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Research Article

Some Operations of Complex Interval-valued Pythagorean Fuzzy Set and its Application

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Abstract. In this article, we discuss the notion of the complex interval-valued Pythagorean fuzzy set (CIVPyFS). We present the algebraic operators and aggregation operators of CIVPyFS. A suitable score function for CIVPyFS is developed to rank the alternatives. Finally, two case studies are given. In the first case, we have used CIVPyFWA operator and in the second case CIVPyFWG. Both the results are compared to bring out the effectiveness, reliability and validity of the proposed method.

Keywords. Complex fuzzy set; Complex interval-valued fuzzy set; Complex Pythagorean fuzzy set; Complex interval-valued Pythagorean fuzzy set

Mathematics Subject Classification (2020). 47S40

1. Introduction

In *fuzzy set* (FS) was introduced by Zadeh [12]. Later, Zadeh [13] also discussed the concept of *interval-valued FS* (IVFS). Attanassov [2] defined the concept of *intuitionistic FS* (IFS) as a generalization of FS and discussed its properties. *Interval valued intuitionistic FS* (IVIFS) was presented by Atanassov and Gargov [3] the concept of to illustrate the uncertainty in a broadly manner than the FS. *Pythagorean FS* (PFS) [11] was developed by Yager with a condition that the square sum of its membership value and non-membership value is less than or equal to one. Peng and Yang [8] presented the concept of interval-valued PFS, a generalization of PFS and IVIFS. Ramot *et al*. [9] developed the notion of *complex FS* (CFS) a mathematical

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framework for providing membership value in terms of complex number. Greenfield *et al*. [6] extended the notion of *interval-valued CFS* (IVIFS). Alkouri and Salleh [1] defined the notion of CFS to *complex IFS* (CIFS) by adding the degree of non-membership and explain their basic operations. Garg and Dimple [7] coined the definition of *complex IVIFS* (CIVIFS) to act for the time-periodic problems. Ullah *et al*. [10] discussed the concept of *complex PFS* (CPFS) and its properties. Chinnadurai *et al*. [5] introduced the notion of *complex cubic set* (CSS) and *complex cubic intuitionistic fuzzy set* (CIFS) [4]. In this manuscript, we present the notion of *complex interval-valued PFS* (CIVPyFS). The reliability of the proposed method is established with examples.

In Section 2, we discuss the basic concepts required for this study. In Section 3, we present the concept of CIVPyFS. In Section 4, we provide the aggregation operators of CIVPYFS. Section 5 shows the reliability of the proposed method with an illustrated example. Finally, Section 6 ends with a conclusion.

2. Preliminaries

In this section, we discuses the basic concepts of CFS, ICVFS, and CPFS for all contains in universal set U.

Definition 2.1 ([9]). A CFS $\mathcal{A}_t$ represented as $\mathcal{A}_t = \{(x, P_{\mathcal{A}_t}(x)) | x \in U\}$, where $P_{\mathcal{A}_t}(x) : U \to \{\grave{a} : \grave{a} \in C : |\grave{a}| \le 1\}$ is a membership function which assigns a grade of membership.

The membership value $P_{\mathcal{A}_t}(x)$ is deceive unit circle in the complex plane and given as $\gamma_t(x) \cdot e^{i\theta_{\gamma_t}(x)}$ where $i = \sqrt{-1}$, $\gamma_t(x) \in [0,1]$ and $\theta_{\gamma_t}(x) \in [0, 2\pi]$.

Definition 2.2 ([13]). A IVCFS $\mathcal{A}_t$ represented as $\mathcal{A}_t = \left\{(x, \left[\underline{P}_{\mathcal{A}_t}(x), \overline{P}_{\mathcal{A}_t}(x)\right] | x \in U\right\}$, where $\underline{P}_{\mathcal{A}_t}(x), \overline{P}_{\mathcal{A}_t}(x) : U \to \{\grave{a} : \grave{a} \in C : |\grave{a}| \le 1\}$. Let $\left[\underline{P}_{\mathcal{A}_t}(x), \overline{P}_{\mathcal{A}_t}(x)\right]$ is called the complex degree of lower and upper bound membership value then $\underline{P}_{\mathcal{A}_t}(x) = \underline{\alpha}_t(x) \cdot e^{i\underline{\theta}_{\alpha_t}(x)}$, $\overline{P}_{\mathcal{A}_t}(x) = \overline{\alpha}_t(x) \cdot e^{i\overline{\theta}_{\alpha_t}(x)}$ which satisfy the condition $0 \le \underline{\alpha}_t(x) \le \overline{\alpha}_t(x) \le 1$ and $0 \le \underline{\theta}_{\alpha_t}(x) \le \overline{\theta}_{\alpha_t}(x) \le 2\pi$.

Definition 2.3 ([10]). A CPFS $\mathcal{F}_t$ represented as $\mathcal{F}_t = \left\{(x, P_{\mathcal{F}_t}(x), Q_{\mathcal{F}_t}(x)) | x \in U\right\}$, where $P_{\mathcal{F}_t} : U \to \{\grave{z}_1 : \grave{z}_1 \in C : |\grave{z}_1| \le 1\}$, $Q_{\mathcal{F}_t} : U \to \{\grave{z}_2 : \grave{z}_2 \in C : |\grave{z}_2| \le 1\}$ provided that $0 \le |\grave{z}_1|^2 + |\grave{z}_2|^2 \le 1$ or $P_{\mathcal{F}_t}(x) = \gamma_t(x) \cdot e^{i2\pi\theta_{\gamma_t}(x)}$ and $Q_{\mathcal{F}_t}(x) = \kappa_t(x) \cdot e^{i2\pi\theta_{\kappa_t}(x)}$.

Satisfying the condition $0 \le \gamma_t^2(x) + \kappa_t^2(x) \le 1$ and $0 \le \theta_{\gamma_t}^2(x) + \theta_{\kappa_t}^2(x) \le 1$. The degree of hesitancy function $H_t = \eta_t(x) \cdot e^{i2\pi\theta_{\eta_t}(x)}$, such that $\eta_t(x) = \sqrt{1 - \gamma_t^2(x) - \kappa_t^2(x)}$ and $\theta_{\eta_t}(x) = \sqrt{1 - \theta_{\gamma_t}^2(x) - \theta_{\kappa_t}^2(x)}$. Then $\mathcal{F}_t = (\gamma_t \cdot e^{i2\pi\theta_{\gamma_t}}, \kappa_t \cdot e^{i2\pi\theta_{\kappa_t}})$ is called CPFN.

3. Complex Interval-Valued Pythagorean Fuzzy Set (CIVPyFS)

In this section, we introduce the new notion of CIVPyS and its properties.

Definition 3.1. Let U be the universal set. A CIVPyFS represented as

$$\mathcal{F}_t = \left\{x, \left[\underline{P}_{\mathcal{F}_t}(x), \overline{P}_{\mathcal{F}_t}(x)\right] \left[\underline{Q}_{\mathcal{F}_t}(x), \overline{Q}_{\mathcal{F}_t}(x)\right] / x \in U\right\}$$

in this function $\left[\underline{P}_{\mathcal{F}_t}(x), \overline{P}_{\mathcal{F}_t}(x)\right] : U \to \{\underline{z}_1, \overline{z}_1 : \underline{z}_1, \overline{z}_1 \in \mathcal{F}_t : |\underline{z}_1|, |\overline{z}_1| \le 1\}$ and $\left[\underline{Q}_{\mathcal{F}_t}(x), \overline{Q}_{\mathcal{F}_t}(x)\right] : U \to \{\underline{z}_2, \overline{z}_2 : \underline{z}_2, \overline{z}_2 \in \mathcal{F}_t : |\underline{z}_2|, |\overline{z}_2| \le 1\}$. Such that $\underline{P}_{\mathcal{F}_t}(x) = \underline{z}_1 = \underline{\gamma}_t(x) \cdot e^{i2\pi\underline{\theta}_{\gamma_t}(x)}$, $\overline{P}_{\mathcal{F}_t}(x) = \overline{z}_1 = \overline{\gamma_t}(x) \cdot e^{i2\pi\overline{\theta}_{\gamma_t}(x)}$ and $\underline{Q}_{\mathcal{F}_t}(x) = \underline{z}_2 = \underline{\kappa_t}(x) \cdot e^{i2\pi\underline{\theta}_{\kappa_t}(x)}$, $\overline{Q}_{\mathcal{F}_t}(x) = \overline{z}_2 = \overline{\kappa_t}(x) \cdot e^{i2\pi\overline{\theta}_{\kappa_t}(x)}$.

Satisfying the condition $0 \le (\overline{\gamma_t}(x))^2 + (\overline{\kappa_t}(x))^2 \le 1$ and $0 \le (\overline{\theta}_{\gamma_t}(x))^2 + (\overline{\theta}_{\kappa_t}(x)^2 \le 1$. The hesitancy function can be $H_t = \left[\underline{\vartheta_t}(x), \overline{\vartheta_t}(x)\right] \cdot e^{i2\pi[\underline{\theta}_{\vartheta_t}(x), \overline{\theta}_{\vartheta_t}(x)]}$. Such that

$$\underline{\vartheta_t}(x) = \sqrt{1 - (\overline{\gamma_t}(x))^2 - (\overline{\kappa_t}(x))^2}, \quad \overline{\vartheta_t}(x) = \sqrt{1 - (\underline{\gamma_t}(x))^2 - (\underline{\kappa_t}(x))^2} \quad \text{and}$$

$$\underline{\theta}_{\vartheta_t}(x) = \sqrt{1 - (\overline{\theta}_{\gamma_t}(x))^2 - (\overline{\theta}_{\kappa_t}(x))^2}, \quad \overline{\theta}_{\vartheta_t}(x) = \sqrt{1 - (\underline{\theta}_{\gamma_t}(x))^2 + (\underline{\theta}_{\kappa_t}(x))^2}.$$

Therefore, mathematically CIVPyFS $\mathcal{F}_t$ defined on U can be represented as

$$\mathcal{F}_t = \left\{x, \left[\underline{\gamma_t}(x), \overline{\gamma_t}(x)\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_t}(x), \overline{\theta}_{\gamma_t}(x)]}, \left[\underline{\kappa_t}(x), \overline{\kappa_t}(x)\right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_t}(x), \overline{\theta}_{\kappa_t}(x)]} / x \in U\right\}$$

is called CIVPyFS.

The amplitude terms $\left[\underline{\gamma_t}(x), \overline{\gamma_t}(x), \underline{\kappa_t}(x), \overline{\kappa_t}(x)\right] \subset [0,1]$ and the real valued phase terms lie within the interval $\left[\underline{\theta}_{\gamma_t}(x), \overline{\theta}_{\gamma_t}(x), \underline{\theta}_{\kappa_t}(x), \overline{\theta}_{\kappa_t}(x)\right] \subset [0,1]$ and the condition,

$$(\overline{\gamma_t}(x))^2 + (\overline{\kappa_t}(x))^2 \le 1, (\overline{\theta}_{\gamma_t}(x))^2 + (\overline{\theta}_{\kappa_t}(x))^2 \le 1. \tag{3.1}$$

Furthermore, $\mathcal{F}_t = \left\langle \left[\underline{\gamma_t}, \overline{\gamma_t}\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_t}, \overline{\theta}_{\gamma_t}]}, \left[\underline{\kappa_t}, \overline{\kappa_t}\right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_t}, \overline{\theta}_{\kappa_t}]} \right\rangle$ is called CIVPyFN.

