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Research Article

On Poly-Euler Polynomials and Arakawa-Kaneko Type Zeta Functions of Parameters a,b,c

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Abstract. In this paper, we investigate a class of generalized poly-Euler polynomials with a,b,c parameters, a generalization of the classical Euler numbers and polynomials. Various properties of these generalized polynomials are established. We also introduce the Arakawa-Kaneko type zeta functions for the poly-Euler polynomials with a,b,c parameters and obtain an interpolation formula for the generalization of poly-Euler numbers and polynomials with a,b,c parameters. Furthermore, we establish the relationship between the Arakawa-Kaneko type zeta functions for generalized poly-Euler polynomials and the Arakawa-Kaneko zeta functions for generalized poly-Bernoulli polynomials defined in [1].

Keywords. Euler numbers and polynomials; Bernoulli numbers and polynomials; Riemann zeta functions; Arakawa-Kaneko zeta functions; Poly-Euler numbers and polynomials; Poly-Bernoulli numbers and polynomials; Generalized poly-Euler numbers and polynomials; Generalized poly-Bernoulli numbers and polynomials; Generalized Arakawa-Kaneko zeta functions; Polylogarithm; Stirling numbers of the second kind

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1. Introduction

For an integer k, the polylogarithm function $\mathrm{Li}_k(x)$ is defined via the formal power series

$$\mathrm{Li}_k(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^k}.$$

If $k \le 0$, say $k = -s$, then it converges for $|x| < 1$ and is given by

$$\mathrm{Li}_{-s}(x) = \frac{\sum_{j=0}^{s} \left\langle {s \atop j} \right\rangle x^{s-j}}{(1-x)^{s+1}},$$

where $\left\langle {s \atop j} \right\rangle$ are the Eulerian numbers. The number $\left\langle {s \atop j} \right\rangle$ is the number of permutations of $\{1, 2, \cdots, s\}$ with j permutation ascents. Moreover,

$$\left\langle {s \atop j} \right\rangle = \sum_{l=0}^{j+1} \binom{s+1}{l} (j-l+1)^s .$$

For more properties of Eulerian numbers (see [9]).

In [4], Bayad and Hamahata introduced poly-Bernoulli polynomials $B_n^{(k)}(x)$ by means of the following exponential generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} . \tag{1.1}$$

The numbers $B_n^{(k)} := B_n^{(k)}(0)$ are called the poly-Bernoulli numbers which were introduced by Kaneko [13] as generalizations of the classical Bernoulli numbers B_n.

It can be seen from the generating function (1.1) that, for any $n \ge 0$,

$$(-1)^n B_n^{(1)}(-x) = B_n(x),$$

where $B_n(x)$ are the classical Bernoulli polynomials given by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} .$$

The poly-Bernoulli numbers $B_n^{(k)}$ satisfy the relation (see [3])

$$B_n^{(-k)} = \sum_{m \ge 0} m! S_2(n+1, m+1) m! S_2(k+1, n+1),$$

where $S_2(n,m)$ are the Stirling numbers of the second kind. For a detailed discussion of these numbers, one may see [6].

Arakawa and Kaneko [2] introduced the so-called Arakawa-Kaneko zeta function $\xi_k(s)$ defined for any integer $k \ge 1$ by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1} t^{s-1} dt .$$

The above integral converges for $\Re(s) > 0$ and the function ξ_k can be analytically continued to the entire function of the whole s-plane. Note that

$$\xi_1(s) = s\zeta(s+1) \quad \text{and} \quad \xi_k(1) = \zeta(k+1),$$

where $\zeta(s)$ is the Riemann zeta function.

Moreover, Arakawa and Kaneko [2] have expressed the special values of function $\xi_k(s)$ at the negative integers with the aid of the poly-Bernoulli numbers $B_n^{(k)}$ through

$$\xi_k(-m) = \sum_{l=0}^{m} (-1)^l \binom{m}{l} B_l^{(k)}, \quad m = 0, 1, 2, \cdots .$$

A generalization of Arakawa-Kaneko zeta function was introduced by Coppo and Candelpergher [7] defined for $\Re(s) > 0$ and $x > 0$ by

$$\xi_k(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1} e^{-xt} t^{s-1}\, dt.$$

This is a very natural extension of the Arakawa-Kaneko zeta function the same way the as the Hurwitz zeta function $\zeta(s,x)$ generalizes the Riemann zeta function $\zeta(s)$. In particular, $\xi_k(s,1) = \xi_k(s)$ and $\xi_1(s,x) = s\zeta(s+1,x)$.

On the other hand, in [10] Hamahata defined poly-Euler polynomials $E_n^{(k)}(x)$ via the generating function

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{t(1+e^t)} e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}.$$

These polynomials satisfy the explicit formula

$$E_n^{(k)}(x) = \frac{1}{n+1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} E_{n+1}(x-j), \tag{1.2}$$

where $E_n(x) := E_n^{(1)}(x)$ are the classical Euler polynomials defined via the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

When $x = 0$, $E_n^{(k)} := E_n^{(k)}(0)$ are called the poly-Euler numbers.

Moreover, Hamahata introduced Arakawa-Kaneko type zeta function for poly-Euler polynomials defined for any integer k by

$$Z_{E,k}(s,x) = \frac{2}{\Gamma(s)} \int_0^\infty \frac{\mathrm{Li}_k(1-e^{-t})}{e^t+1} e^{-xt} t^{s-2}\, dt,$$

and showed that the function $s \to Z_{E,k}(s,x)$ has analytic continuation to an entire function on the whole complex s-plane and

$$Z_{E,k}(-n,x) = (-1)^n E_n^{(k)}(-x), \quad n \ge 0 \tag{1.3}$$

given that $x > 0$ if $k > 1$, and $x > |k| + 1$ if $k \le 1$.

