac{k}{c}\right)\frac{1}{1!} + \left(\frac{k}{c}\right)^2\frac{1}{2!} - \left(\frac{k}{c}\right)^3\frac{1}{3!} + \dots\right\} \\
&= \gamma_0 + (\gamma_{\max} - \gamma_0)\left(\frac{k}{c}\right)\frac{1}{1!} - (\gamma_{\max} - \gamma_0)\left(\frac{k}{c}\right)^2\frac{1}{2!} + (\gamma_{\max} - \gamma_0)\left(\frac{k}{c}\right)^3\frac{1}{3!} \\
&= \gamma_0 + (\gamma_{\max} - \gamma_0)\left\{\left(\frac{k}{c}\right)\frac{1}{1!} + \left(\frac{k}{c}\right)^3\frac{1}{3!} + \dots\right\} - (\gamma_{\max} - \gamma_0)\left\{\left(\frac{k}{c}\right)^2\frac{1}{2!} + \left(\frac{k}{c}\right)^4\frac{1}{4!} + \dots\right\} \\
&= \gamma_0 + (\gamma_{\max} - \gamma_0)\sinh\left(\frac{k}{c}\right) - (\gamma_{\max} - \gamma_0)\left\{\cosh\left(\frac{k}{c}\right) - 1\right\}
\end{aligned}$$

$$= \gamma_0 + \gamma_{\max} \sinh\left(\frac{k}{c}\right) - \gamma_0 \sinh\left(\frac{k}{c}\right) - \gamma_{\max} \cosh\left(\frac{k}{c}\right) + \gamma_{\max} + \gamma_0 \cosh\left(\frac{k}{c}\right) - \gamma_0$$

$$= \gamma_{\max} + (\gamma_{\max} - \gamma_0)\sinh\left(\frac{k}{c}\right) - (\gamma_{\max} - \gamma_0)\cosh\left(\frac{k}{c}\right), \tag{3.1}$$

where $\gamma_0 \geq 0$ indicate the minimum execute no-availability rate and $\gamma_0 < \gamma_{\max} = 1$ maximum execute no-availability rate; c is a no-availability parameter; k is the value of patient's appointment at the time when a patient joins the queue.

3.1 Probability of a Patient Finishes His/her Service

If a probability of a patient finishes his/her service in the time interval between t and $t + \Delta t$ also drop behind k patients for new appointment, for this we use $D(k, t, t + \Delta t)$, where $0 = k = K - 1$. Similarly, $D(t, t + \Delta t) = D(0, t, t + \Delta t) + D(1, t, t + \Delta t) + D(2, t, t + \Delta t) \ldots D(K - 1, t, t + \Delta t)$.

3.2 Departure Rate

Now, the corresponding departure rate for patients be

$$d(k,t) = \lim_{\Delta t \to 0} D(k, t, t + \Delta t), \text{ where } 0 = k = K - 1,$$

$$d(t) = \lim_{\Delta t \to 0} D(t, t + \Delta t).$$

Let $P(k, t)$ be the probability of new appointment with includes k patients at the time t. Now, assume the time intervals $\varphi_n = \{t : (n-1)T \leq t \leq nT\}$, $n \in N$. Numerically, if time $t = 0$ then appointment system is empty. If time $t - 1$ then the time period $\varphi_1 = \{t : 0 \leq t < T\}$ that means no patients departures for the completeness we report here some result.

Theorem 3.1. *Let $p(k,t)$, $k = 0, 1, 2, 3, \ldots, K$ be the probability that the appointment backlog include k patient at time t with time interval.*

Proof. $p(0,0) = 1$ that means no patients appointment when $t = 0$,
$p(k, 0) = 0$, $k = 1, 2, 3, \ldots, K$ when $t = 0$,
and $d(k, 0) = 0$, $k = 01, 2, 3, \ldots, K - 1$ when $t = 0$.
So probability distribution $p(k,t)$ obey the following distribution:

$$\frac{dp(0,t)}{dt} = -\lambda N p(0,t).$$

Thus, we get

$$\frac{\frac{d}{dt} p(0,t)}{p(0,t)} = -\lambda N.$$

After integrating with respect to t, we get

$$\log p(0,t) = -\lambda N t + C_1,$$

where C_1 is constant and put $t = 0$ in above we get $C_1 = 0$, so we get

$$\log p(0,t) = -\lambda N t.$$

Finally, we get

$$p(0,t) = e^{-\lambda N t}. \tag{3.2}$$

and

$$\frac{d}{dt}p(k,t) = -\lambda N p(k,t) + \lambda N p(K-1,t)$$

where $k = 1,2,3,\dots,K-1$

$$\frac{d}{dt}p(k,t) + \lambda N p(k,t) = \lambda N p(K-1,t).$$

Integrating with respect to t, we get

$$p(k,t) + \lambda N \int p(k,t)dt = \lambda N \int p(K-1,t)dt,$$

$$p(k,t) = \lambda N \left[\int p(K-1,t)dt - \int p(k,t)dt \right] \tag{3.3}$$

and for $k = K$

$$\frac{d}{dt}p(k,t) = -\lambda N p(K-1,t). \tag{3.4}$$

□

Now, the following result explain the evolution of the appointment system.