Example 3.1. Let us consider an example in CIVIFS form
$\mathcal{F}_1 = \langle [0.2, 0.3] \cdot e^{i2\pi[0.2,0.4]}, [0.4, 0.5] \cdot e^{i2\pi[0.3,0.5]} \rangle$.
It is clear that $0 \le 0.3 + 0.5 \le 1$ and $0 \le 0.4 + 0.5 \le 1$ then satisfied the given condition another example is CIVIFS form $\mathcal{F}_2 = \langle [0.5, 0.6] \cdot e^{i2\pi[0.2,0.4]}, [0.7, 0.8] \cdot e^{i2\pi[0.5,0.7]} \rangle$.
It is solve that $0.6 + 0.8 \neq 1$ and $0.4 + 0.7 \neq 1$ in this set does not satisfy the given condition of CIVIFS. This shows that CIVIFS is not sufficient to deal under the condition. However, it is evident that CIVPyFS can satisfy the condition $0 \le 0.6^2 + 0.8^2 \le 1$ and $0.4^2 + 0.7^2 \le 1$.

Definition 3.2. Let $\mathcal{F}_t = \left\langle \left[\underline{\gamma_t}, \overline{\gamma_t}\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_t}, \overline{\theta}_{\gamma_t}]}, \left[\underline{\kappa_t}, \overline{\kappa_t}\right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_t}, \overline{\theta}_{\kappa_t}]} \right\rangle$,
$\mathcal{F}_1 = \left\langle \left[\underline{\gamma}_1, \overline{\gamma}_1\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma 1}, \overline{\theta}_{\gamma 1}]}, \left[\underline{\kappa}_1, \overline{\kappa}_1\right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_1}, \overline{\theta}_{\kappa_1}]} \right\rangle$ and
$\mathcal{F}_2 = \left\langle \left[\underline{\gamma}_2, \overline{\gamma}_2\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma 2}, \overline{\theta}_{\gamma 2}]}, \left[\underline{\kappa}_2, \overline{\kappa}_2\right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_2}, \overline{\theta}_{\kappa_2}]} \right\rangle$ be CIVPyFNs then,

(i) $\mathcal{F}^c = \left\langle \left[\underline{\kappa}, \overline{\kappa}\right] \cdot e^{i2\pi[\underline{\theta}_{\kappa}, \overline{\theta}_{\kappa}]}, \left[\underline{\gamma}, \overline{\gamma}\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma}, \overline{\theta}_{\gamma}]} \right\rangle$

(ii) $\mathcal{F}_1 \wedge \mathcal{F}_2 = \Big\langle \left[\min[\underline{\gamma}_1 \underline{\gamma}_2], \min[\overline{\gamma}_1, \overline{\gamma}_2]\right] \cdot e^{i2\pi[\min(\underline{\theta}_{\gamma 1}, \underline{\theta}_{\gamma 2}), \min(\overline{\theta}_{\gamma 1}, \overline{\theta}_{\gamma 2})]},$

$\qquad [\max[\underline{\kappa}_1 \underline{\kappa}_2], \max[\overline{\kappa}_1, \overline{\kappa}_2]] \cdot e^{i2\pi[\max(\underline{\theta}_{\kappa_1}, \underline{\theta}_{\kappa_2}), \max(\overline{\theta}_{\kappa_1}, \overline{\theta}_{\kappa_2})]} \Big\rangle$

(iii) $\mathcal{F}_1 \vee \mathcal{F}_2 = \Big\langle \Big[\max[\underline{\gamma}_1 \underline{\gamma}_2], \max[\overline{\gamma}_1, \overline{\gamma}_2]\Big] \cdot e^{i2\pi[\max(\underline{\theta}_{\gamma_1}, \underline{\theta}_{\gamma_2}), \max(\overline{\theta}_{\gamma_1}, \overline{\theta}_{\gamma_2})]},$

$\qquad [\min[\underline{\kappa}_1 \underline{\kappa}_2], \min[\overline{\kappa}_1, \overline{\kappa}_2]] \cdot e^{i2\pi[\min(\underline{\theta}_{\kappa_1}, \underline{\theta}_{\kappa_2}), \min(\overline{\theta}_{\kappa_1}, \overline{\theta}_{\kappa_2})]} \Big\rangle$

(iv) $\mathcal{F}_1 \oplus \mathcal{F}_2 = \Big\langle \Big[\sqrt{\underline{\gamma}_1^2 + \underline{\gamma}_2^2 - \underline{\gamma}_1^2 \underline{\gamma}_2^2}, \sqrt{\overline{\gamma}_1^2 + \overline{\gamma}_2^2 - \overline{\gamma}_1^2 \overline{\gamma}_2^2}\Big] \cdot e^{i2\pi\Big[\sqrt{\underline{\theta}_{\gamma_1}^2 + \underline{\theta}_{\gamma_2}^2 - \underline{\theta}_{\gamma_1}^2 \underline{\theta}_{\gamma_2}^2}, \sqrt{\overline{\theta}_{\gamma_1}^2 + \overline{\theta}_{\gamma_2}^2 - \overline{\theta}_{\gamma_1}^2 \overline{\theta}_{\gamma_2}^2}\Big]},$

$\qquad [[\underline{\kappa}_1 \underline{\kappa}_2], [\overline{\kappa}_1 \overline{\kappa}_2]] \cdot e^{i2\pi[(\underline{\theta}_{\kappa_1} \underline{\theta}_{\kappa_2}), (\overline{\theta}_{\kappa_1} \overline{\theta}_{\kappa_2})]} \Big\rangle$

(v) $\mathcal{F}_1 \otimes \mathcal{F}_2 = \Big\langle \Big[[\underline{\gamma}_1 \underline{\gamma}_2], [\overline{\gamma}_1 \overline{\gamma}_2]\Big] \cdot e^{i2\pi[(\underline{\theta}_{\gamma_1} \underline{\theta}_{\gamma_2}), (\overline{\theta}_{\gamma_1} \overline{\theta}_{\gamma_2})]},$

$\qquad \Big[\sqrt{\underline{\kappa}_1^2 + \underline{\kappa}_2^2 - \underline{\kappa}_1^2 \underline{\kappa}_2^2}, \sqrt{\overline{\kappa}_1^2 + \overline{\kappa}_2^2 - \overline{\kappa}_1^2 \overline{\kappa}_2^2}\Big] \cdot e^{i2\pi\Big[\sqrt{\underline{\theta}_{\kappa_1}^2 + \underline{\theta}_{\kappa_2}^2 - \underline{\theta}_{\kappa_1}^2 \underline{\theta}_{\kappa_2}^2}, \sqrt{\overline{\theta}_{\kappa_1}^2 + \overline{\theta}_{\kappa_2}^2 - \overline{\theta}_{\kappa_1}^2 \overline{\theta}_{\kappa_2}^2}\Big]} \Big\rangle$

(vi) $\lambda \cdot \mathcal{F} = \Big\langle \Big[\sqrt{1 - (1 - \underline{\gamma}^2)^\lambda}, \sqrt{1 - (1 - \overline{\gamma}^2)^\lambda}\Big] \cdot e^{i2\pi\Big[\sqrt{1 - (1 - \underline{\theta}_\gamma^2)^\lambda}, \sqrt{1 - (1 - \overline{\theta}_\gamma^2)^\lambda}\Big]},$

$\qquad [(\underline{\kappa})^\lambda, (\overline{\kappa})^\lambda] \cdot e^{i2\pi\big[(\underline{\theta}_\kappa)^\lambda, (\overline{\theta}_\kappa)^\lambda\big]} \Big\rangle, \quad \lambda > 0$

(vii) $\mathcal{F}^\lambda = \Big\langle \Big[(\underline{\gamma})^\lambda, (\overline{\gamma})^\lambda\Big] \cdot e^{i2\pi\big[(\underline{\theta}_\gamma)^\lambda, (\overline{\theta}_\gamma)^\lambda\big]}, \Big[\sqrt{1 - (1 - \underline{\kappa}^2)^\lambda}, \sqrt{1 - (1 - \overline{\kappa}^2)^\lambda}\Big]$

$\qquad \cdot e^{i2\pi\Big[\sqrt{1 - (1 - \underline{\theta}_\kappa^2)^\lambda}, \sqrt{1 - (1 - \overline{\theta}_\kappa^2)^\lambda}\Big]} \Big\rangle, \quad \lambda > 0.$

Definition 3.3. For any CIVPyFN, $\mathcal{F} = \Big\langle \Big[\underline{\gamma}, \overline{\gamma}\Big] \cdot e^{i2\pi[\underline{\theta}_\gamma, \overline{\theta}_\gamma]}, [\underline{\kappa}, \overline{\kappa}] \cdot e^{i2\pi[\underline{\theta}_\kappa, \overline{\theta}_\kappa]} \Big\rangle$, we define the score function (Ş) as:

$$Ş(\mathcal{F}) = \frac{1}{2}\left[(\underline{\gamma}^2 + \overline{\gamma}^2) - (\underline{\kappa}^2 + \overline{\kappa}^2) + \frac{1}{2\pi}\left[(\underline{\theta}_\gamma^2 + \overline{\theta}_\gamma^2) - (\underline{\theta}_\gamma^2 + \overline{\theta}_\gamma^2)\right]\right].$$

It is clear that $Ş(\mathcal{F}) \in [-2, 2]$, and the accuracy function Å of $\mathcal{F}$ is defined as follows:

$$Å(\mathcal{F}) = \frac{1}{2}\left[(\underline{\gamma}^2 + \overline{\gamma}^2 + \underline{\kappa}^2 + \overline{\kappa}^2) + \frac{1}{2\pi}\left[(\underline{\theta}_\gamma^2 + \overline{\theta}_\gamma^2 + \underline{\theta}_\gamma^2 + \overline{\theta}_\gamma^2)\right]\right].$$

Based on these function, a comparison method for any two CIVPyFNs $\mathcal{F}_1, \mathcal{F}_2$ is defined as follows

Definition 3.4. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two CIVPyFNs corresponding to CIVPyFS, then the relation between two CIVPyFNs is: if $Ş(\mathcal{F}_1) < Ş(\mathcal{F}_2)$ then $\mathcal{F}_1$ is inferior to $\mathcal{F}_2$ and if $Ş(\mathcal{F}_1) = Ş(\mathcal{F}_2)$ then, if $Å(\mathcal{F}_1) < Å(\mathcal{F}_2)$ then $\mathcal{F}_1$ is inferior to $\mathcal{F}_2$ and if $Å(\mathcal{F}_1) = Å(\mathcal{F}_2)$ then $\mathcal{F}_1$ and $\mathcal{F}_2$ be the same information indicted by $\mathcal{F}_1 \sim \mathcal{F}_2$.