2. Generalized Poly-Euler Polynomials of Parameters a, b, c

Recently, generalized poly-Bernoulli polynomials of parameters a, b, c were introduced by Jolany *et al*. [11] (see also [8, 12]) via the generating function

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t - a^{-t}} c^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x;a,b,c) \frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln b - \ln a|}. \tag{2.1}$$

The numbers $B_n^{(k)}(a,b,c) := B_n^{(k)}(0;a,b,c)$ are called the poly-Bernoulli numbers of parameters a, b, c. When $k = 1$ in (2.1),

$$B_n^{(1)}(x;a,b,c) = (\ln ab) B_n(x;a^{-1},b,c),$$

where $B_n(x,a,b,c)$ are the generalized Bernoulli polynomials with parameters a, b, c defined by Luo *et al*. [14] using the generating function

$$\frac{t}{b^t - a^t} c^{xt} = \sum_{n=0}^{\infty} B_n(x;a,b,c) \frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln b - \ln a|}.$$

Thus, we have

$$B_n(x;1,e) := B_n(x),\ B_n(0;a,b) := B_n(a,b) \text{ and } B_n(0;1,e) := B_n,$$

where $B_n(a,b)$ are called the generalized Bernoulli numbers with a,b parameters.

In this section, parallel to the above generalization of poly-Bernoulli polynomials, we introduce $E_n^{(k)}(x;a,b,c)$, the generalized poly-Euler polynomials of parameters a,b,c as a generalization of Hamahata's work in [10] and establish various properties of these polynomials. Moreover, we also obtain several identities involving $E_n^{(k)}(x;a,b,c)$, poly-Euler polynomials $E_n^{(k)}(x)$, and poly-Bernoulli polynomials $B_n^{(k)}(x)$.

Definition 2.1. For $a,b,c>0$ and $k\in\mathbb{Z}$, we define the *generalized poly-Euler polynomials of parameters* a,b,c by means of the generating function

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(a^t+b^{-t})}c^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}. \tag{2.2}$$

When $c=e$ in (2.2), we define $E_n^{(k)}(x;a,b) := E_n^{(k)}(x;a,b,e)$, the *poly-Euler polynomials of parameters* a,b. Setting further $x=0$, the numbers $E_n^{(k)}(a,b) := E_n^{(k)}(0;a;b)$ are called the *generalized poly-Euler numbers of parameters* a,b. In particular,

$$E_n^{(k)}(x;e,1,e) = E_n^{(k)}(x).$$

Moreover, when $k=1$ in (2.2)

$$E_n^{(1)}(x;a,b,c) = (\ln ab)E_n(x,a,b^{-1},c) \text{ and } E_n^{(1)}(x;e,1,e) = E_n(x),$$

where $E_n(x;a,b,c)$ are the generalized Euler polynomials of parameters a,b,c obtained by Luo *et al*. in [15] defined through the generating function

$$\frac{2}{b^t+a^t}c^{xt} = \sum_{n=0}^{\infty} E_n(x;a,b,c)\frac{t^n}{n!}, \quad |t| < \frac{\pi}{|\ln b - \ln a|}.$$

Here, we have

$$E_n(x;e,1) := E_n(x),\ E_n(0;e,1) := E_n, \text{ and } E_n(0;a,b) := E_n(a,b),$$

where $E_n(a,b)$ are called the *generalized Euler numbers with* a,b *parameters*.

The next theorem follows directly from the generating function (2.2).

Theorem 2.2. *The generalized poly-Euler polynomials satisfy the following relations:*

$$E_n^{(k)}(x;a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}E_i^{(k)}(a,b)x^{n-i}, \quad x\neq 0, \tag{2.3}$$

$$E_n^{(k)}(x+y;a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}E_i^{(k)}(y;a,b,c)x^{n-i}, \quad x\neq 0,$$

$$E_n^{(k)}(x;a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}E_i^{(k)}(y;a,b,c)(x-y)^{n-i}, \quad x\neq y.$$

For the basic derivative and integral properties of $F_n^{(\alpha)}(x,y;a,b,c)$, we have the following theorem:

Theorem 2.3. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$\frac{d}{dx}E_{n+1}^{(k)}(x;a,b,c) = (n+1)\ln c \cdot E_n^{(k)}(x;a,b,c), \tag{2.4}$$

$$\int E_n^{(k)}(x;a,b,c)dx = \frac{1}{(n+1)\ln c}E_{n+1}^{(k)}(x;a,b,c). \tag{2.5}$$

Proof. From (2.3), we have

$$E_{n+1}^{(k)}(x;a,b,c) = \sum_{i=0}^{n+1}\binom{n+1}{i}(\ln c)^{n+1-i}E_i^{(k)}(a,b)x^{n+1-i}. \tag{2.6}$$

Differentiating both side of (2.6), we obtain

$$\begin{aligned}\frac{d}{dx}E_{n+1}^{(k)}(x;a,b,c) &= \sum_{i=0}^{n}\binom{n+1}{i}(n+1-i)(\ln c)^{n+1-i}E_i^{(k)}(a,b)x^{n-i}\\ &= (n+1)\ln c\sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}E_i^{(k)}(a,b)x^{n-i}\\ &= (n+1)\ln c \cdot E_n^{(k)}(x;a,b,c).\end{aligned}$$

Equation (2.5) follows directly from (2.4). □

The next identity gives the relation between $E_n^{(k)}(x;a,b,c)$ and $E_n^{(k)}(x)$.

