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Research Article

Invariant Approximation Property for Subgroups

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Abstract. Analytic properties of invariant approximation property, studies analytic techniques from operator theory that encapsulate geometric properties of a group. We will study the invariant approximation property in various contexts. We shall show that it passes to subgroups.

Keywords. Uniform Roe algebras; Invariant approximation property; Rapid decay property

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1. Introduction

The purpose of this paper is to provide an illustration of an interesting and nontrivial interaction between analytic and geometric properties of a group. We provide approximation property of operator algebras associated with discrete groups. There are various notions of finite dimensional approximation properties for C^*-algebras and more generally operator algebras. Some of these (approximation properties) notations will be defined in this paper, the reader is referred to [1–8, 10, 11] for these a beautiful concept: Haagerup discovery that that the reduced C^*-algebra $\mathbb{F}_n$ has the metric approximation property, Higson and Kasparov's resolution of the Baum-connes conjecture for the Haagerup groups. We studies analytic techniques from operator theory that encapsulate geometric properties of a group. On approximation properties of group C^*-algebras is everywhere; it is powerful, important, backbone of countless breakthroughs.

Roe [9] considered the discrete group of the reduced group C^*-algebra of $C_r^*(G)$ is the fixed point algebra $\{Ad\rho(t) : t \in G\}$ acting on the uniform Roe algebra $C_u^*(G)$. A discrete group G

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has natural coarse structure which allows us to define the the uniform Roe algebra, $C_u^*(G)$. According to Roe [9] G has the *Invariant Approximation Property* (IAP) if

$$C_\lambda^*(G) = C_u^*(G)^G.$$

We give a general exposition of *Invariant Approximation Property* (IAP), which was initiated by Roe. The main result of paper is the following (see Theorem 3.1). We give a general exposition of *Invariant Approximation Property* (IAP), which was initiated by Roe [9].

This paper is organized as follows: Section 2 contains basic definitions and results concering IAP. In section 3, the IAP passes to subgroup is studied in detail.

2. Preliminaries

In this section, we shall establish the basic definitions and notations for the category of coarse metric spaces [1–8, 10, 11].

Example 2.1. Let G be a finitely generated group. Then the bounded coarse structure associated to any word metric on G is generated by the diagonals

$$\Delta_g = \{(h, hg) : h \in G\}.$$

Let X be a discrete metric space.

Definition 2.2. We say that discrete metric space X has *bounded geometry* if for all R there exists N in $\mathbb{N}$ such that for all $x \in X$, $|B_R(x)| < N$, where $B(x,r) = \{x \in X : d(y,x) \le r\}$.

Definition 2.3. A kernel $\phi : X \times X \longrightarrow \mathbb{C}$

- is *bounded* if there, exists $M > 0$ such that $|\phi(s,t)| < M$ for all $s, t \in X$.
- has *finite propagation* if there exists $R > 0$ such that $\phi(s,t) = 0$ if $d(s,t) > R$.

Let $B(X)$ be a set of bounded finite propagation kernels on $X \times X$. Each such ϕ defines a bounded operator on $\ell^2(X)$ via the usual formula for matrix multiplication

$$\phi * \zeta(s) = \sum_{r \in G} \phi(s,r)\zeta(r), \quad \text{for } \zeta \in \ell^2(X).$$

Next, we show the operator associated with a bounded kernel is bounded.

Lemma 2.4. *Let X be bounded geometry metric space. An operator associated with a bounded finite propagation kernel is bounded.*

We shall denote the finite propagation kernels on X by $A^\infty(X)$.

Definition 2.5. The uniform Roe algebra of a metric space X is the closure of $A^\infty(X)$ in the algebra $B(\ell^2(X))$ of bounded operators on X.

If a discrete group G is equipped with its bounded coarse structure introduced in Example 2.1 then one can associated with it uniform Roe algebra $C_u^*(G)$ by repeating the above. In this section, we will give definition of invariant approximation property. A discrete group G has a natural coarse structure which allows us to define the uniform Roe algebra $C_u^*(G)$. A group G can be equipped with either the left or right-invariant of the metric. A choice of one of the determines whether $C_\lambda^*(G)$ or $C_\rho^*(G)$ is a sublagebra of the uniform Roe algebra $C_u^*(G)$ of G. Hence any element of $\mathbb{C}[G]$ will give use to finite propagation and this assignment extends to an inclusion

$$C_\lambda^*(G) \hookrightarrow C_u^*(G).$$

Next if the metric on G is left-invariant then

$$C_\rho^*(G) \subset C_u^*(G).$$

Let d_1 be the left-invariant metric on G

$$d_1(x,y) = d_1(gx,gy), \quad \text{for all } g \in G.$$

Let us now choose a right invariant metric for G so that $C_\lambda^*(G) \hookrightarrow C_u^*(G)$. The right regular representation ρ gives use to the adjoint action on $C_u^*(G)$ defined by

$$Ad\rho(g)T = \rho(g)T\rho(g)^* = \rho(g)T\rho(g)^{-1},$$

for all $t \in G$, $T \in C_u^*(G)$. Our remarks above show that elements of $C_\lambda^*(G)$ are invariant with respect to this action and so $C_\lambda^*(G)$ is contained in invariant subalgebra $C_u^*(G)^G$.