Theorem 3.1. *All the operational results is Definition* 3.2 *are CIVPyFNs.*

Proof. (i): Since $\mathcal{F} = \Big\langle \Big[\underline{\gamma}, \overline{\gamma}\Big] \cdot e^{i2\pi[\underline{\theta}_\gamma, \overline{\theta}_\gamma]}, [\underline{\kappa}, \overline{\kappa}] \cdot e^{i2\pi[\underline{\theta}_\kappa, \overline{\theta}_\kappa]} \Big\rangle$ is an CIVPyFN. So it is satisfies the equation (3.1) and hence $\mathcal{F}^c = \Big\langle [\underline{\kappa}, \overline{\kappa}] \cdot e^{i2\pi[\underline{\theta}_\kappa, \overline{\theta}_\kappa]}, \Big[\underline{\gamma}, \overline{\gamma}\Big] \cdot e^{i2\pi[\underline{\theta}_\gamma, \overline{\theta}_\gamma]} \Big\rangle$ also satisfies this condition.

Thus $\mathcal{F}^c$ is CIVPyFN.

(ii): Since $\mathcal{F}_1 = \left\langle \left[\underline{\gamma}_1, \overline{\gamma}_1\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_1}, \overline{\theta}_{\gamma_1}]}, [\underline{\kappa}_1, \overline{\kappa}_1] \cdot e^{i2\pi[\underline{\theta}_{\kappa_1}, \overline{\theta}_{\kappa_1}]} \right\rangle$ and $\mathcal{F}_2 = \left\langle \left[\underline{\gamma}_2, \overline{\gamma}_2\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_2}, \overline{\theta}_{\gamma_2}]}, [\underline{\kappa}_2, \overline{\kappa}_2] \cdot e^{i2\pi[\underline{\theta}_{\kappa_2}, \overline{\theta}_{\kappa_2}]} \right\rangle$ are CIVPyFNs, so $\mathcal{F}_1, \mathcal{F}_2$ satisfy the condition (3.1), i.e., $[\underline{\gamma}_1, \overline{\gamma}_1] \subset [0,1], [\underline{\kappa}_1, \overline{\kappa}_1] \subset [0,1]$ and $[\underline{\theta}_{\gamma_1}, \overline{\theta}_{\gamma_1}] \subset [0,1], [\underline{\theta}_{\kappa_1}, \overline{\theta}_{\kappa_1}] \subset [0,1]$, $\overline{\gamma}_1^2 + \overline{\kappa}_1^2 \leq 1$ and $\overline{\theta}_{\gamma_1}^2 + \overline{\theta}_{\kappa_1}^2 \leq 1$. Such that $\min[\overline{\gamma_1}, \overline{\gamma_2}] + \max[\overline{\kappa_1}, \overline{\kappa_2}] \leq 1$, $\min[\overline{\theta}_{\gamma_1}, \overline{\theta}_{\gamma_2}] + \max[\overline{\theta}_{\kappa_1}, \overline{\theta}_{\kappa_2}] \leq 1$ then $\mathcal{F}_1 \wedge \mathcal{F}_2$ satisfy the condition (3.1), i.e., $\mathcal{F}_1 \wedge \mathcal{F}_2$ is an CIVPyFN.

(iii): Similar to (ii), we can prove that $\mathcal{F}_1 \vee \mathcal{F}_2$ is an CIVPyFN.

(iv): Since both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the condition (3.1), it follows that,

$$\sqrt{\underline{\gamma}_1^2 + \underline{\gamma}_2^2 - \underline{\gamma}_1^2\underline{\gamma}_2^2} = \sqrt{\underline{\gamma}_1^2(1 - \underline{\gamma}_2^2) + \underline{\gamma}_2^2} \geq \sqrt{\underline{\gamma}_2^2} \geq \underline{\gamma}_2 \geq 0$$

and $\underline{\kappa}_1, \underline{\kappa}_2 \geq 0$ and the lower phase terms

$$\sqrt{\underline{\theta}_{\gamma_1}^2 + \underline{\theta}_{\gamma_2}^2 - \underline{\theta}_{\gamma_1}^2 \underline{\theta}_{\gamma_2}^2} = \sqrt{\underline{\theta}_{\gamma_1}^2(1 - \underline{\theta}_{\gamma_2}^2) + \underline{\theta}_{\gamma_2}^2} \geq \sqrt{\underline{\theta}_{\gamma_2}^2} \geq \underline{\theta}_{\gamma_2} \geq 0$$

and $\underline{\theta}_{\kappa_1}^2, \underline{\theta}_{\kappa_1}^2 \geq 0$. Also

$$\overline{\gamma}_1^2 + \overline{\gamma}_2^2 - \overline{\gamma}_1^2\overline{\gamma}_2^2 + \overline{\kappa}_1^2\overline{\kappa}_2^2 \leq \overline{\gamma}_1^2 + \overline{\gamma}_2^2 - \overline{\gamma}_1^2\overline{\gamma}_2^2 + (1 - \overline{\gamma}_1^2)(1 - \overline{\gamma}_2^2) = 1$$

and the upper phase terms

$$\overline{\theta}_{\gamma_1}^2 + \overline{\theta}_{\gamma_2}^2 - \overline{\theta}_{\gamma_1}^2 \overline{\theta}_{\gamma_2}^2 + \overline{\theta}_{\kappa_1}^2 \overline{\theta}_{\kappa_2}^2 \leq \overline{\theta}_{\gamma_1}^2 + \overline{\theta}_{\gamma_2}^2 - \overline{\theta}_{\gamma_1}^2 \overline{\theta}_{\gamma_2}^2 + (1 - \overline{\theta}_{\gamma_1}^2)(1 - \overline{\theta}_{\gamma_2}^2) = 1.$$

Therefore, the value of $\mathcal{F}_1 \bigoplus \mathcal{F}_2$ satisfy the condition of equation (3.1) and hence it is a CIVPyFN. In the similar way (v) can be proven.

(vi): Since $\sqrt{1 - (1 - \underline{\gamma}^2)^\lambda} \geq 0$, $\sqrt{1 - (1 - \overline{\gamma}^2)^\lambda} \geq 0$, $(\underline{\kappa})^\lambda, (\overline{\kappa})^\lambda \geq 0$ and the phase terms $\sqrt{1 - (1 - \underline{\theta}_\gamma^2)^\lambda} \geq 0$, $\sqrt{1 - (1 - \overline{\theta}_\gamma^2)^\lambda} \geq 0$, $(\underline{\theta}_\kappa)^\lambda, (\overline{\theta}_\kappa)^\lambda \geq 0$ and $1 - (1 - \overline{\gamma}^2)^\lambda + (\overline{\kappa}^2)^\lambda \leq 1 - (1 - \overline{\gamma}^2)^\lambda + (1 - \overline{\gamma}^2)^\lambda = 1$ and the phase terms $1 - (1 - \overline{\theta}_\gamma^2)^\lambda + (\overline{\theta}_\kappa^2)^\lambda \leq 1 - (1 - \overline{\theta}_\gamma^2)^\lambda + (1 - \overline{\theta}_\gamma^2)^\lambda = 1$. Thus, the value of $\lambda \cdot \mathcal{F}$ is a CIVPyFN.

(vii): Can be proven similarly. This completes the proof. □

Theorem 3.2. *Let* $\lambda, \lambda_1, \lambda_2 \geq 0$ *then*

(i) $\mathcal{F}_1 \bigoplus \mathcal{F}_2 = \mathcal{F}_2 \bigoplus \mathcal{F}_1$

(ii) $\mathcal{F}_1 \bigotimes \mathcal{F}_2 = \mathcal{F}_2 \bigotimes \mathcal{F}_1$

(iii) $\lambda \cdot (\mathcal{F}_1 \bigoplus \mathcal{F}_2) = \lambda \cdot \mathcal{F}_1 \bigoplus \lambda \cdot \mathcal{F}_2$

(iv) $(\mathcal{F}_1 \bigotimes \mathcal{F}_2)^\lambda = \mathcal{F}_1^\lambda \bigotimes \mathcal{F}_2^\lambda$

(v) $\lambda_1 \cdot \mathcal{F} \bigoplus \lambda_2 \cdot \mathcal{F} = (\lambda_1 + \lambda_2).\mathcal{F}$

(vi) $\mathcal{F}^{\lambda_1} \bigotimes \mathcal{F}^{\lambda_2} = \mathcal{F}^{\lambda_1 + \lambda_2}$.

Proof. (i): By (vi) of the Definition 3.2, we have

$$\mathcal{F}_1 \bigoplus \mathcal{F}_2 = \Bigg\langle \left[\sqrt{\underline{\gamma}_1^2 + \underline{\gamma}_2^2 - \underline{\gamma}_1^2\underline{\gamma}_2^2}, \sqrt{\overline{\gamma}_1^2 + \overline{\gamma}_2^2 - \overline{\gamma}_1^2\overline{\gamma}_2^2}\right] \cdot e^{i2\pi\left[\sqrt{\underline{\theta}_{\gamma_1}^2 + \underline{\theta}_{\gamma_2}^2 - \underline{\theta}_{\gamma_1}^2\underline{\theta}_{\gamma_2}^2}, \sqrt{\overline{\theta}_{\gamma_1}^2 + \overline{\theta}_{\gamma_2}^2 - \overline{\theta}_{\gamma_1}^2\overline{\theta}_{\gamma_2}^2}\right]},$$
$$\left[[\underline{\kappa}_1\underline{\kappa}_2], [\overline{\kappa}_1\overline{\kappa}_2]\right] \cdot e^{i2\pi[(\underline{\theta}_{\kappa_1}\underline{\theta}_{\kappa_2}), (\overline{\theta}_{\kappa_1}\overline{\theta}_{\kappa_2})]} \Bigg\rangle$$

$$= \left\langle \left[\sqrt{\underline{\gamma}_2^2+\underline{\gamma}_1^2-\underline{\gamma}_2^2\underline{\gamma}_1^2}, \sqrt{\overline{\gamma}_2^2+\overline{\gamma}_1^2-\overline{\gamma}_2^2\overline{\gamma}_1^2}\right] \cdot e^{i2\pi\left[\sqrt{\underline{\theta}_{\gamma_2}^2+\underline{\theta}_{\gamma_1}^2-\underline{\theta}_{\gamma_2}^2\underline{\theta}_{\gamma_1}^2}, \sqrt{\overline{\theta}_{\gamma_2}^2+\overline{\theta}_{\gamma_1}^2-\overline{\theta}_{\gamma_2}^2\overline{\theta}_{\gamma_1}^2}\right]},\right.$$

$$\left.[[\underline{\kappa}_2\underline{\kappa}_1],[\overline{\kappa}_2\overline{\kappa}_1]] \cdot e^{i2\pi[(\underline{\theta}_{\kappa_2}\underline{\theta}_{\kappa_2}),(\overline{\theta}_{\kappa_2}\overline{\theta}_{\kappa_1})]}\right\rangle$$

$$= \mathcal{F}_2 \bigoplus \mathcal{F}_1.$$

(ii): Based on the proof of (i), similarly we can prove (ii).