Theorem 2.4. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$E_n^{(k)}(x;a,b,c) = (\ln a + \ln b)^{n+1}E_n^{(k)}\left(\frac{x\ln c + \ln b}{\ln a + \ln b}\right).$$

Proof. From (2.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(1+(ab)^t)b^{-t}}c^{xt} = \frac{\ln ab \cdot 2\mathrm{Li}_k(1-e^{-t\ln ab})}{t\ln ab\cdot(1+e^{t\ln ab})}e^{t\ln ab\left(\frac{x\ln c+\ln b}{\ln ab}\right)}\\ &= \sum_{n=0}^{\infty}(\ln a+\ln b)^{n+1}E_n^{(k)}\left(\frac{x\ln c+\ln b}{\ln a+\ln b}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we get the desired result. □

Using Theorem 2.4 and (1.2), we obtain the explicit formula of $E_n^{(k)}(x;a,b,c)$ in terms of the classical Euler polynomials.

Theorem 2.5. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$E_n^{(k)}(x;a,b,c) = \frac{(\ln a+\ln b)^{n+1}}{n+1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}E_{n+1}\left(\frac{x\ln c+\ln b}{\ln a+\ln b}-j\right).$$

The next result gives a recursive formula for $E_n^{(k)}(x;a,b,c)$ in terms of the poly-Bernoulli numbers of parameters a,b and Euler polynomials of parameters a,b,c.

Theorem 2.6. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$E_n^{(k)}(x;a,b,c) = (\ln ab)\sum_{m=0}^{n}\binom{n}{m}B_{n-m}^{(k-1)}(b,a)\sum_{l=0}^{m}\binom{m}{l}\frac{(-\ln b)^{m-l}}{n-l+1}E_l(x;a,b^{-1},c).$$

Proof. Note that

$$\mathrm{Li}_{k+1}(t)=\int_0^t \frac{\mathrm{Li}_k(s)}{s}\,ds.$$

Thus,

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}c^{xt}=\frac{2c^{xt}}{t(b^{-t}+a^t)}\int_0^t \frac{\mathrm{Li}_{k-1}(1-(ab)^{-s})}{1-(ab)^{-s}}(\ln ab)e^{-s\ln ab}\,ds.$$

Consequently,

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}\\
&=(\ln ab)\left(\sum_{n=0}^{\infty} E_n(x;a,b^{-1},c)\frac{t^{n-1}}{n!}\right)\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-\ln a)^n s^n}{n!}\cdot\sum_{n=0}^{\infty} B_n^{(k-1)}(a,b)\frac{s^n}{n!}\right)ds\\
&=(\ln ab)\left(\sum_{n=0}^{\infty} E_n(x;a,b^{-1},c)\frac{t^{n-1}}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(-\ln a)^{n-m}B_m^{(k-1)}(a,b)\frac{t^{n+1}}{(n+1)!}\right)\\
&=(\ln ab)\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} E_{n-l}(x;a,b^{-1},c)\sum_{m=0}^{l}\binom{l}{m}(-\ln a)^{l-m}B_m^{(k-1)}(a,b)\right)\frac{t^n}{(l+1)!(n-l)!}\\
&=(\ln ab)\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{E_{n-l}(x;a,b^{-1},c)}{l+1}\sum_{m=0}^{l}\binom{l}{m}(-\ln a)^{l-m}B_m^{(k-1)}(a,b)\right)\frac{t^n}{n!}.
\end{aligned}$$

Applying the identity

$$\binom{n}{l}\binom{l}{m}=\binom{n}{m}\binom{n-m}{n-l},$$

we obtain

$$\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}=(\ln ab)\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_m^{(k-1)}(a,b)\sum_{l=m}^{n}\binom{n-m}{n-l}\frac{(-\ln a)^{l-m}}{l+1}E_{n-l}(x;a,b^{-1},c)\right)\frac{t^n}{n!}.$$

Setting $l'=n-l$, we get

$$\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}=(\ln ab)\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_m^{(k-1)}(a,b)\sum_{l'=0}^{n-m}\binom{n-m}{l'}\frac{(-\ln a)^{n-l'-m}}{n-l'+1}E_{l'}(x;a,b^{-1},c)\right)\frac{t^n}{n!}.$$

Setting $m'=n-m$, gives us

$$\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}=(\ln ab)\sum_{n=0}^{\infty}\left(\sum_{m'=0}^{n}\binom{n}{m'}B_{n-m'}^{(k-1)}(a,b)\sum_{l'=0}^{m'}\binom{m'}{l'}\frac{(-\ln a)^{m'-l'}}{n-l'+1}E_{l'}(x;a,b^{-1},c)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain

$$E_n^{(k)}(x;a,b,c)=(\ln ab)\sum_{m=0}^{n}\binom{n}{m}B_{n-m}^{(k-1)}(a,b)\sum_{l=0}^{m}\binom{m}{l}\frac{(-\ln a)^{m-l}}{n-l+1}E_l(x;a,b^{-1},c).$$ □

Theorem 2.7. *For $k\in\mathbb{Z}$ and $n\ge 0$,*

$$E_n^{(k)}(x;a,b,c)=\sum_{j=0}^{n}\binom{n}{j}E_{n-j}(x;a,b^{-1},c)c_j,$$

where $c_j=\sum_{m=0}^{j}\frac{(-1)^{m+j}(\ln ab)^{j+1}m!}{(m+1)^{k-1}(j+1)}S_2(j+1,m+1)$.

