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Inclusion of Loops into Lie Groups: Infinitesimal Characteristics

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Abstract. The infinitesimal characteristics for some class of smooth loops generalizing Bruck loops to be embedded into a Lie group are given.

The interest to the problem of inclusion of loops into Lie groups has been shown by various recent publications dedicated to representations of loops as transversals on homogeneous spaces.

There is presented an example of a smooth left Bol loop, satisfying the generalized Bruck identity, realized as a section on some reductive homogeneous space.

1. **Introduction**

1.1. *Inclusion of loops into groups*

The inclusion of loops into groups has been considered by many authors.

M. Kikkawa considered the construction suggested by Yamaguty in order to obtain a loop (homogeneous Lie loop) [4].

The book [15] is dedicated to the theory of loops presented from the point of view of the Group Theory, i.e., loops are represented as "sharply" transitive sections in Lie groups. The general theory developed there and examples are mostly concern the class of Bol loops [1], Bruck loops (see [2, p. 262]) and Moufang loops [8].

We consider so called Generalized Bruck loops, i.e., left Bol loops satisfying the generalized Bruck identity [14, p. 351 (7)]. The class of smooth Generalized Bruck loops originated in the theory of generalized symmetric spaces (see [10, (1.7) p. 340]) treated in the frames of non associative algebra [9, (A.2.21), p. 192].

The interest to study Generalized Bruck loops is also motivated by applications to Special Relativity, due to the non-associative algebraic system describing the gyrocommutative gyrogroups introduced by A. Ungar [16], the former turns out to be a Bruck loop (see [11]).

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Some examples of non-gyrocommutative gyrogroups by A. Ungar (see also [2, p. 266]) correspond to the case when the gyrocommutative property is not valid. In order to construct an example of non-Bruck loop, T. Fogel ([2, p. 263]) applied the algebraic construction of inclusion of loops into groups which is due to L. Sabinin [9].

Following this construction for the case of smooth loops we present an example of a Generalized Bruck loop realized as a cross-section (transversal) on some reductive homogeneous space, thus it may be related to a non-gyrocommutative gyrogroup.

In Subsection 1.2 we give a brief description of the Baer-Sabinin construction ([9, Ch. 2]) in terms corresponding to our consideration.

Section 2 is dedicated to the conditions, necessary and sufficient, for a cross-section on a homogeneous space G/H to be a loop with given properties.

In Section 3 we construct a transversal (cross-section) Q on a reductive homogeneous space G/H ([5, p. 184]), and verify the necessary and sufficient conditions for the loop Q to be a Generalized Bruck loop.

1.2. *General construction: loops as transversals on homogeneous spaces*

Let us give the necessary preliminaries.

A left loop $\langle Q, \cdot, \varepsilon \rangle$ can be defined as a set Q with a binary operation $\cdot$ such that the equation

$$a \cdot x = b$$

has a unique solution $\forall a, b \in Q$, and there exist a fixed element ε (*right identity*), so that $x \cdot \varepsilon = x$.

Thus the equation

$$(a \cdot b) \cdot x = a \cdot (b \cdot c)$$

possesses the unique solution

$$x = l_{a,b}\, c = L^{-1}_{a \cdot b} \circ L_a \circ L_b\, c\,,$$

where $L_a x = a \cdot x$.

The group $\mathrm{as}_l(Q)$ generated by $l_{a,b}$ is a subgroup in $\mathrm{Lmult}(Q) = \langle L_q \rangle_{q \in Q}$, and it is called the left associant of the left loop Q.

By means of the construction of the *semidirect product* $Q \boxtimes H$ of a left loop Q by its transassociant H ([9, Proposition 2.10, p. 40]) a left homogeneous space $Q \boxtimes H / i(H)$ may be obtained in a unique way ([9, Proposition 2.13, p. 41]). (A left tranassociant of a left loop Q is a group $H \subset \mathfrak{S}_Q$ which contains the associant $as_l(Q)$, $H\{\varepsilon\} = \{\varepsilon\}$, and $\mathrm{m}_Q H \subset H$, where $\mathrm{m}_q(h) = L^{-1}_{hq} \circ h \circ L_q \circ h^{-1}$, $q \in Q$, $h \in H$, describes the 'deviation' of h from being an automorphism).

Furthermore, there may be constructed its cross-section endowed with a binary operation, by projecting along the left cosets ([9, Proposition 2.12, p. 41]). Then the Proposition 2.13 in [9, p. 41] shows that $Q \boxtimes H/i(H)$ can be identified with the loop $\tilde{Q} = (Q, \mathrm{id}_Q) \cong Q$, that is, the homogeneous space thus obtained is canonically equipped with the structure of loop, isomorphic to the initial one.