(iii): It follows from (iv) in Definition 3.2 that,

$$\mathcal{F}_1 \bigoplus \mathcal{F}_2 = \left\langle \left[\sqrt{\underline{\gamma}_1^2+\underline{\gamma}_2^2-\underline{\gamma}_1^2\underline{\gamma}_2^2}, \sqrt{\overline{\gamma}_1^2+\overline{\gamma}_2^2-\overline{\gamma}_1^2\overline{\gamma}_2^2}\right] \cdot e^{i2\pi\left[\sqrt{\underline{\theta}_{\gamma_1}^2+\underline{\theta}_{\gamma_2}^2-\underline{\theta}_{\gamma_1}^2\underline{\theta}_{\gamma_2}^2}, \sqrt{\overline{\theta}_{\gamma_1}^2+\overline{\theta}_{\gamma_2}^2-\overline{\theta}_{\gamma_1}^2\overline{\theta}_{\gamma_2}^2}\right]},\right.$$

$$\left.[[\underline{\kappa}_1\underline{\kappa}_2],[\overline{\kappa}_1\overline{\kappa}_2]] \cdot e^{i2\pi[(\underline{\theta}_{\kappa_1}\underline{\theta}_{\kappa_2}),(\overline{\theta}_{\kappa_1}\overline{\theta}_{\kappa_2})]}\right\rangle.$$

According to (vi) of Definition 3.2, we get

$$\lambda \cdot (\mathcal{F}_1 \bigoplus \mathcal{F}_2) = \left\langle \left[\sqrt{1-\left(1-\left(\underline{\gamma}_1^2+\underline{\gamma}_2^2-\underline{\gamma}_1^2\underline{\gamma}_2^2\right)^{\lambda}\right)}, \sqrt{1-\left(1-\left(\overline{\gamma}_1^2+\overline{\gamma}_2^2-\overline{\gamma}_1^2\overline{\gamma}_2^2\right)^{\lambda}\right)}\right]\right.$$

$$\cdot e^{i2\pi\left[\sqrt{1-\left(1-\left(\underline{\theta}_{\gamma_1}^2+\underline{\theta}_{\gamma_2}^2-\underline{\theta}_{\gamma_1}^2\underline{\theta}_{\gamma_2}^2\right)^{\lambda}\right)}, \sqrt{1-\left(1-\left(\overline{\theta}_{\gamma_1}^2+\overline{\theta}_{\gamma_2}^2-\overline{\theta}_{\gamma_1}^2\overline{\theta}_{\gamma_2}^2\right)^{\lambda}\right)}\right]},$$

$$\left.\left[\left[[\underline{\kappa}_1\underline{\kappa}_2]^{\lambda},[\overline{\kappa}_1\overline{\kappa}_2]\right]^{\lambda} . e^{i2\pi\left[[\underline{\theta}_{\kappa_1}\underline{\theta}_{\kappa_2}]^{\lambda},[\overline{\theta}_{\kappa_1}\overline{\theta}_{\kappa_2}]^{\lambda}\right]}\right]\right\rangle$$

$$= \left\langle \left[\sqrt{1-\left(1-\underline{\gamma}_1^2\right)^{\lambda}\left(1-\underline{\gamma}_2^2\right)^{\lambda}}, \sqrt{1-\left(1-\overline{\gamma}_1^2\right)^{\lambda}\left(1-\overline{\gamma}_2^2\right)^{\lambda}}\right]\right.$$

$$\cdot e^{i2\pi\left[\sqrt{1-\left(1-\underline{\theta}_{\gamma_1}^2\right)^{\lambda}\left(1-\underline{\theta}_{\gamma_2}^2\right)^{\lambda}}, \sqrt{1-\left(1-\overline{\theta}_{\gamma_1}^2\right)^{\lambda}\left(1-\overline{\theta}_{\gamma_2}^2\right)^{\lambda}}\right]},$$

$$\left.\cdot\left[[\underline{\kappa}_1\underline{\kappa}_2]^{\lambda},[\overline{\kappa}_1\overline{\kappa}_2]^{\lambda}\right] \cdot e^{i2\pi\left[[\underline{\theta}_{\kappa_1}\underline{\theta}_{\kappa_2}]^{\lambda},[\overline{\theta}_{\kappa_1}\overline{\theta}_{\kappa_2}]^{\lambda}\right]}\right\rangle$$

$$\lambda \cdot \mathcal{F}_1 = \left\langle \left[\sqrt{1-(1-\underline{\gamma}_1^2)^{\lambda}}, \sqrt{1-(1-\overline{\gamma}_1^2)^{\lambda}}\right] \cdot e^{i2\pi\left[\sqrt{1-(1-\underline{\theta}_{\gamma_1}^2)^{\lambda}}, \sqrt{1-(1-\overline{\theta}_{\gamma_1}^2)^{\lambda}}\right]},\right.$$

$$\left.\left[(\underline{\kappa}_1)^{\lambda},(\overline{\kappa}_1)^{\lambda}\right] \cdot e^{i2\pi\left[(\underline{\theta}_{\kappa_1})^{\lambda},(\overline{\theta}_{\kappa_1})^{\lambda}\right]}\right\rangle$$

$$\lambda \cdot \mathcal{F}_2 = \left\langle \left[\sqrt{1-(1-\underline{\gamma}_2^2)^{\lambda}}, \sqrt{1-(1-\overline{\gamma}_2^2)^{\lambda}}\right] \cdot e^{i2\pi\left[\sqrt{1-(1-\underline{\theta}_{\gamma_2}^2)^{\lambda}}, \sqrt{1-(1-\overline{\theta}_{\gamma_2}^2)^{\lambda}}\right]},\right.$$

$$\left.\left[(\underline{\kappa}_2)^{\lambda},(\overline{\kappa}_2)^{\lambda}\right] \cdot e^{i2\pi\left[(\underline{\theta}_{\kappa_2})^{\lambda},(\overline{\theta}_{\kappa_2})^{\lambda}\right]}\right\rangle$$

$$\lambda \cdot \mathcal{F}_1 \bigoplus \lambda \cdot \mathcal{F}_2 = \left\langle \left[\sqrt{1-\left(1-\underline{\gamma}_1^2\right)^{\lambda}\left(1-\underline{\gamma}_2^2\right)^{\lambda}}, \sqrt{1-\left(1-\overline{\gamma}_1^2\right)^{\lambda}\left(1-\overline{\gamma}_2^2\right)^{\lambda}}\right]\right.$$

$$\cdot e^{i2\pi\sqrt{1-\left(1-\underline{\theta}_{\gamma_1}^2\right)^{\lambda}\left(1-\underline{\theta}_{\gamma_2}^2\right)^{\lambda}}, \sqrt{1-\left(1-\overline{\theta}_{\gamma_1}^2\right)^{\lambda}\left(1-\overline{\theta}_{\gamma_2}^2\right)^{\lambda}}},$$

$$\left[\left[\underline{\kappa}_1\underline{\kappa}_2\right]^{\lambda},[\overline{\kappa}_1\overline{\kappa}_2]^{\lambda}\right]\cdot e^{i2\pi\left[[\underline{\theta}_{\kappa_1}\underline{\theta}_{\kappa_2}]^{\lambda},[\overline{\theta}_{\kappa_1}\overline{\theta}_{\kappa_2}]^{\lambda}\right]}\Big\rangle$$

$$\lambda\cdot(\mathcal{F}_1\bigoplus\mathcal{F}_2)=\lambda\cdot\mathcal{F}_1\bigoplus\lambda\cdot\mathcal{F}_2.$$

Similarly, we can prove (iv).

(iv): Based on the proof of (iii), similarly we can prove (iv).

(v): Since

$$\lambda_1\cdot\mathcal{F}_1=\Big\langle\left[\sqrt{1-(1-\underline{\gamma}_1^2)^{\lambda_1}},\sqrt{1-(1-\overline{\gamma}_1^2)^{\lambda_1}}\right]\cdot e^{i2\pi\left[\sqrt{1-(1-\underline{\theta}^2_{\gamma_1})^{\lambda_1}},\sqrt{1-(1-\overline{\theta}^2_{\gamma_1})^{\lambda_1}}\right]},$$
$$\left[(\underline{\kappa}_1)^{\lambda_1},(\overline{\kappa_1})^{\lambda_1}\right]\cdot e^{i2\pi\left[(\underline{\theta}_{\kappa_1})^{\lambda_1},(\overline{\theta}_{\kappa_1})^{\lambda_1}\right]}\Big\rangle$$

$$\lambda_2\cdot\mathcal{F}_1=\Big\langle\left[\sqrt{1-(1-\underline{\gamma}_1^2)^{\lambda_2}},\sqrt{1-(1-\overline{\gamma}_1^2)^{\lambda_2}}\right]\cdot e^{i2\pi\left[\sqrt{1-(1-\underline{\theta}^2_{\gamma_1})^{\lambda_2}},\sqrt{1-(1-\overline{\theta}^2_{\gamma_2})^{\lambda_1}}\right]},$$
$$\left[(\underline{\kappa}_1)^{\lambda_2},(\overline{\kappa_2})^{\lambda_2}\right]\cdot e^{i2\pi\left[(\underline{\theta}_{\kappa_1})^{\lambda_2},(\overline{\theta}_{\kappa_1})^{\lambda_2}\right]}\Big\rangle$$

$$\lambda_1\cdot\mathcal{F}_1\bigoplus\lambda_2\cdot\mathcal{F}_1=\Big\langle\left[\sqrt{1-\left(1-\underline{\gamma}_1^2\right)^{\lambda_1}\left(1-\underline{\gamma}_1^2\right)^{\lambda_2}},\sqrt{1-\left(1-\overline{\gamma}_1^2\right)^{\lambda_1}\left(1-\overline{\gamma}_1^2\right)^{\lambda_2}}\right]$$
$$\cdot e^{i2\pi\left[\sqrt{1-\left(1-\underline{\theta}^2_{\gamma_1}\right)^{\lambda_1}\left(1-\underline{\theta}^2_{\gamma_1}\right)^{\lambda_2}},\sqrt{1-\left(1-\overline{\theta}^2_{\gamma_1}\right)^{\lambda_1}\left(1-\overline{\theta}^2_{\gamma_1}\right)^{\lambda_2}}\right]},$$
$$\left[\left[\underline{\kappa}_1\right]^{\lambda_1+\lambda_2},[\overline{\kappa}_1]^{\lambda_1+\lambda_2}\right]\cdot e^{i2\pi\left[[\underline{\theta}_{\kappa_1}]^{\lambda_1+\lambda_2},[\overline{\theta}_{\kappa_1}]^{\lambda_1+\lambda_2}\right]}\Big\rangle$$
$$=\Big\langle\left[\sqrt{1-\left(1-\underline{\gamma}_1^2\right)^{\lambda_1+\lambda_2}},\sqrt{1-\left(1-\overline{\gamma}_1^2\right)^{\lambda_1+\lambda_2}}\right]$$
$$\cdot e^{i2\pi\left[\sqrt{1-\left(1-\underline{\theta}^2_{\gamma_1}\right)^{\lambda_1+\lambda_2}},\sqrt{1-\left(1-\overline{\theta}^2_{\gamma_1}\right)^{\lambda_1+\lambda_2}}\right]},$$
$$\left[\left[\underline{\kappa}_1\right]^{\lambda_1+\lambda_2},[\overline{\kappa}_1]^{\lambda_1+\lambda_2}\right]\cdot e^{i2\pi\left[[\underline{\theta}_{\kappa_1}]^{\lambda_1+\lambda_2},[\overline{\theta}_{\kappa_1}]^{\lambda_1+\lambda_2}\right]}\Big\rangle$$

$$\lambda_1\cdot\mathcal{F}\bigoplus\lambda_2\cdot\mathcal{F}=(\lambda_1+\lambda_2).\mathcal{F}.$$