Proof. Using (2.2), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{2c^{xt}}{t(a^t+b^{-t})}\sum_{m=0}^{\infty}\frac{(1-e^{-t\ln ab})^{m+1}}{(m+1)^k}\\
&= \frac{2c^{xt}}{t(a^t+b^{-t})}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}m!}{(m+1)^{k-1}}\left\{\frac{(e^{-t\ln ab}-1)^{m+1}}{m+1)!}\right\}\\
&= \frac{2c^{xt}}{t(a^t+b^{-t})}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}m!}{(m+1)^{k-1}}\sum_{j=m+1}^{\infty}S_2(j,m+1)\frac{(-t\ln ab)^j}{j!}\\
&= \frac{2c^{xt}}{a^t+b^{-t}}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}m!}{(m+1)^{k-1}}\sum_{j=m+1}^{\infty}(-1)^j(\ln ab)^j S_2(j,m+1)\frac{t^{j-1}}{j!}\\
&= \frac{2c^{xt}}{a^t+b^{-t}}\sum_{m=0}^{\infty}\sum_{j=m}^{\infty}\frac{(-1)^{m+j}m!(\ln ab)^{j+1}}{(m+1)^{k-1}(j+1)}S_2(j+1,m+1)\frac{t^j}{j!}\\
&= \frac{2c^{xt}}{a^t+b^{-t}}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{m+j}m!(\ln ab)^{j+1}}{(m+1)^{k-1}(j+1)}S_2(j+1,m+1)\frac{t^j}{j!}\\
&= \left(\sum_{n=0}^{\infty}E_n(x;a,b^{-1},c)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}c_j\frac{t^j}{j!}\right),
\end{aligned}$$

where

$$c_j = \sum_{m=0}^{j}\frac{(-1)^{m+j}(\ln ab)^{j+1}m!}{(m+1)^{k-1}(j+1)}S_2(j+1,m+1).$$

Hence,

$$\sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}E_{n-j}(x;a,b^{-1},c)c_j\frac{t^n}{n!}. \qquad \square$$

Theorem 2.8. *For $k \in \mathbb{Z}$ and $n > 0$,*

$$E_{n-1}^{(k)}(x;a,b,c) = \frac{2}{n}\sum_{s=0}^{n}\binom{n}{s}\left(B_{n-s}^{(k)}\left(x+2\log_c b;a,b,c\right) - B_{n-s}^{(k)}\left(x+\log_c(b/a);a,b,c\right)\right)\alpha_s,$$

where $\alpha_s = \sum_{j=0}^{\infty}(-1)^j(j\ln ab)^s$.

Proof. It follows from the generating function (2.2) that

$$\sum_{n=1}^{\infty}\frac{1}{2}nE_{n-1}^{(k)}(x;a,b,c)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^{-t}+a^t}c^{xt}. \tag{2.7}$$

Expanding the right-hand side of (2.7), we obtain

$$\begin{aligned}
\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^{-t}+a^t}c^{xt} &= \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}\cdot\frac{(b^t-a^{-t})}{(a^t+b^{-t})}c^{xt}\\
&= \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}(b^t-a^{-t})c^{(x+\log_c b)t}(1+e^{t\ln ab})^{-1}\\
&= \left(\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+2\log_c b)t} - \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+\log_c(b/a))t}\right)\left(\sum_{j=0}^{\infty}(-1)^j e^{tj\ln ab}\right)
\end{aligned}$$

$$= \left(\sum_{n=0}^{\infty} \left[B_n^{(k)} \left(x + 2\log_c b; a, b, c\right) - B_n^{(k)} \left(x + \log_c (b/a); a, b, c\right) \right] \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j (j \ln ab)^n \frac{t^n}{n!} \right)$$
$$= \sum_{n=0}^{\infty} \left\{ \sum_{s=0}^{n} \binom{n}{s} \left(\sum_{j=0}^{\infty} (-1)^j (j \ln ab)^s \right) \left[B_{n-s}^{(k)} \left(x + 2\log_c b; a, b, c\right) - B_{n-s}^{(k)} \left(x + \log_c (b/a); a, b, c\right) \right] \right\} \frac{t^n}{n!}. \quad \square$$

Expressing $B_n^{(k)}(x,a,b,c)$ in terms of $B_n^k(x)$ in Theorem 2.8 using the relation,

$$B_n^{(k)}(x,a,b,c) = (\ln ab)^n B_n^{(k)} \left(\frac{x \ln c - \ln b}{\ln ab} \right). \quad \text{(see [8, Theorem 3.5])}$$

We obtain the following expression of $E_n^{(k)}(x,a,b,c)$ in terms of the poly-Bernoulli polynomials $B_n^{(k)}(x)$.