3.3 New Appointment Probabilities

The following result explains the evolution of the appointment system. Suppose $\rho = \lambda N T$, where λN is doctor's service rate and T is service time.

$$\begin{aligned}
\alpha(k) &= e^{-\rho}\frac{\rho^k}{k!} \\
&= \left(1 - \frac{\rho}{1!} + \frac{\rho^2}{2!} - \frac{\rho^3}{3!} + \dots + (-1)^k \frac{\rho^k}{k!} + \dots \right)\frac{\rho^k}{k!} \\
&= \left(\frac{\rho}{k!} - \frac{\rho^{K+1}}{1!k!} + \frac{\rho^{K+2}}{2!k!} - \frac{\rho^{K+3}}{3!k!} + \dots + (-1)^k \frac{\rho^K \rho^K}{k!k!} + \dots \right) \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{\rho^{k+m}}{r!k!} k = 0 \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda N)^{k+m}}{r!k!} k = 0
\end{aligned}$$

then for any time interval

$$\begin{aligned}
d(k,t) = e^{-\lambda N(t-T)}\lambda N &\left[\{1 - r\gamma(k)\} \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda N)^{k+m}}{m!k!} + r\gamma(k-1) \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda N)^{k-1+m}}{m!(k-1)!} \right] \\
&+ \{1 - r\gamma(k)\} \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda N)^m}{m!} d(k+1, t-T) \\
&+ \sum_{i=1}^{k} \left[\{1 - r\gamma(k)\} \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda N)^{k+1-i+m}}{m!(k+1-i)!} + r\gamma(k-1) \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda N)^{k-i+m}}{m!(k-i)!} \right] d(i, t-T).
\end{aligned}$$

The above equation is new appointment probabilities.

4. Result and Numerical Illustration of the Model

If n = total number of patient in the our model, then,

(i) Expected queue length for waiting patient in a queue is measured by

$$\begin{aligned} L_q &= \left[\frac{1}{(s-1)!}\left(\frac{\lambda}{\mu}\right)^s \frac{\mu\lambda}{(\mu s-\lambda)^2}\right] p_0 \\ &= \left[\frac{1}{(s-1)!}\frac{\lambda^{s+1}}{\mu^{s-1}}\frac{\mu\lambda}{\mu s\left(1-\frac{\lambda}{\mu s}\right)^2}\right] p_0 \\ &= \left[\frac{1}{(s-1)!}\frac{\lambda^{s+1}}{\mu^s}\frac{1}{s(1-\lambda)^2}\right] p_0 \\ &= \left[\frac{1}{s(s-1)!}\frac{\lambda^{s+1}}{\mu^s}\frac{1}{(1-\lambda)^2}\right] p_0 \\ &= \left[\frac{1}{s!}\frac{\lambda^{s+1}}{\mu^s}\frac{1}{(1-\lambda)^2}\right] p_0 . \end{aligned} \tag{4.1}$$

(ii) Expected waiting time for the patient in the queue

$$W_q = \frac{L_q}{\lambda}. \tag{4.2}$$

(iii) Expected queue length of waiting patient in the model

$$L_m = L_q + \frac{\lambda}{\mu}. \tag{4.3}$$

(iv) Expected waiting time of the patient in the model

$$W_m = \frac{L_m}{\lambda}. \tag{4.4}$$

The performance measures using queuing analysis and queuing simulation of single server and multiple server queuing model at a government hospital using arrival rate λ, service rate μ and number of server.

	Measure	PRD (for two window)	OPD	Pharmacy (only single window)
(i)	Arrival rate (λ) of patient	70/hour	22/hour	42
(ii)	Service rate (μ) of patient	38/hour	15/hour	45
(iii)	Model Utilization (ρ)	92.10%	73.34%	93.33%
(iv)	Probability (P_0) when model is idle	4.46%	15.55%	9.56%
(v)	mean (L_m)	8.18	0.517	7.609
(vi)	mean (L_q)	8.16	0.516	7.60
(vii)	mean time of patient spend in queue (W_m-hour)	0.108	2.35	0.181
(viii)	mean time of patient spend in queue (W_q-hour)	0.109	2.28	0.180

The queued features in a district hospital of Raipur C.G. are analyzed using the queue analysis and linear simulation in three departments. Single servers and multiple server queue models have been used for these analyzes. The present paper compared to other departments, waiting time for patients in the queue in both days is more in the pharmacy department. It has also been observed that waiting time for patients can be reduced by using multiple servers instead of single server queued model. Lastly, numerical Illustration of the model has been provided. The proposed result would be useful for academic literature, queuing scientists and practitioners.

Notations

If n = total number of patient in the our model;
S = Number of doctor's;
λ = Arrival rate /hour;
μ = Serving rate/ hour;
$s\mu$ = Service rate when $s > 1$;
ρ = model utilization, i.e., $\rho = \dfrac{\text{Arrival rate/hour}}{\text{Service rate}}$ or $\rho = \frac{\lambda}{s\mu}$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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