(vi): Can be proven similarly. This complete the proof. □

Theorem 3.3. *$\mathcal{F}_1=\Big\langle\left[\underline{\gamma}_1,\overline{\gamma}_1\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_1},\overline{\theta}_{\gamma_1}]},\left[\underline{\kappa}_1,\overline{\kappa}_1\right]\cdot e^{i2\pi[\underline{\theta}_{\kappa_1},\overline{\theta}_{\kappa_1}]}\Big\rangle$ and $\mathcal{F}_2=\Big\langle\left[\underline{\gamma}_2,\overline{\gamma}_2\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_2},\overline{\theta}_{\gamma_2}]},\left[\underline{\kappa}_2,\overline{\kappa}_2\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_2},\overline{\theta}_{\gamma_2}]}\Big\rangle$ be two CIVPyFNs then,*

(i) $\mathcal{F}_1^c\wedge\mathcal{F}_2^c=(\mathcal{F}_1\vee\mathcal{F}_2)^c$

(ii) $\mathcal{F}_1^c\vee\mathcal{F}_2^c=(\mathcal{F}_1\wedge\mathcal{F}_2)^c$

(iii) $\mathcal{F}_1^c\oplus\mathcal{F}_2^c=(\mathcal{F}_1\otimes\mathcal{F}_2)^c$

(iv) $\mathcal{F}_1^c\otimes\mathcal{F}_2^c=(\mathcal{F}_1\oplus\mathcal{F}_2)^c$

(v) $(\mathcal{F}_1\vee\mathcal{F}_2)\oplus(\mathcal{F}_1\wedge\mathcal{F}_2)=\mathcal{F}_1\oplus\mathcal{F}_2$

(vi) $(\mathcal{F}_1 \vee \mathcal{F}_2) \bigotimes (\mathcal{F}_1 \wedge \mathcal{F}_2) = \mathcal{F}_1 \bigotimes \mathcal{F}_2$.

Proof. Straight forward by using Definition 3.2. □

4. Complex Interval-Valued Pythagorean Fuzzy Aggregation Operators

In this section, we discuss some operators for aggregating CIVPyFNs.

Definition 4.1. Let $\mathcal{F}_t = \left\langle \left[\underline{\gamma}_t, \overline{\gamma}_t\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_t}, \overline{\theta}_{\gamma_t}]}, [\underline{\kappa}_t, \overline{\kappa}_t] \cdot e^{i2\pi[\underline{\theta}_{\kappa_t}, \overline{\theta}_{\kappa_t}]} \right\rangle$, $(t = 1, 2, \ldots n)$ a collection of CIVPyFNs and let $\text{CIVPyFWA}_\chi : \aleph^n \to \aleph$.

$$\text{If } \text{CIVPyFWA}_\chi(\mathcal{F}_1, \mathcal{F}_2, \ldots \mathcal{F}_n) = \chi_1 \cdot \mathcal{F}_1 \bigoplus \chi_2 \cdot \mathcal{F}_1 \bigoplus, \ldots \bigoplus \chi_n . \mathcal{F}_n, \tag{4.1}$$

where $\aleph$ is the collection of all CIVPyFNs, χ_t is the weight of $\mathcal{F}_t$ $(t = 1, 2, \ldots n)$, $\chi_t \in [0, 1]$ and $\sum_{t=1}^{n} \chi_t = 1$ then the function CIVPyFWA is called an CIVPyF weighted averaging operator.

In particular, if $\chi_t = \frac{1}{n}$ for all j then CIVPyFWA_χ operator reduces to CIVPyF averaging operator (CIVPyFA) $\text{CIVPyFA}(\mathcal{F}_1, \mathcal{F}_2, \ldots \mathcal{F}_n) = \frac{1}{n}(\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \ldots \oplus \mathcal{F}_n)$.

Theorem 4.1. *Let* $\mathcal{F}_t = \left\langle \left[\underline{\gamma}_t, \overline{\gamma}_t\right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_t}, \overline{\theta}_{\gamma_t}]}, [\underline{\kappa}_t, \overline{\kappa}_t] \cdot e^{i2\pi[\underline{\theta}_{\kappa_t}, \overline{\theta}_{\kappa_t}]} \right\rangle, (t = 1, 2, \ldots n)$ *be CIVPyFNs then the aggregated value by using equation* (4.1) *is also an CIVPyFNs and*

$$\begin{aligned} &\text{CIVPyFWA}_\chi(\mathcal{F}_1, \mathcal{F}_2, \ldots \mathcal{F}_n) \\ &= \left\langle \left[\sqrt{1 - \prod_{t=1}^{n}(1 - \underline{\gamma}_t^2)^{\chi_t}}, \sqrt{1 - \prod_{t=1}^{n}(1 - \overline{\gamma}_t^2)^{\chi_t}}\right] \cdot e^{i2\pi\left[\sqrt{1 - \prod_{t=1}^{n}(1 - \underline{\theta}_{\gamma_t}^2)^{\chi_t}}, \sqrt{1 - \prod_{t=1}^{n}(1 - \overline{\theta}_{\gamma_t}^2)^{\chi_t}}\right]}, \right. \\ &\qquad \left. \left[\prod_{t=1}^{n} \underline{\kappa}_t^{\chi_t}, \prod_{t=1}^{n} \overline{\kappa}_t^{\chi_t}\right] \cdot e^{i2\pi\left[\prod_{t=1}^{n} \underline{\theta}_{\kappa_t}^{\chi_t}, \prod_{t=1}^{n} \overline{\theta}_{\kappa_t}^{\chi_t}\right]} \right\rangle, \end{aligned} \tag{4.2}$$

where χ_t *is the weight of* $\mathcal{F}_t$ $(t = 1, 2, \ldots n)$, $\chi_t \in [0, 1]$ *and* $\sum_{t=1}^{n} \chi_t = 1$.

Proof. We prove equation (4.2), when $n = 2$. $\text{CIVPyFWA}_\chi(\mathcal{F}_1, \mathcal{F}_2) = \chi_1 \mathcal{F}_1 \oplus \chi_1 \mathcal{F}_1$.
According to Theorem 3.1, we can see that both $\chi_1 \mathcal{F}_1$ and $\chi_1 \mathcal{F}_1$ are CIVPyFNs, and the value of $\chi_1 \mathcal{F}_1 \oplus \chi_1 \mathcal{F}_1$ is an CIVPyFN. By Definition 3.2 law (vi), we have

$$\begin{aligned} \chi_1 \cdot \mathcal{F}_1 = &\left\langle \left[\sqrt{1 - (1 - \underline{\gamma_1}^2)^{\chi_1}}, \sqrt{1 - (1 - \overline{\gamma_1}^2)^{\chi_1}}\right] \cdot e^{i2\pi\left[\sqrt{1 - (1 - \underline{\theta}_{\gamma_1}^2)^{\chi_1}}, \sqrt{1 - (1 - \overline{\theta}_{\gamma_1}^2)^{\chi_1}}\right]}, \right. \\ &\left. \left[(\underline{\kappa_1})^{\chi_1}, (\overline{\kappa_1})^{\chi_1}\right] \cdot e^{i2\pi(\underline{\theta}_{\kappa_1})^{\chi_1}, (\overline{\theta}_{\kappa_1})^{\chi_1}} \right\rangle, \end{aligned}$$

$$\begin{aligned} \chi_2 \cdot \mathcal{F}_2 = &\left\langle \left[\sqrt{1 - (1 - \underline{\gamma_2}^2)^{\chi_2}}, \sqrt{1 - (1 - \overline{\gamma_2}^2)^{\chi_2}}\right] \cdot e^{i2\pi\left[\sqrt{1 - (1 - \underline{\theta}_{\gamma_2}^2)^{\chi_2}}, \sqrt{1 - (1 - \overline{\theta}_{\gamma_2}^2)^{\chi_2}}\right]}, \right. \\ &\left. \left[(\underline{\kappa_2})^{\chi_2}, (\overline{\kappa_2})^{\chi_2}\right] \cdot e^{i2\pi(\underline{\theta}_{\kappa_2})^{\chi_2}, (\overline{\theta}_{\kappa_2})^{\chi_2}} \right\rangle. \end{aligned}$$

Then

$$\begin{aligned}\text{CIVPyFWA}_\chi(\mathcal{F}_1,\mathcal{F}_2) &= \chi_1\mathcal{F}_1 \bigoplus \chi_1\mathcal{F}_1\\ &= \Big\langle \left[\sqrt{1-\left(1-\underline{\gamma}_1^2\right)^{\chi_1}\left(1-\underline{\gamma}_1^2\right)^{\chi_2}}, \sqrt{1-\left(1-\overline{\gamma}_1^2\right)^{\chi_1}\left(1-\overline{\gamma}_1^2\right)^{\chi_2}}\right]\\ &\quad \cdot e^{i2\pi\left[\sqrt{1-\left(1-\underline{\theta}_{\gamma_1}^2\right)^{\chi_1}\left(1-\underline{\theta}_{\gamma_1}^2\right)^{\chi_2}}, \sqrt{1-\left(1-\overline{\theta}_{\gamma_1}^2\right)^{\chi_1}\left(1-\overline{\theta}_{\gamma_1}^2\right)^{\chi_2}}\right]},\\ &\quad \left[\underline{\kappa}_1^{\chi_1}\underline{\kappa}_2^{\chi_2}, \overline{\kappa}_1^{\chi_1}\overline{\kappa}_2^{\chi_2}\right]\cdot e^{i2\pi\left[(\underline{\theta}_{\kappa_1}^{\chi_1}\underline{\theta}_{\kappa_2}^{\chi_2}),(\overline{\theta}_{\kappa_1}^{\chi_1}\overline{\theta}_{\kappa_2}^{\chi_2})\right]}\Big\rangle.\end{aligned}$$

Thus, result $n=2$ is true.