Corollary 2.9. *For $k \in \mathbb{Z}$ and $n > 0$,*

$$E_{n-1}^{(k)}(x;a,b,c) = \frac{2(\ln ab)^n}{n} \sum_{s=0}^{n} \binom{n}{s} \left(\sum_{j=0}^{\infty} (-1)^j j^s \right) \left[B_{n-s}^{(k)} \left(\frac{x \ln c + \ln b}{\ln ab} \right) - B_{n-s}^{(k)} \left(\frac{x \ln c - \ln a}{\ln ab} \right) \right].$$

Theorem 2.10. *For $k \in \mathbb{Z}$ and $n > 0$,*

$$nE_{n-1}^{(k)}(x + \log_c a; a,, b, c) + nE_{n-1}^{(k)}(x - \log_c b; a, b, c)$$
$$= B_n^{(k)}(x + \log_c b; a, b, c) - B_n^{(k)}(x - \log_a c; a, b, c).$$

Proof. Consider the equation

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^t + b^{-t})}(a^t + b^{-t})c^{xt} t = \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t - a^{-t}}(b^t - a^{-t})c^{xt}. \tag{2.8}$$

Expanding the left-hand side of (2.8), we obtain

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(a^t + b^{-t})}(a^t + b^{-t})c^{xt} t$$
$$= \sum_{n=0}^{\infty} \left[E_n^{(k)}(x + \log_c a; a, b, c) + E_n^{(k)}(x - \log_c b; a, b, c) \right] \frac{t^{n+1}}{n!}$$
$$= \sum_{n=1}^{\infty} \left[nE_{n-1}^{(k)}(x + \log_c a; a, b, c) + nE_{n-1}^{(k)}(x - \log_c b; a, b, c) \right] \frac{t^n}{n!}. \tag{2.9}$$

Similarly, expanding into series, the right-hand side of (2.8) is equal to

$$\sum_{n=0}^{\infty} \left[B_n^{(k)}(x + \log_c b; a, b, c) + B_n^{(k)}(x - \log_c a; a, b, c) \right] \frac{t^n}{n!}. \tag{2.10}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (2.9) and (2.10) completes the proof. $\square$

Theorem 2.11. *For $k \in \mathbb{Z}$ and $n \geq 0$, the generalized poly-Euler polynomials $E_n^{(k)}(x;a,b,c)$ satisfy the following relations:*

$$E_n^{(k)}(x;a,b,c) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} (\ln c)^l S_2(l,m) E_{n-l}^{(k)}(-m \ln c; a, b) x^{(m)}, \tag{2.11}$$

$$E_n^{(k)}(x;a,b,c) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} (\ln c)^l S_2(l,m) \binom{n}{m} E_{n-l}^{(k)}(a,b)(x)_m, \tag{2.12}$$

$$E_n^{(k)}(x;a,b,c) = \sum_{m=0}^{\infty}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{l}}S_2(l+s,s)E_{n-m-l}^{(k)}(a,b)B_m^{(s)}(x\ln c), \tag{2.13}$$

$$E_n^{(k)}(x;a,b,c) = \sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k)}(j;a,b)H_m^{(s)}(x\ln c;\lambda), \tag{2.14}$$

where

$$\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n=0}^{\infty}B_n^{(s)}(x)\frac{t^n}{n!} \quad \textit{and} \quad \left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt} = \sum_{n=0}^{\infty}H_n^{(s)}(x;\lambda)\frac{t^n}{n!}.$$

Here, $(x)_m$ *and* $x^{(m)}$ *are the falling and rising factorials respectively, defined as*

$$(x)_m = x(x-1)\cdots(x-m+1) \textit{ and } x^{(m)} = x(x+1)\cdots(x+m-1) \textit{ for } m\ge 1, \textit{ and } (x)_0 = x^{(0)} = 1.$$

Proof. For relation (2.11), we note that (2.2) can be written as

$$\sum_{n=0}^{\infty}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}(1-(1-e^{-t\ln c}))^{-x}.$$

Applying Newton's binomial theorem

$$(A+w)^{-x} = \sum_{m=0}^{\infty}\binom{x+m-1}{m}A^{-x-m}(-w)^m, \quad (|w|<|A|)$$

and

$$(e^t-1)^m = m!\sum_{n=0}^{\infty}S_2(n,m)\frac{t^n}{n!}, \tag{2.15}$$

where $S_2(n,m)$ are the Stirling numbers of the second kind, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}\sum_{m=0}^{\infty}\binom{x+m-1}{m}(1-e^{-t\ln c})^m \\
&= \sum_{m=0}^{\infty}x^{(m)}\frac{(e^{t\ln c}-1)^m}{m!}\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}e^{-mt\ln c} \\
&= \sum_{m=0}^{\infty}x^{(m)}\left(\sum_{n=0}^{\infty}S_2(n,m)\frac{(t\ln c)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k)}(-m\ln c;a,b)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{l=m}^{n}(\ln c)^l S_2(l,m)\binom{n}{l}E_{n-l}^{(k)}(-m\ln c;a,b)x^{(m)}\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives relation (2.11).

For relation (2.12), we can express (2.2) as

$$\sum_{n=0}^{\infty}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} = \frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}((e^{t\ln c}-1)+1)^x.$$

Again, using binomial theorem and (2.15), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} &= \frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}\sum_{m=0}^{\infty}\binom{x}{m}(e^{t\ln c}-1)^m \\
&= \sum_{m=0}^{\infty}(x)_m\frac{(e^{t\ln c}-1)^m}{m!}\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}
\end{aligned}$$

$$= \sum_{m=0}^{\infty}(x)_m\left(\sum_{n=0}^{\infty}S_2(n,m)\frac{(t\ln c)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k)}(a,b)\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{l=m}^{n}(\ln c)^l S_2(l,m)\binom{n}{l}E_{n-l}^{(k)}(a,b)(x)_m\right)\frac{t^n}{n!}.$$

Comparing the coefficients completes the proof of (2.12).