Lemma 2.6. *If $T \in C_u^*(G)$ has kernel $A(x,y)$, then $Ad\rho(t)T$ has kernel $A(xt,yt)$.*

In general, if $T \in C_u^*(X)$ then $\forall\ x,y \in G$:

$$\begin{aligned}\langle Ad(\rho(t))T\delta_x,\delta_y\rangle &= \langle \rho(t)T\rho(t^{-1})\delta_x,\delta_y\rangle\\ &= \langle T\rho(t^{-1})\delta_x,\rho(t^{-1})\delta_y\rangle\\ &= \langle T\delta_{xt},\delta_{yt}\rangle.\end{aligned}$$

So the operator T is $Ad\rho$-invariant if and only if

$$\forall\ x,y \in X,\ \ \forall\ t \in G,\ \ \langle T\delta_{xt},\delta_{yt}\rangle = \langle T\delta_x,\delta_y\rangle.$$

We now define the invariant approximation property (IAP)

Definition 2.7. We say that G has the *Invariant Approximation Property* (*IAP*) if

$$C_\lambda^*(G) = C_u^*(G)^G.$$

3. The IAP Passes to Subgroups

The main result of paper is the following:

Theorem 3.1. *Any subgroup H of a discrete group G with the invariant approximation property has the invariant approximation property.*

Proof. Let us fix a set of representatives R of the right cosets G/H so that for every element $g \in G$ there is a unique representation $g = h_g r_g$ where $h_g \in H$ and $r_g \in R$. We then have the isomorphism of Hilbert spaces:

$$\ell^2(G) \cong \ell^2(H) \otimes \ell^2(G/H),$$

given by

$$\delta_g \longmapsto \delta_{h_g} \otimes \delta_{r_g},$$

with the converse map given by,

$$\delta_h \otimes \delta_r \longmapsto \delta_{hr}.$$

The uniform Roe algebra $C_u^*(H)$ acts on this space by $T \otimes 1$ for every $T \in C_u^*(H)$, which gives an embedding, i.e.

$$C_u^*(H) \hookrightarrow C_u^*(G) \text{ by } T \longmapsto T \otimes 1.$$

Using this inclusion, we shall show that

$$C_u^*(H)^H \cong C_u^*(H)^G.$$

First, it is clear that a G-invariant operator in $C_u^*(H)$ is also H-invariant operator, restricting the $Ad\rho$ action from G to H. To show the converse,

$$C_u^*(H)^H \subseteq C_u^*(H)^G,$$

we proceed as follows. We want to extended an kernel on $H \times H$ which is invariant with respect to the $Ad\rho_H$ action to a kernel on $G \times G$ which is invariant with respect to the $Ad\rho_G$ action. Given $a(h,h')$ we define $A : G \times G \longrightarrow \mathbb{C}$ as follows: for every $s,t \in G$

$$A(s,t) = \begin{cases} a(h,h'), & \text{if there exists } r \in R \text{ s.t. } (s,t) = (hr,h'r), \\ 0, & \text{otherwise.} \end{cases}$$

Now, we need to show that $A(s,t)$ is $Ad\rho_G$-invariant. If we write

$$rt = h_1 r_1, \quad \text{for } h_1 \in H,\ r, r_1 \in R$$

we get,

$$\begin{aligned} Ad\rho_G(t)A(hr,h'r) &= A(hrt,h'rt) \\ &= A(hh_1r_1,h'h_1r_1) \\ &= a(hh_1,h'h_1) \\ &= a(h,h') \\ &= A(hr,h'r). \end{aligned}$$

Given that invariant Roe kernels form a dense subset of $C_u^*(H)^H$, it follows that

$$C_u^*(H)^H \subseteq C_u^*(H)^G,$$

and so we have an isomorphism,

$$C_u^*(H)^H \cong C_u^*(H)^G.$$

Let $T \in C_u^*(H)^G$. Then $T \in C_u^*(G)^G$ and $T \in C_u^*(H)$, and we have

$$C_u^*(H)^G \subseteq C_u^*(G)^G \cap C_u^*(H).$$

Since

$$C_u^*(G)^G \cap C_u^*(H) \subseteq C_u^*(H)^G,$$

we have

$$C_u^*(H)^G = C_u^*(G)^G \cap C_u^*(H).$$

We now want to show that a similar isomorphism holds for the regular C^*- algebras:

$$C_\lambda^*(H) \cong C_\lambda^*(G) \cap C_u^*(H).$$

First there is an inclusion

$$\mathbb{C}[G] \longrightarrow A^\infty(G),$$

$$g \longmapsto U_g(x,y),$$

where,

$$U_g(x,y) = \begin{cases} 1, & \text{if } gx = y, \\ 0, & \text{otherwise.} \end{cases}$$

This extends to a ring homomorphism so we have

$$\mathbb{C}[G] \hookrightarrow A^\infty(G) \hookrightarrow C_u^*(G),$$

where, $A^\infty(G)$ is the uniform translation algebra. Since H is normal subgroup of G. We have an inclusion

$$\mathbb{C}[H] \hookrightarrow \mathbb{C}[G].$$

Then

$$\Phi : \mathbb{C}[H] \xrightarrow{\cong} \mathbb{C}[G] \cap A^\infty(H)$$

By taking completion of both sides, we have

$$C_\lambda^*(H) \cong C_\lambda^*(G) \cap C_u^*(H).$$

We now suppose that G has IAP. Then

$$C_u^*(G)^G = C_\lambda^*(G),$$

and using the above results we have that,

$$\begin{aligned} C_u^*(H)^H &\cong C_u^*(H)^G \\ &= C_u^*(G)^G \cap C_u^*(H) \\ &= C_\lambda^*(G) \cap C_u^*(H) \\ &\cong C_\lambda^*(H). \end{aligned}$$

Hence

$$C_u^*(H)^H \cong C_\lambda^*(H)$$

and so the IAP passes to subgroups. □

4. Conclusion

Any subgroup of a discrete group with the invariant approximation property has the invariant approximation property.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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