Equation (4.2) holds, then the result is true for $n=r$, i.e.,

$$\begin{aligned}&\text{CIVPyFWA}_\chi(\mathcal{F}_1,\mathcal{F}_2,\dots\mathcal{F}_r)\\ &= \Big\langle \left[\sqrt{1-\prod_{t=1}^{r}(1-\underline{\gamma}_t^2)^{\chi_t}}, \sqrt{1-\prod_{t=1}^{r}(1-\overline{\gamma}_t^2)^{\chi_t}}\right]\cdot e^{i2\pi\left[\sqrt{1-\prod_{t=1}^{r}(1-\underline{\theta}_{\gamma_t}^2)^{\chi_t}}, \sqrt{1-\prod_{t=1}^{r}(1-\overline{\theta}_{\gamma_t}^2)^{\chi_t}}\right]},\\ &\quad \left[\prod_{t=1}^{r}\underline{\kappa}_t^{\chi_t}, \prod_{t=1}^{r}\overline{\kappa}_t^{\chi_t}\right]\cdot e^{i2\pi\left[\prod_{t=1}^{r}\underline{\theta}_{\kappa_t}^{\chi_t}, \prod_{t=1}^{r}\overline{\theta}_{\kappa_t}^{\chi_t}\right]}\Big\rangle.\end{aligned}$$

Then, when $n=r+1$, by (iv) and (vi) in Definition 3.2 we get,

$$\begin{aligned}&\text{CIVPyFWA}_\chi(\mathcal{F}_1,\mathcal{F}_2,\dots\mathcal{F}_{r+1})\\ &= \text{CIVPyFWA}_\chi(\mathcal{F}_1,\mathcal{F}_2,\dots\mathcal{F}_r) \bigoplus \chi_{r+1}.\mathcal{F}_{r+1}\\ &= \Big\langle \left[\sqrt{1-\prod_{t=1}^{r}(1-\underline{\gamma}_t^2)^{\chi_t}}, \sqrt{1-\prod_{t=1}^{r}(1-\overline{\gamma}_t^2)^{\chi_t}}\right]\cdot e^{i2\pi\left[\sqrt{1-\prod_{t=1}^{r}(1-\underline{\theta}_{\gamma_t}^2)^{\chi_t}}, \sqrt{1-\prod_{t=1}^{r}(1-\overline{\theta}_{\gamma_t}^2)^{\chi_t}}\right]},\\ &\quad \left[\prod_{t=1}^{r}\underline{\kappa}_t^{\chi_t}, \prod_{t=1}^{r}\overline{\kappa}_t^{\chi_t}\right]\cdot e^{i2\pi\left[\prod_{t=1}^{r}\underline{\theta}_{\gamma_t}^{\chi_t}, \prod_{t=1}^{r}\overline{\theta}_{\gamma_t}^{\chi_t}\right]}\Big\rangle\\ &\quad \bigoplus \Big\langle \left[\sqrt{1-(1-\underline{\gamma}_{r+1}^2)^{\chi_{r+1}}}, \sqrt{1-(1-\overline{\gamma}_{r+1}^2)^{\chi_{r+1}}}\right]\cdot e^{i2\pi\left[\sqrt{1-(1-\underline{\theta}_{\gamma_{r+1}}^2)^{\chi_{r+1}}}, \sqrt{1-(1-\overline{\theta}_{\gamma_{r+1}}^2)^{\chi_{r+1}}}\right]},\\ &\quad \left[\underline{\kappa}_{r+1}^{\chi_{r+1}}, \overline{\kappa}_{r+1}^{\chi_{r+1}}\right]\cdot e^{i2\pi\left[(\underline{\theta}_\kappa)_{r+1}^{\chi_{r+1}}, (\overline{\theta}_\kappa)_{r+1}^{\chi_{r+1}}\right]}\Big\rangle\\ &= \Big\langle \left[\sqrt{1-\prod_{t=1}^{r+1}(1-\underline{\gamma}_t^2)^{\chi_t}}, \sqrt{1-\prod_{t=1}^{r+1}(1-\overline{\gamma}_t^2)^{\chi_t}}\right]\cdot e^{i2\pi\left[\sqrt{1-\prod_{t=1}^{r+1}(1-\underline{\theta}_{\gamma_t}^2)^{\chi_t}}, \sqrt{1-\prod_{t=1}^{r+1}(1-\overline{\theta}_{\gamma_t}^2)^{\chi_t}}\right]},\\ &\quad \left[\prod_{t=1}^{r+1}\underline{\kappa}_t^{\chi_t}, \prod_{t=1}^{r+1}\overline{\kappa}_t^{\chi_t}\right]\cdot e^{i2\pi\left[\prod_{t=1}^{r+1}\underline{\theta}_{\kappa_t}^{\chi_t}, \prod_{t=1}^{r+1}\overline{\theta}_{\kappa_t}^{\chi_t}\right]}\Big\rangle\end{aligned}$$

i.e., when $n=r+1$, equation (4.2) also holds. Next in order to show CIVPyWA$_\chi$ is an CIVPyFN. As $\mathcal{F}_t = \left\langle \left[\underline{\gamma}_t, \overline{\gamma}_t\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_t}, \overline{\theta}_{\gamma_t}]}, \left[\underline{\kappa}_t, \overline{\kappa}_t\right]\cdot e^{i2\pi[\underline{\theta}_{\kappa_t}, \overline{\theta}_{\kappa_t}]}\right\rangle$ for all t is an CIVPyFN, thus $0 \le \underline{\gamma}_t, \overline{\gamma}_t, \underline{\kappa}_t$, $\overline{\kappa}_t \le 1$ then satisfy the condition $\overline{\gamma}_t^2 + \overline{\kappa}_t^2 \le 1$ and the phase terms are $0 \le \underline{\theta}_{\gamma_t}, \overline{\theta}_{\gamma_t}, \underline{\theta}_{\kappa_t}, \overline{\theta}_{\kappa_t} \le 1$ and the condition $\overline{\theta}_{\gamma_t}^2 + \overline{\theta}_{\kappa_t}^2 \le 1$. Thus $0 \le \prod_{t=1}^{n}(1-\underline{\gamma}_t^2)^{\chi_t} \le 1$ and hence $0 \le \sqrt{1-\prod_{t=1}^{n}(1-\underline{\gamma}_t^2)^{\chi_t}} \le 1$,

$0 \le \prod_{t=1}^{n} \underline{\kappa}_t^{\chi_t} \le 1$ and $0 \le \sqrt{1-\prod_{t=1}^{r}(1-\underline{\theta}^2_{\gamma_t})^{\chi_t}} \le 1$, $0 \le \prod_{t=1}^{r} \underline{\theta}^{\chi_t}_{\kappa_t} \le 1$. Similarly, $0 \le \sqrt{1-\prod_{t=1}^{n}(1-\overline{\gamma}^2_t)^{\chi_t}} \le 1$ and $0 \le \prod_{t=1}^{n} \overline{\kappa}_t^{\chi_t} \le 1$ and the phase terms are, $0 \le \sqrt{1-\prod_{t=1}^{r}(1-\overline{\theta}^2_{\gamma_t})^{\chi_t}} \le 1$, $0 \le \prod_{t=1}^{r} \overline{\theta}^{\chi_t}_{\kappa_t} \le 1$.

Again,

$$\left(\sqrt{1-\prod_{t=1}^{n}(1-\overline{\gamma}^2_t)^{\chi_t}}\right)^2 + \left(\prod_{t=1}^{n}\overline{\kappa}_t^{\chi_t}\right)^2 = 1-\prod_{t=1}^{n}(1-\overline{\gamma}^2_t)^{\chi_t} + \left(\prod_{t=1}^{n}\overline{\kappa}_t^{2\chi_t}\right)$$
$$\le 1-\left(\prod_{t=1}^{n}\overline{\kappa}_t^{2\chi_t}\right) + \left(\prod_{t=1}^{n}\overline{\kappa}_t^{2\chi_t}\right) = 1$$

and the phase terms are

$$\left(\sqrt{1-\prod_{t=1}^{n}(1-\overline{\theta}^2_{\gamma_t})^{\chi_t}}\right)^2 + \left(\prod_{t=1}^{n}\overline{\theta}_{\kappa_t}^{\chi_t}\right)^2 = 1-\prod_{t=1}^{n}(1-\overline{\theta}^2_{\gamma_t})^{\chi_t} + \prod_{t=1}^{n}\overline{\theta}_{\kappa_t}^{2\chi_t}$$
$$\le 1-\left(\prod_{t=1}^{n}\overline{\theta}_{\kappa_t}^{2\chi_t}\right) + \left(\prod_{t=1}^{n}\overline{\theta}_{\kappa_t}^{2\chi_t}\right) = 1.$$

Hence, CIVPyWA$_\chi$ is an CIVPyFN and therefore proof is completed. □

Definition 4.2. Let CIVPyFWG$_\chi : \aleph^n \to \aleph$. If

$$\mathrm{CIVPyFWG}_\chi(\mathcal{F}_1,\mathcal{F}_2,\dots\mathcal{F}_n) = \mathcal{F}_1^{\chi_1} \bigotimes \mathcal{F}_1^{\chi_2} \bigotimes,\dots \bigotimes \mathcal{F}_n^{\chi_n} \tag{4.3}$$

then the function CIVPyFWG$_\chi$ is called an CIVPyF weighted geometric operator. In particular, if $\chi = \left(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}\right)^T$ then the CIVPyFWG$_\chi$ operator reduces to an complex interval-valued Pythagorean fuzzy geometric operator $\mathrm{CIVPyFWG}_\chi(\mathcal{F}_1,\mathcal{F}_2,\dots\mathcal{F}_n) = (\mathcal{F}_1 \otimes \mathcal{F}_1 \otimes,\dots \otimes \mathcal{F}_n)^{\frac{1}{n}}$.

Theorem 4.2. *The aggregated value by using equation* (4.3) *is also an CIVPyFN, and*

$$\mathrm{CIVPyFWG}_\chi(\mathcal{F}_1,\mathcal{F}_2,\dots\mathcal{F}_n)$$
$$= \left\langle \left[\prod_{t=1}^{n}\underline{\gamma}_t^{\chi_t}, \prod_{t=1}^{n}\overline{\gamma}_t^{\chi_t}\right] \cdot e^{i2\pi\left[\prod_{t=1}^{n}\underline{\theta}_{\gamma_t}^{\chi_t}, \prod_{t=1}^{n}\overline{\theta}_{\gamma_t}^{\chi_t}\right]}, \left[\sqrt{1-\prod_{t=1}^{n}(1-\underline{\kappa}_t^2)^{\chi_t}}, \sqrt{1-\prod_{t=1}^{n}(1-\overline{\kappa}_t^2)^{\chi_t}}\right]\right.$$
$$\left. \cdot e^{i2\pi\left[\sqrt{1-\prod_{t=1}^{n}(1-\underline{\theta}^2_{\kappa_t})^{\chi_t}}, \sqrt{1-\prod_{t=1}^{n}(1-\overline{\theta}^2_{\kappa_t})^{\chi_t}}\right]}\right\rangle. \tag{4.4}$$