For relation (2.13), we express (2.2) as

$$\sum_{n=0}^{\infty}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}$$
$$= \frac{(e^t-1)^s}{s!}\cdot\frac{t^s e^{xt\ln c}}{(e^t-1)^s}\cdot\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}\cdot\frac{s!}{t^s}.$$
$$= \left(\sum_{n=0}^{\infty}S_2(n+s,s)\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}B_m^{(s)}(x\ln c)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k)}(a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s}$$
$$= \left(\sum_{n=0}^{\infty}S_2(n+s,s)\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}B_m^{(s)}(x\ln c)\frac{t^m}{m!}E_{n-m}^{(k)}(a,b)\frac{t^{n-m}}{(n-m)!}\right)\frac{s!}{t^s}$$
$$= \sum_{m=0}^{\infty}\left\{\sum_{n=m}^{\infty}\sum_{l=0}^{n-m}S_2(l+s,s)\frac{t^{l+s}}{(l+s)!}B_m^{(s)}(x\ln c)E_{n-m-l}^{(k)}(a,b)\frac{t^{n-m-l}}{(n-m-l)!}\frac{t^m}{m!}\frac{s!}{t^s}\right\}$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{s!}}S_2(l+s,s)E_{n-m-l}^{(k)}(a,b)B_m^{(s)}(x\ln c)\right\}\frac{t^n}{n!}.$$

Comparing the coefficients completes the proof of (2.13).

For relation (2.14), we express (2.2) as

$$\sum_{n=0}^{\infty}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!} = \frac{(1-\lambda)^s}{(e^t-\lambda)^s}e^{xt\ln c}\cdot\frac{(e^t-\lambda)^s}{(1-\lambda)^s}\cdot\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}$$
$$= \frac{1}{(1-\lambda)^s}\left(\sum_{n=0}^{\infty}H_n^{(s)}(x\ln c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{t(b^{-t}+a^t)}e^{jt}\right)$$
$$= \frac{1}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}\left(\sum_{n=0}^{\infty}H_m^{(s)}(x\ln c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k)}(j;a,b)\frac{t^n}{n!}\right)$$
$$= \frac{1}{(1-\lambda)^s}\left(\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}H_m^{(s)}(x\ln c;\lambda)E_{n-m}^{(k)}(j;a,b)\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k)}(j;a,b)H_m^{(s)}(x\ln c;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients completes the proof of (2.14). □

In particular, when $c=e$ in Theorem 2.11, we have the following identities for the generalized poly-Euler polynomials of parameters a,b.

Corollary 2.12. *For $k\in\mathbb{Z}$ and $n\ge 0$, the generalized poly-Euler polynomials $E_n^{(k)}(x;a,b)$ satisfy the following relations:*

$$E_n^{(k)}(x;a,b) = \sum_{m=0}^{\infty}\sum_{l=m}^{n}S_2(l,m)E_{n-l}^{(k)}(-m;a,b)x^{(m)},$$

$$E_n^{(k)}(x;a,b)=\sum_{m=0}^{\infty}\sum_{l=m}^{n}S_2(l,m)\binom{n}{m}E_{n-l}^{(k)}(a,b)(x)_m\,,$$
$$E_n^{(k)}(x;a,b)=\sum_{m=0}^{\infty}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{l}}S_2(l+s,s)E_{n-m-l}^{(k)}(a,b)B_m^{(s)}(x),$$
$$E_n^{(k)}(x;a,b)=\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k)}(j;a,b)H_m^{(s)}(x;\lambda).$$

3. Arakawa-Kaneko Type Zeta Functions

In [10], Hamahata defined the Arakawa-Kaneko type zeta functions $Z_{E,k}$ for poly-Euler polynomials by means of the Laplace-Mellin integral

$$Z_{E,k}(s,x)=\frac{2}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{Li}_k(1-e^{-t})}{1+e^t}e^{-xt}t^{s-2}\,dt, \tag{3.1}$$

where $\Re(s)>1$ and $x>0$ if $k\geq 1$, and $\Re(s)>1$ and $x>|k|+1$ if $k\leq 0$.

For $k=1$,

$$Z_{E,1}(s,x)=\frac{2}{\Gamma(s)}\int_0^{\infty}\frac{e^{-xt}}{e^t+1}t^{s-1}dt=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+x)^s}=\zeta_E(s,x+1),$$

where $\zeta_E(s,x)$ is the Euler zeta function of Hurwitz type defined by

$$\zeta_E(s,x)=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+x)^s}\,.$$

In this section, we give a generalized Arakawa-Kaneko type zeta functions for the poly-Euler polynomials of parameters a,b,c and obtain an interpolation formula between these generalized zeta functions and the poly-Euler polynomials with parameters a,b,c. Moreover, we also establish relation between $Z_{E,k}(s,x;a,b,c)$ and the generalized Arakawa-Kaneko zeta functions $Z_{B,k}(s,x;a,b,c)$ (see [1]) for poly-Bernoulli polynomials of parameters a,b,c (see [11]).

Definition 3.1. For $k\in\mathbb{Z}$, we define the generalized Arakawa-Kaneko type zeta functions with parameters a,b,c via the Laplace-Mellin type integral

$$Z_{E,k}(s,x;a,b,c)=\frac{2}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^{-t}+a^t}c^{-xt}t^{s-2}\,dt. \tag{3.2}$$

It can be seen that $Z_{E,k}(s,x;e,1,e)$ are just the Arakawa-Kaneko zeta type functions $Z_{E,k}(x,s)$ defined by Hamahata [10].

The following lemma gives a relation between the generalized Arakawa-Kaneko type zeta functions with parameters a,b,c and Arakawa-Kaneko type zeta functions $Z_{E,k}(x,s)$.