Moreover, an arbitrary homogeneous space G/R can be equipped with a left loop structure Q such that $Q \boxtimes H/i(H) \cong G/R$ for some choice of the transassociant H.

In general, in Chapter 2 of [9] the equivalence of the categories of left loops and left homogeneous spaces is established.

As a result, an arbitrary left loop $\langle Q, \cdot, \varepsilon \rangle$ with two-sided neutral element ε can be included into some group.

Namely, there exists a group $\langle G, \boxdot, \varepsilon \rangle$, a subgroup $H \subset G$ and the inclusion map $i : Q \to G$ such that Q, being identified with $i(Q)$, turns out to be a section of the set of left cosets G/H (i.e., for any element $g \in G$ there exist the unique elements $q \in Q$ and $h \in H$ such that $g = q \boxdot h$) and the operations are related by $a \cdot b = \pi_Q(a \boxdot b)$, where $\pi_Q : G \to Q \cong G/H$ is a projection on Q along H.

A cross-section (transversal) $\tilde{Q}$ is called a reductant if

$$h\tilde{Q}h^{-1} \subset \tilde{Q}, \quad \forall\, h \in H, \tag{1.1}$$

which is called reductivity condition.

In the case of a smooth left A-loop the corresponding left homogeneous space turns out to be reductive.

A left loop $\langle Q, \cdot, \varepsilon \rangle$ is said to be a left A-loop if $\mathrm{as}_l(Q) \subset \mathrm{Aut}(Q)$, i.e.,

$$l_{a,b}(x \cdot y) = (l_{a,b}x \cdot l_{a,b}y). \tag{1.2}$$

Let us note that in the case of a smooth A-loop the group $H = \mathrm{as}_l(Q)$ generated by $l_{a,b}$ is always a Lie group.

Based on Definition 2.14 and Propositions 2.16–2.19 in [9] we formulate

Proposition 1.1. *If Q is a left A-loop and $H = \mathrm{as}_l(Q)$ is its transassociant, then the homogeneous space $Q \boxtimes H/i(H)$ $(i : H \to Q \boxtimes H)$ is reductive and the cross-section $\tilde{Q} = (Q, \mathrm{Id}_Q)$ of the space of left cosets is its reductant.*

Conversely, if G/R is a left reductive homogeneous space with a reductant Q, then Q is a left A-loop with respect to the operation induced on Q, and a suitable transassociant $\tilde{R}$ is a subgroup of Aut(Q).

Proof. See ([9, Proposition 2.20, p. 44]). □

Remark 1.2. Passing to the corresponding Lie algebras we obtain in this case the reductive decomposition, this explains the terminology.

Remark 1.3. If a left A-loop is a left loop $\langle Q, \cdot, \varepsilon \rangle$ with two-sided neutral element ε, then the corresponding reductant is homogeneous, i.e., it contains εH, and vice versa. This is the case of homogeneous Lie loops considered by M. Kikkawa [4].

Remark 1.4. In [2] the author refers to the reductivity condition (1.1) as the normality condition because for the case when Q is a group it leads to a normal subgroup $H \subset \mathrm{Aut}(Q)$.

If Q is a group then $Q \boxtimes H$ turns to be a classic semidirect product of Q and the subgroup H of the group of its automorphisms.

2. Generalized smooth Bruck Loops as transversals on reductive homogeneous spaces

2.1. *Identities in loops: Conditions in terms of the group operation*

Since an arbitrary homogeneous space G/R can be equipped with a left loop structure Q the problem arises to find necessary and sufficient conditions in terms of the group operation so that a cross-section on G/H appears to be a loop with certain identities.

For example, the necessary and sufficient conditions for a local analytic left Bol loop to be embedded into a local Lie group are well known.

We shall describe the corresponding construction following [6, p. 63, p. 64], (see [7, p. 427, p. 416]).

Let $\langle G, \triangle, \varepsilon\rangle$ be a local Lie group with the Lie algebra $\mathfrak{g}$, and let H be a connected Lie subgroup with the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Consider a vector subspace $\mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$.

Let $\pi : G \to G/H$ be the canonical projection on the local space of left cosets, and let ψ be the restriction of the mapping composition $\pi \circ \exp$ to $\mathfrak{h}$. Then ψ is a diffeomorphism of a suitable neighborhood U of the point O in $\mathfrak{h}$ with a neighborhood of the coset $\pi(e)$ in G/H (see [3, Chapter II]).