Proof. The proof of this theorem is similar to Theorem 4.1, so we omit here. □

Example 4.1. Suppose that $\mathcal{F}_1 = \langle [0.5,0.6]e^{i2\pi[0.4,0.5]}, [0.1,0.5]e^{i2\pi[0.3,0.4]}\rangle$, $\mathcal{F}_2 = \langle [0.4,0.6]e^{i2\pi[0.5,0.6]}, [0.2,0.5]e^{i2\pi[0.7,0.8]}\rangle$ and $\mathcal{F}_3 = \langle [0.1,0.3]e^{i2\pi[0.3,0.4]}, [0.5,0.6]e^{i2\pi[0.3,0.4]}\rangle$ and $\chi = (0.3,0.5,0.2)^T$ is weight vector of $\mathcal{F}_t (t = 1,2,3)$. Then

$$\mathrm{CIVPyFWA}_\chi(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$$
$$= \langle [\sqrt{1-(1-0.5^2)^{0.3}(1-0.4^2)^{0.5}(1-0.1^2)^{0.2}}, \sqrt{1-(1-0.6^2)^{0.3}(1-0.6^2)^{0.5}(1-0.3^2)^{0.2}}]$$
$$e^{i2\pi\sqrt{1-(1-0.4^2)^{0.3}(1-0.5^2)^{0.5}(1-0.3^2)^{0.2}}, \sqrt{1-(1-0.5^2)^{0.3}(1-0.6^2)^{0.5}(1-0.4^2)^{0.2}}},$$
$$[(0.1)^{0.3}(0.2)^{0.5}(0.5)^{0.2}, (0.5)^{0.3}(0.5)^{0.5}(0.6)^{0.2}] \cdot e^{i2\pi(0.3)^{0.3}(0.7)^{0.5}(0.3)^{0.2}, (0.4)^{0.3}(0.8)^{0.5}(0.4)^{0.2}}\rangle$$

$$= \langle [0.4, 0.55] \cdot e^{i2\pi[0.44, 0.53]}, [0.19, 0.51] \cdot e^{i2\pi[0.45, 0.6]} \rangle,$$

$$\mathrm{CIVPyFWG}_{\chi}(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$$

$$= \langle [(0.5)^{0.3}(0.4)^{0.5}(0.1)^{0.2}, (0.6)^{0.3}(0.6)^{0.5}(0.3)^{0.2}] \cdot e^{i2\pi(0.4)^{0.3}(0.5)^{0.5}(0.3)^{0.2}, (0.5)^{0.3}(0.6)^{0.5}(0.4)^{0.2}},$$

$$[\sqrt{1-(1-0.1^2)^{0.3}(1-0.2^2)^{0.5}(1-0.5^2)^{0.2}}, \sqrt{1-(1-0.5^2)^{0.3}(1-0.5^2)^{0.5}(1-0.6^2)^{0.2}}]$$

$$\cdot e^{i2\pi\sqrt{1-(1-0.3^2)^{0.3}(1-0.7^2)^{0.5}(1-0.3^2)^{0.2}}, \sqrt{1-(1-0.4^2)^{0.3}(1-0.8^2)^{0.5}(1-0.4^2)^{0.2}}} \rangle$$

$$= \langle [0.32, 0.47] \cdot e^{i2\pi[0.45, 0.52]}, [0.27, 0.55] \cdot e^{i2\pi[0.56, 0.67]} \rangle.$$

5. Multi-Criteria Decision Making (MCDM) using Complex Interval-Valued Pythagorean Fuzzy Numbers (CIVPyFNs)

Let $\mathcal{F}_t = \{\mathcal{F}_1, \mathcal{F}_2, \dots \mathcal{F}_m\}$ be alternatives and let $\mathcal{H}_j = \{\mathcal{H}_1, \mathcal{H}_2, \dots \mathcal{H}_n\}$ be given criteria. Each alternative can be represented in CIVPyFN form as

$$\mathcal{F}_t = \left\{ \left\langle \mathcal{H}_j, \left[\underline{\gamma}_t(\mathcal{H}_j), \overline{\gamma}_t(\mathcal{H}_j) \right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_t}(\mathcal{H}_j), \overline{\theta}_{\gamma_t}(\mathcal{H}_j)]}, \left[\underline{\kappa}_t(\mathcal{H}_j), \overline{\kappa}_t(\mathcal{H}_j) \right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_t}(\mathcal{H}_j), \overline{\theta}_{\kappa_t}(\mathcal{H}_j)]} \right\rangle | \mathcal{H}_j \in \mathcal{H} \right\}$$

with a condition that $0 \le (\overline{\gamma}_t(\mathcal{H}_j))^2 + (\overline{\kappa}_t(\mathcal{H}_j))^2 \le 1$ and $0 \le (\overline{\theta}_{\gamma_t}(\mathcal{H}_j))^2 + (\overline{\theta}_{\kappa_t}(\mathcal{H}_j))^2 \le 1$.

The CIVPyFN value is given by

$$\delta_{tj} = \left\langle \left[\underline{\gamma}_{tj}, \overline{\gamma}_{tj} \right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_{tj}}, \overline{\theta}_{\gamma_{tj}}]}, \left[\underline{\kappa}_{tj}, \overline{\kappa}_{tj} \right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_{tj}}, \overline{\theta}_{\kappa_{tj}}]} \right\rangle$$

where $\left[\underline{\gamma}_{tj}, \overline{\gamma}_{tj} \right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_{tj}}, \overline{\theta}_{\gamma_{tj}}]}$ represent the membership value given by the *decision maker* (DM) for the alternative $\mathcal{F}_t$ corresponding to the criteria $\mathcal{H}_j$. Similarly, $\left[\underline{\kappa}_{tj}, \overline{\kappa}_{tj} \right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_{tj}}, \overline{\theta}_{\kappa_{tj}}]}$ represents the non-membership value given by the DM for the alternative $\mathcal{F}_t$ corresponding to the criteria $\mathcal{H}_j$. $\left[\underline{\gamma}_{tj}, \overline{\gamma}_{tj}, \underline{\kappa}_{tj}, \overline{\kappa}_{tj} \right] \subset [0,1]$ and $\left[\underline{\theta}_{\gamma_{tj}}, \overline{\theta}_{\gamma_{tj}}, \underline{\theta}_{\kappa_{tj}}, \overline{\theta}_{\kappa_{tj}} \right] \subset [0,1]$. Decision matrix can be represented as $M = (\delta_{tj})_{m \times n}$. So based on this, an aggregation CIVPyFN, δ_t for $\mathcal{F}_t (t = 1, 2, \dots, m)$ is given as, $\delta_t = \left\langle \left[\underline{\gamma}_t, \overline{\gamma}_t \right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_t}, \overline{\theta}_{\gamma_t}]}, \left[\underline{\kappa}_t, \overline{\kappa}_t \right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_t}, \overline{\theta}_{\kappa_t}]} \right\rangle = \mathrm{CIVPyFWA}(\delta_{t1}, \delta_{t2}, \dots, \delta_{tn})$ or $\mathrm{CIVPyFWG}(\delta_{t1}, \delta_{t2}, \dots, \delta_{tn})$. Finally, calculate the score value $\c{S}(\delta_t)$ of CIVPyFNδ_t $(t = 1, 2, \dots, m)$ to rank the alternatives $\mathcal{F}_t$ $(t = 1, 2, \dots, m)$ and then select the best one. Let us summarize the steps for computing MCDM as below:

Step 1: Let $\mathcal{F}_t = \{\mathcal{F}_1, \mathcal{F}_2, \dots \mathcal{F}_m\}$ be alternatives and $\mathcal{H}_j = \{\mathcal{H}_1, \mathcal{H}_2, \dots \mathcal{H}_n\}$ be given criteria with known weights. Now, let us assume that the decision matrix of the form,

$$M_{m \times n}(\delta_{tj}) = \delta_{tj} = \left\langle \left[\underline{\gamma}_{tj}, \overline{\gamma}_{tj} \right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_{tj}}, \overline{\theta}_{\gamma_{tj}}]}, \left[\underline{\kappa}_{tj}, \overline{\kappa}_{tj} \right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_{tj}}, \overline{\theta}_{\kappa_{tj}}]} \right\rangle$$

where $\left[\underline{\gamma}_{tj}, \overline{\gamma}_{tj} \right] \cdot e^{i2\pi[\underline{\theta}_{\gamma_{tj}}, \overline{\theta}_{\gamma_{tj}}]}$ represents the membership value provided by the DM for the alternative $\mathcal{F}_t$ corresponding to the criteria $\mathcal{H}_j$. Similarly, $\left[\underline{\kappa}_{tj}, \overline{\kappa}_{tj} \right] \cdot e^{i2\pi[\underline{\theta}_{\kappa_{tj}}, \overline{\theta}_{\kappa_{tj}}]}$ represents the non-membership value provided by the DM for the alternative $\mathcal{F}_t$ corresponding to the criteria $\mathcal{H}_j$. We know that $\left[\underline{\gamma}_{tj}, \overline{\gamma}_{tj}, \underline{\kappa}_{tj}, \overline{\kappa}_{tj} \right] \subset [0,1]$ and $[\underline{\theta}_{\gamma_{tj}}, \overline{\theta}_{\gamma_{tj}}, \underline{\theta}_{\kappa_{tj}}, \overline{\theta}_{\kappa_{tj}}] \subset [0,1]$, with a condition that $0 \le \overline{\gamma}_{tj} + \overline{\kappa}_{tj} \le 1$, $0 \le \overline{\theta}_{\gamma_{tj}} + \overline{\theta}_{\kappa_{tj}} \le 1$, $(t = 1, 2, \dots, m)$, $(j = 1, 2, \dots, n)$.

Therefore, we can represent the CIVPyF matrix as

$$M_{m\times n}(\delta_{tj}) = \begin{pmatrix} \left\langle \left[\underline{\gamma}_{11},\overline{\gamma}_{11}\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_{11}},\overline{\theta}_{\gamma_{11}}]},\left[\underline{\kappa}_{11},\overline{\kappa}_{11}\right]\cdot e^{i2\pi[\underline{\theta}_{\kappa_{11}},\overline{\theta}_{\kappa_{11}}]}\right\rangle & \cdots & \left\langle \left[\underline{\gamma}_{1n},\overline{\gamma}_{1n}\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_{1n}},\overline{\theta}_{\gamma_{1n}}]},\left[\underline{\kappa}_{1n},\overline{\kappa}_{1n}\right]\cdot e^{i2\pi[\underline{\theta}_{\kappa_{1n}},\overline{\theta}_{\kappa_{1n}}]}\right\rangle \\ \left\langle \left[\underline{\gamma}_{21},\overline{\gamma}_{21}\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_{21}},\overline{\theta}_{\gamma_{tj}}]},\left[\underline{\kappa}_{21},\overline{\kappa}_{21}\right]\cdot e^{i2\pi[\underline{\theta}_{\kappa_{21}},\overline{\theta}_{\kappa_{t21}}]}\right\rangle & \cdots & \left\langle \left[\underline{\gamma}_{2n},\overline{\gamma}_{2n}\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_{2n}},\overline{\theta}_{\gamma_{2n}}]},\left[\underline{\kappa}_{2n},\overline{\kappa}_{2n}\right]\cdot e^{i2\pi[\underline{\theta}_{\kappa_{2n}},\overline{\theta}_{\kappa_{2n}}]}\right\rangle \\ \vdots & \vdots & \vdots \\ \left\langle \left[\underline{\gamma}_{m1},\overline{\gamma}_{m1}\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_{m1}},\overline{\theta}_{\gamma_{m1}}]},\left[\underline{\kappa}_{m1},\overline{\kappa}_{m1}\right]\cdot e^{i2\pi[\underline{\theta}_{\kappa_{m1}},\overline{\theta}_{\kappa_{m1}}]}\right\rangle & \cdots & \left\langle \left[\underline{\gamma}_{mn},\overline{\gamma}_{mn}\right]\cdot e^{i2\pi[\underline{\theta}_{\gamma_{mn}},\overline{\theta}_{\gamma_{mn}}]},\left[\underline{\kappa}_{mn},\overline{\kappa}_{mn}\right]\cdot e^{i2\pi[\underline{\theta}_{\kappa_{mn}},\overline{\theta}_{\kappa_{mn}}]}\right\rangle \end{pmatrix}$$

Step 2: By using definition of CIVPyFWA or CIVPyFWG, aggregate the value (δ_{tj}) $(j = 1,2,\dots,n)$ of the ith line and get the overall values (δ_t) corresponding to the alternative $\mathcal{F}_t$ $(t = 1,2,\dots,m)$.

Step 3: Compute the score value by using Definition 3.3.

Step 4: Rank all the alternatives according to the highest value and select the best alternatives.