Lemma 3.2. *For $k\in\mathbb{Z}$,*

$$Z_{E,k}(s,x;a,b,c)=(\ln a+\ln b)^{1-s}Z_{E,k}\left(s,\frac{x\ln c-\ln b}{\ln a+\ln b}\right).$$

Proof. Using the generating function (3.2),

$$Z_{E,k}(s,x;a,b,c)=\frac{2}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{Li}_k(1-e^{-t\ln ab})}{1+e^{t\ln ab}}e^{-t(x\ln c-\ln b)}t^{s-2}\,dt. \tag{3.3}$$

By changing variables $z = (\ln a + \ln b)t$, we obtain

$$\begin{aligned} Z_{E,k}(s,x;a,b,c) &= \frac{1}{(\ln a+\ln b)^{s-1}}\frac{2}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_k(1-e^{-z})}{1+e^z}e^{-z\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)}z^{s-2}\,dz \\ &= \frac{1}{(\ln a+\ln b)^{s-1}}Z_{E,k}\left(s,\frac{x\ln c-\ln b}{\ln a+\ln b}\right). \end{aligned}$$ □

Theorem 3.3 (Interpolation Formula). *The function $s \to \xi_k(s,x;a,b,c)$ has analytic continuation to an entire function on the whole complex s-plane and for any positive integer n,*

$$Z_{E,k}(-n,x;a,b,c) = (-1)^n E_n^{(k)}(-x;a,b,c).$$

Proof. To prove that $s \to Z_{E,k}(s,x;a,b,c)$ has analytic continuation to an entire function on the whole complex s-plane, it is sufficient to show that $s \to Z_{E,k}(s,x)$ has such a property which was already shown in [10, Theorem 4.3]. Hence by Lemma 3.2, equation (1.3) and Theorem 2.4, we obtain

$$\begin{aligned} Z_{E,k}(-n,x;a,b,c) &= (\ln a+\ln b)^{1-(-n)}Z_{E,k}\left(-n,\frac{x\ln c-\ln b}{\ln a+\ln b}\right) \\ &= (\ln a+\ln b)^{n+1}(-1)^n E_n^{(k)}\left(\frac{-x\ln c+\ln b}{\ln a+\ln b}\right) \\ &= (-1)^n E_n^{(k)}(-x;a,b,c). \end{aligned}$$ □

We now give an explicit formulas of $Z_{E,k}(s,x;a,b,c)$ in terms of $\zeta(s,x)$.

Theorem 3.4. *The Arakawa-Kaneko type zeta function $Z_{E,k}(s,x;a,b,c)$ can be expressed as follows: For $s \neq 1$,*

(i) *If $k \in \mathbb{Z}$, then*

$$Z_{E,k}(s,x;a,b,c) = \frac{(\ln a+\ln b)^{1-s}}{s-1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^j\zeta_E\left(s-1,\frac{x\ln c-\ln b}{\ln a+\ln b}+j+1\right).$$

(ii) *If $k \le 0$, then*

$$Z_{E,k}(s,x;a,b,c) = \frac{(\ln a+\ln b)^{1-s}}{s-1}\sum_{j=0}^{|k|}\left\langle {|k| \atop j} \right\rangle\sum_{i=0}^{|k|-j}(-1)^j\binom{|k|-j}{i}\zeta_E\left(s-1,\frac{x\ln c-\ln b}{\ln a+\ln b}+i-|k|\right).$$

Proof. (i) It follows from (3.2) that

$$\begin{aligned} Z_{E,k}(s,x;a,b,c) &= (\ln a+\ln b)^{1-s}\frac{2}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_k(1-e^{-z})}{1+e^z}e^{-z\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)}z^{s-2}\,dz \\ &= (\ln a+\ln b)^{1-s}\frac{2}{\Gamma(s)}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\int_0^\infty \frac{(1-e^{-z})^{m+1}}{1+e^z}e^{-z\left(\frac{x\ln c-\ln b}{\ln a+\ln b}\right)}z^{s-2}\,dz. \end{aligned}$$

For $s \neq 1$,

$$\begin{aligned} Z_{E,k}(s,x;a,b,c) &= \frac{(\ln a+\ln b)^{1-s}}{s-1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^j\frac{2}{\Gamma(s-1)}\int_0^\infty\frac{e^{-z\left(\frac{x\ln c-\ln b}{\ln a+\ln b}+j\right)}}{1+e^z}z^{s-2}\,dz \\ &= \frac{(\ln a+\ln b)^{1-s}}{s-1}\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^j\zeta_E\left(s-1,\frac{x\ln c-\ln b}{\ln a+\ln b}+j+1\right). \end{aligned}$$

(ii) Note that for $k \le 0$, we have

$$\mathrm{Li}_k(1-e^{-t\ln ab}) = \frac{\sum_{j=0}^{|k|} \left\langle {|k| \atop j} \right\rangle \left(1-e^{-t\ln ab}\right)^{|k|-j}}{e^{-t\ln ab(|k|+1)}} = e^{t\ln ab(|k|+1)} \sum_{j=0}^{|k|} \left\langle {|k| \atop j} \right\rangle \sum_{i=0}^{|k|-j} \binom{|k|-j}{i} (-1)^i e^{-it\ln ab}.$$