Then on the points of the local cross-section $Q = \exp U$ of left cosets G/H there can be introduced a local law of composition

$$a \times b = \pi_Q(a \triangle b),$$

where $\pi_Q = \exp \circ (\psi)^{-1} \circ \pi : G \to Q$ is the local projection on Q parallel to H, which establishes a correspondence between each element $g \in G$ and a unique element $a \in Q$ so that $g = a \triangle p$, where $p \in H$.

Then the corresponding cross-section $\langle Q, \times, \varepsilon\rangle$ is a local analytic loop with the left monoalternative property [7, p. 417]:

$$a^k \times (a^l \times b) = a^{k+l} \times b. \tag{2.1}$$

Proposition 2.1. *Suppose that*

$$a \triangle b \triangle a \in Q, \quad Q = \exp U, \tag{2.2}$$

for all $a, b \in Q$ sufficiently close to the point ε.

Then the local analytic loop $\langle Q, \times, \varepsilon\rangle$ satisfies the left Bol identity

$$a \times (b \times (a \times c)) = (a \times (b \times a)) \times c\,. \tag{2.3}$$

Moreover, if the subgroup H does not contain any nontrivial subgroup normal in G then the following stronger version holds:

Let $Q = \exp U$ be a local cross-section for the left cosets G/H. Then the local analytic loop $\langle Q, \times, \varepsilon\rangle$ satisfies the left Bol identity if and only if $a\triangle b\triangle a \in Q$, for any $a, b \in Q$ sufficiently close to ε.

Definition 2.2. A smooth local left loop $\langle Q, \cdot, \varepsilon\rangle$ with a local automorphism $s \in \mathrm{Aut}(Q)$ is called a Generalized Bruck Loop if the following two identities are satisfied:

$$x\cdot(y\cdot(x\cdot z)) = (x\cdot(y\cdot x))\cdot z \qquad \text{(Left Bol Identity)}, \tag{2.4}$$

$$(x\cdot y)\cdot \mathrm{I}s(x\cdot y) = x\cdot((y\cdot \mathrm{I}s\,y)\cdot \mathrm{I}s\,x) \quad \text{(Generalized Bruck Identity)}, \tag{2.5}$$

where $(x\cdot \mathrm{I}x) = \varepsilon$, or, equivalently, $\mathrm{I}w = w^{-1}$.

Smooth local left loops $\langle Q, \cdot, \varepsilon\rangle$ with a local automorphism $s \in \mathrm{Aut}(Q)$ satisfying the above conditions admit inclusion into local Lie groups $\langle G, \triangle, \varepsilon\rangle$.

We need the following Theorem (see [6, p. 68], [13])

Theorem 2.3. *Let us consider a smooth left loop $\langle Q, \cdot, \varepsilon\rangle$ with the identity:*

$$x\cdot(y\cdot(sx^{-1}\cdot z)) = (x\cdot(y\cdot(sx^{-1}))\cdot z, \tag{2.6}$$

where $s(x\cdot y) = (sx)\cdot(sy)$, $s: Q\to Q$, such that $s_{*,\varepsilon} - \mathrm{id}_{*,\varepsilon}$ is invertible, and the left monoalternative identity (2.1) *is satisfied.*

Then there exists a local Lie group $\langle G, \triangle, \varepsilon\rangle$ and an automorphism $\tilde{s} \in \mathrm{Aut}(G)$ such that in Q, being identified with a local cross-section of the left cosets G/H, $H = \{x \in G \mid \tilde{s}(x) = x\}$, the following relations are satisfied:

(i) $\tilde{s}|_Q = s$ *has no fixed points on Q except for the point ε,*

(ii) $h\triangle Q\triangle h^{-1} \subset Q, \quad \forall\, h \in H,$

(iii) $x\triangle y\triangle sx^{-1} \in Q, \quad \forall\, x, y \in Q,$ (2.7)

where

$$x\cdot y = \pi_Q(a\triangle b), \quad \forall\, x, y \in Q,$$

as in the construction which has been described above in 1.2*.*

The conditions (i), (ii) and (iii) are necessary and sufficient.

Proof. First, we shall give the proof for (iii).

Following the General Construction one can take G/H as $L(Q)/\mathrm{as}_l(Q)$, (see 1.2)

Let us take the set $L_Q = \{L_a\}_{a\in Q}$ of all left translations of the left loop Q. Then it can be considered as a reductant (after the identification (see 1.2)). For every element T of the group $L(Q)$ there is the unique decomposition $T = L\circ h$, where $L = L_{Te} \in Q$, (see 1.2), $h \in \mathrm{as}_l(Q)$.