Case Studies

In this section, we discuss two case studies. In *Case* I we study the growth of sugarcane using CIVPyFWA operator and in the case-II we use CIVPyFWG operator.

***Case* I:** The aim of this study is to monitor the growth of sugarcane by using residual lignite fly ash(LFA) and press mud collected from the sugar mill. Let the expert conduct a field experiment $\mathcal{F}_t$ $(t = 1,2,3,4)$ to evaluate the effects of LFA and press mud. Let $\mathcal{F}_1$ represent the usage of LFA, $\mathcal{F}_2$ represent the usage of press mud, $\mathcal{F}_3$ represent the combination of LFA press mud and fertilizers, $\mathcal{F}_4$ represent the combination of LFA and press mud.

Let the parameters be represented as e_1 = soil erosion analysis, e_2 = risk analysis, e_3 = growth analysis, e_4 = environment analysis.

Let us assume the weight of e_1, e_2, e_3 and e_4 be 0.2, 0.3, 0.4 and 0.1, respectively.

The Method

To find out the best alternative, we adopt the proposed method.

Step 1: The decision maker provides the information for the alternative ω_i $(i = 1,2,3,4)$ in CIVPyF matrix as

$$M_{4\times 4}(\delta_{tj}) = \begin{pmatrix} \left\langle[0.15,0.2]\cdot e^{i2\pi[0.1,0.2]},[0.5,0.7]\cdot e^{i2\pi[0.7,0.8]}\right\rangle & \left\langle[0.32,0.51]\cdot e^{i2\pi[0.3,0.4]},[0.7,0.8]\cdot e^{i2\pi[0.6,0.7]}\right\rangle & \left\langle[0.17,0.25]\cdot e^{i2\pi[0.2,0.3]},[0.71,0.8]\cdot e^{i2\pi[0.8,0.9]}\right\rangle & \left\langle[0.41,0.52]\cdot e^{i2\pi[0.5,0.6]},[0.4,0.5]\cdot e^{i2\pi[0.7,0.8]}\right\rangle \\ \left\langle[0.23,0.54]\cdot e^{i2\pi[0.4,0.5]},[0.6,0.7]\cdot e^{i2\pi[0.5,0.7]}\right\rangle & \left\langle[0.1,0.22]\cdot e^{i2\pi[0.8,0.9]},[0.8,0.9]\cdot e^{i2\pi[0.17,0.21]}\right\rangle & \left\langle[0.4,0.73]\cdot e^{i2\pi[0.17,0.32]},[0.3,0.52]\cdot e^{i2\pi[0.53,0.71]}\right\rangle & \left\langle[0.05,0.12]\cdot e^{i2\pi[0.42,0.6]},[0.8,0.9]\cdot e^{i2\pi[0.7,0.75]}\right\rangle \\ \left\langle[0.73,0.81]\cdot e^{i2\pi[0.9,0.92]},[0.2,0.3]\cdot e^{i2\pi[0.1,0.15]}\right\rangle & \left\langle[0.25,0.47]\cdot e^{i2\pi[0.5,0.6]},[0.3,0.5]\cdot e^{i2\pi[0.35,0.7]}\right\rangle & \left\langle[0.2,0.4]\cdot e^{i2\pi[0.5,0.6]},[0.6,0.7]\cdot e^{i2\pi[0.15,0.21]}\right\rangle & \left\langle[0.27,0.38]\cdot e^{i2\pi[0.51,0.7]},[0.7,0.8]\cdot e^{i2\pi[0.2,0.4]}\right\rangle \\ \left\langle[0.6,0.7]\cdot e^{i2\pi[0.4,0.55]},[0.54,0.72]\cdot e^{i2\pi[0.6,0.7]}\right\rangle & \left\langle[0.18,0.51]\cdot e^{i2\pi[0.25,0.5]},[0.34,0.5]\cdot e^{i2\pi[0.22,0.4]}\right\rangle & \left\langle[0.6,0.8]\cdot e^{i2\pi[0.1,0.2]},[0.1,0.4]\cdot e^{i2\pi[0.8,0.9]}\right\rangle & \left\langle[0.8,0.9]\cdot e^{i2\pi[0.6,0.7]},[0.1,0.2]\cdot e^{i2\pi[0.5,0.6]}\right\rangle \end{pmatrix}$$

Step 2: By using Definition 4.1 of CIVPyFWA, we aggregate the values δ_{tj} $(j = 1,2,3,4)$ the over all values corresponding to each alternative $\mathcal{F}_t$ are as blow:

$$\mathcal{F}_1 = \left\langle[0.25,0.38]\cdot e^{i2\pi[0.27,0.36]},[0.62,0.74]\cdot e^{i2\pi[0.7,0.8]}\right\rangle,$$

$$\mathcal{F}_2 = \left\langle [0.28, 0.56] \cdot e^{i2\pi[0.51,0.68]}, [0.55, 0.68] \cdot e^{i2\pi[0.38,0.49]} \right\rangle,$$
$$\mathcal{F}_3 = \left\langle [0.42, 0.74] \cdot e^{i2\pi[0.65,0.72]}, [0.39, 0.54] \cdot e^{i2\pi[0.18,0.38]} \right\rangle,$$
$$\mathcal{F}_4 = \left\langle [0.56, 0.74] \cdot e^{i2\pi[0.20,0.44]}, [0.31, 0.57] \cdot e^{i2\pi[0.48,0.64]} \right\rangle.$$

Step 3: By using Definition 3.3 we define the score value

$$\c{S}(\mathcal{F}_1) = -0.82,$$
$$\c{S}(\mathcal{F}_2) = 0.0249,$$
$$\c{S}(\mathcal{F}_3) = 0.5223 \text{ and}$$
$$\c{S}(\mathcal{F}_4) = 0.2042.$$

Step 4: Ranking the alternative based on the score value $\c{S}(\mathcal{F}_t)$ $(t = 1, 2, 3, 4.)$ we get $\c{S}(\mathcal{F}_3) \geq \c{S}(\mathcal{F}_4) \geq \c{S}(\mathcal{F}_2) \geq \c{S}(\mathcal{F}_1)$ and the best alternative is $\c{S}(\mathcal{F}_3)$. From the above observation it is evident that LFA + press mud + fertilizers enhanced the yield of sugarcane.

***Case* II:**

Step 1: Let us consider the same values mentioned in Case I.

Step 2: By using Definition 4.1 of CIVPyFWA, we aggregate the values δ_{tj} $(j = 1, 2, 3, 4)$, the over all values corresponding to each alternative $\mathcal{F}_t$ are as blow

$$\mathcal{F}_1 = \left\langle [0.21, 0.31] \cdot e^{i2\pi[0.21,0.31]}, [0.65, 0.76] \cdot e^{i2\pi[0.72,0.83]} \right\rangle,$$
$$\mathcal{F}_2 = \left\langle [0.19, 0.4] \cdot e^{i2\pi[0.35,0.5]}, [0.64, 0.77] \cdot e^{i2\pi[0.48,0.63]} \right\rangle,$$
$$\mathcal{F}_3 = \left\langle [0.28, 0.48] \cdot e^{i2\pi[0.56,0.66]}, [0.49, 0.61] \cdot e^{i2\pi[0.23,0.05]} \right\rangle,$$
$$\mathcal{F}_4 = \left\langle [0.43, 0.68] \cdot e^{i2\pi[0.20,0.36]}, [0.32, 0.51] \cdot e^{i2\pi[0.64,0.76]} \right\rangle.$$

Step 3: By using Definition 3.3 we define the score value

$$\c{S}(\mathcal{F}_1) = -0.9603,$$
$$\c{S}(\mathcal{F}_2) = -0.5306,$$
$$\c{S}(\mathcal{F}_3) = 0.0714 \text{ and}$$
$$\c{S}(\mathcal{F}_4) = -0.2664.$$

Step 4: Ranking the alternative based on the score value $\c{S}(\mathcal{F}_t)$ $(t = 1, 2, 3, 4.)$ we get $\c{S}(\mathcal{F}_3) \geq \c{S}(\mathcal{F}_4) \geq \c{S}(\mathcal{F}_2) \geq \c{S}(\mathcal{F}_1)$ and the best alternative is $\c{S}(\mathcal{F}_3)$. From the above observation it is evident that LFA + press mud + fertilizers enhanced the yield of sugarcane.

It is clear form *Case* I and *Case* II that the usage of LFA + press mud + fertilizers in suitable combination enhanced the growth rate of sugarcane.

6. Conclusion

In this article, we have extended Pythagorean fuzzy CIVPyFS. We discuss two aggregation operators namely CIVPyFWA and CIVPyFWG and score function to facilitate the ranking of the alternatives. Finally a multi-criteria decision making method is provided by using the proposed method. Case studies are provided to show the potency of the proposed method. In future, we may apply these operators in other domains.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. S. Alkouri and A. R. Salleh, Complex intuitionistic fuzzy sets, *2nd International Conference on Fundamental and Applied Sciences* **1482** (2012), 464 – 470, DOI: 10.1063/1.4757515.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986), 87 – 96, DOI: 10.1016/S0165-0114(86)80034-3.

[3] K. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **31** (1989), 343 – 349, DOI: 10.1016/0165-0114(89)90205-4.

[4] V. Chinnadurai, S. Thayalan and A. Bobin, Complex cubic intuitionistic fuzzy set and its decision making, *Advances in Mathematics: Scientific Journal* **9** (2020), 7933 – 7946, DOI: 10.37418/amsj.9.10.27.

[5] V. Chinnadurai, S. Thayalan and A. Bobin, Complex cubic set and their properties, *Advances in Mathematics: Scientific Journal* **9** (2020), 1561 – 1567, DOI: 10.37418/amsj.9.4.10.

[6] S. Greenfield, F. Chiclana and S. Dick, Interval-valued complex fuzzy logic, in: *Proceedings of the IEEE International Conference on Fuzzy Systems* **16** (2016), 2014 – 2019, DOI: 10.1109/FUZZ-IEEE.2016.7737939.

[7] H. Garg and D. Rani, Complex interval-valued intuitionistic fuzzy sets and their aggregation operators, *Fundamenta Informaticae* **164** (2019), 61 – 101, DOI: 10.3233/FI-2019-1755.

[8] X. Peng and Y. Yong, Fundamental properties of interval-valued Pythagorean fuzzy aggregation operator, *International Journal of Intelligent System* **31** (2019), 444 – 487, DOI: 10.1002/int.21790.

[9] D. Ramot, R. Milo, M. Friedman and A. Kandel, Complex fuzzy logic, *IEEE Transactions on Fuzzy Systems* **11** (2003), 450 – 461, DOI: 10.1109/TFUZZ.2003.814832.

[10] K. Ullah, T. Mahmood, Z. Ali and N. Jan, On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition, *Complex & Intelligent Systems* **6** (2020), 15 – 27, DOI: 10.1007/s40747-019-0103-6.

[11] R.R. Yager, Pythagorean fuzzy subsets, in: *Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting* **13** (2013), 57 – 61, DOI: 10.1109/IFSA-NAFIPS.2013.6608375.

[12] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338 – 353, DOI: 10.1016/S0019-9958(65)90241-X.

[13] L.A. Zadeh, The concept of a linguistic and application to approximate reasoning — I, *Information Sciences* **8** (1975), 199 – 249, DOI: 10.1016/0020-0255(75)90036-5.