Using (3.2) and (3.3), we get

$$\begin{aligned}
&Z_{E,k}(s,x;a,b,c)\\
&= \frac{2}{\Gamma(s)} \sum_{j=0}^{|k|} \left\langle {|k| \atop j} \right\rangle \sum_{i=0}^{|k|-j} \binom{|k|-j}{i} (-1)^i \int_0^\infty \frac{e^{-t(x\ln c-\ln b+(i-|k|-1)\ln ab}}{1+e^{t\ln ab}} t^{s-2}\,dt\\
&= \frac{(\ln a+\ln b)^{1-s}}{s-1} \sum_{j=0}^{|k|} \left\langle {|k| \atop j} \right\rangle \sum_{i=0}^{|k|-j} \binom{|k|-j}{i} (-1)^i \cdot \frac{2}{\Gamma(s-1)} \int_0^\infty \frac{e^{-z\left(\frac{x\ln c-\ln b}{\ln a+\ln b}+i-|k|-1\right)}}{1+e^z} z^{s-2}\,dz\\
&= \frac{(\ln a+\ln b)^{1-s}}{s-1} \sum_{j=0}^{|k|} \left\langle {|k| \atop j} \right\rangle \sum_{i=0}^{|k|-j} \binom{|k|-j}{i} (-1)^i \zeta_E\left(s-1, \frac{x\ln c-\ln b}{\ln a+\ln b}+i-|k|\right). \qquad \square
\end{aligned}$$

Theorem 3.5 (Addition Formula)**.** *For $k \in \mathbb{Z}$ and $s \neq 1$, we have*

$$\begin{aligned}
&Z_{E,k}(s,x-\log_c ab;a,b,c)+Z_{E,k}(s,x;a,b,c)\\
&\quad = \frac{2}{s-1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j (x\ln c+(j-1)\ln b+j\ln a)^{1-s}.
\end{aligned}$$

Proof. Applying (3.2), we have

$$\begin{aligned}
&Z_{E,k}(s,x-\log_c ab;a,b,c)+Z_{E,k}(s,x;a,b,c)\\
&\quad = \frac{2}{\Gamma(s)} \int_0^\infty \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^{-t}+a^t} \left(c^{-t(x-\log_c(ab))}+c^{-tx}\right) t^{s-2}\,dt\\
&\quad = \frac{2}{\Gamma(s)} \int_0^\infty \frac{\mathrm{Li}_k(1-(ab)^{-t})}{1+(ab)^{-t}} (1+(ab)^{-t}) e^{-t(x\ln c-\ln b)} t^{s-2}\,dt\\
&\quad = \frac{2}{\Gamma(s)} \int_0^\infty \mathrm{Li}_k(1-e^{-t\ln ab}) e^{-t(x\ln c-\ln b)} t^{s-2}\,dt\\
&\quad = \frac{2}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \int_0^\infty \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j e^{-t(x\ln c+(j-1)\ln b+j\ln a)} t^{s-2}\,dt\\
&\quad = 2 \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j \cdot \frac{1}{\Gamma(s)} \int_0^\infty e^{-t(x\ln c+(j-1)\ln b+j\ln a)} t^{s-2}\,dt\\
&\quad = \frac{2}{s-1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j (x\ln c+(j-1)\ln b+j\ln a)^{1-s}. \qquad \square
\end{aligned}$$

3.1 Relation Between $Z_{E,k}(s,x;a,b,c)$ and $Z_{B,k}(s,x;a,b,c)$

Recently, Acala and Aleluya [1] defined the generalized Arakawa-Kaneko type zeta functions with a,b,c parameters $Z_{B,k}(s,x;a,b,c)$ for the poly-Bernoulli polynomials $B_n^{(k)}(x;a,b,c)$ via the Laplace-Mellin type integral

$$Z_{B,k}(s,x;a,b,c) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t - a^{-t}} c^{-xt} t^{s-1}\,dt, \tag{3.4}$$

and showed that the function $s \to Z_{B,k}(s,x;a,b,c)$ has analytic continuation to an entire function on the whole complex s-plane and for any positive integer n,

$$Z_{B,k}(-n,x,a,b,c) = (-1)^n B_n^{(k)}(-x;a,b,c).$$

In the next theorem, we give a relationship between $Z_{B,k}(s,x;a,b,c)$ and $Z_{E,k}(s,x;a,b,c)$.

Theorem 3.6. *For $k \in \mathbb{Z}$,*

$$\begin{aligned} &sZ_{E,k}(s+1,x-\log_c b;a,b,c) + sZ_{E,k}(s+1,x+\log_c a;a,b,c) \\ &\quad = 2_{B,k}(s,x-\log_c b;a,b,c) - 2Z_{B,k}(s,x+\log_c a;a,b,c). \end{aligned}$$

Proof. Note that

$$\begin{aligned} &2Z_{B,k}(s,x-\log_c b;a,b,c) - 2Z_{B,k}(s,x+\log_c a;a,b,c) \\ &\quad = \frac{2}{\Gamma(s)}\int_0^\infty \mathrm{Li}_k(1-(ab)^{-t}) c^{-xt} t^{s-1}\,dt \\ &\quad = \frac{2}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^{-t}+a^t}\left(b^{-t}+a^t\right) c^{-xt} t^{s-1}\,dt \\ &\quad = \frac{2s}{\Gamma(s+1)}\int_0^\infty \left[\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^{-t}+a^t}\left(c^{-t(x+\log_c b)} + c^{-t(x-\log_c a)}\right) t^{(s+1)-2}\right] dt \\ &\quad = sZ_{E,k}(s+1,x+\log_c b;a,b,c) + sZ_{E,k}(s+1,x-\log_c a;a,b,c). \end{aligned}$$

□

4. Conclusion

By introducing a new class of generalized poly-Euler polynomials with a, b, c parameters, we defined the Arakawa-Kaneko type zeta functions for these polynomials and obtained an interpolation formula. Finally, a relationship between the Arakawa-Kaneko type zeta functions for generalized poly-Euler polynomials and the Arakawa-Kaneko for generalized poly-Bernoulli polynomials in [1] was established.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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