Let us introduce the operation

$$L_q \odot L_p = \pi_{L_Q}(L_q \circ L_p) = L_{q\cdot p}, \tag{2.8}$$

where $L_q \circ L_p = L_{q\cdot p} \circ l(q,p)$, so that $\pi_{L_Q}(L_q \circ L_p) = L_{q\cdot p}$ is the projection on L_Q along H.

Furthermore $l(a,b) \circ L_x \circ l(a,b)^{-1} = L_{l(a,b)}x, \ \forall\ a,b \in Q$, so that $\mathrm{as}_l(Q)$ transforms the section L_Q in L_Q (which means that L_Q is a reductant in $L(Q)/\,\mathrm{as}_l(Q)$).

The operation (2.8) defines on the set L_Q a loop, which is isomorphic to the loop $\langle Q,\cdot,\varepsilon\rangle$ with respect to the bijection $L:\ q \to L_q$.

Since the operation of multiplication in the loop Q is related with the law of composition in the group G by $L_a \circ L_b = L_{a\cdot b} \circ l(a,b)$, we have

$$\begin{aligned} L_x \circ L_y \circ L_{sx^{-1}} &= L_x \circ L_{y\cdot sx^{-1}} \circ l(y,sx^{-1}) \\ &= L_{x\cdot(y\cdot sx^{-1})} \circ [l(x, y\cdot sx^{-1}) \circ l(y,sx^{-1})] \\ &= L_{x\cdot(y\cdot sx^{-1})} \circ h. \end{aligned}$$

Acting by both sides on z, we obtain

$$x\cdot(y\cdot(sx^{-1}\cdot z)) = [x\cdot(y\cdot sx^{-1})]\cdot hz\,.$$

If the (2.6) holds then $h = \mathrm{id}$, and we have

$$L_x \circ L_y \circ L_{sx^{-1}} = L_{x\cdot(y\cdot sx^{-1})}\,.$$

This means that

$$x \triangle y \triangle sx^{-1} = x\cdot(y\cdot sx^{-1}) \quad \forall\ x,y \in Q,$$

due to the identifying isomorphism $L: x \mapsto L_x$.
That is

$$x \triangle y \triangle sx^{-1} \in Q, \quad \forall\ x,y \in Q,$$

where x^{-1} is the right inverse element, i.e., $x\cdot x^{-1} = \varepsilon$ in the left loop Q, but because of (2.1) we have $L_x^{-1} = L_{x^{-1}}$, thus it can be considered as the inverse element in the group G.

Conversely, if

$$x \triangle y \triangle sx^{-1} \in Q, \quad \forall\ x,y \in Q,$$

then, taking into account the identifying isomorphism $L: x \mapsto L_x$, we have

$$L_x \circ L_y \circ L_{sx^{-1}} \in \{L_q\}_{q\in Q},$$

where x^{-1} is the right inverse element in the loop as well as in the group. Then

$$L_x \circ L_y \circ L_{sx^{-1}} = L_{x\cdot(y\cdot sx^{-1})},$$

which, after acting on z, gives (2.6).

Let us note, that the automorphism $s : Q \to Q$ may be extended up to the isomorphism of $L(Q)$, and therefore of G, due to identifying isomorphism, if we put by definition $s(L_a \circ h) = L_{sa} \circ h$.

Now, (ii) is the reductivity condition (1.1) and, as is mentioned in the proof of (iii), for the initial left loop the (1.2) is valid, thus the Proposition 1.1 is applied.

The condition (i) is just the invertibility of $s_{*,\varepsilon} - \mathrm{id}_{*,\varepsilon}$, which provides the right division in the given left loop. □

These criteria may be applied to the case of a smooth left loop in Definition 2.2.

In fact, a left Bol loop, i.e., a left loop with two-sided identity element and with the Bol identity (2.4), possesses the right division, and the monoalternativity identity (2.1) is verified. This implies that the conditions (i) and (ii) of Theorem 2.3 hold true (see [7, p. 385], [6, p. 10]).

As well, one may refer to the equivalent characteristics of a smooth loop in Definition 2.2, [14, p. 351, p. 353] (see ([12], [9, p. 192]), wherefrom it follows that a Generalized Bruck loop is a left A-loop, and also the (2.6) holds for the particular case when $\varphi = s$ ([9, p. 215]).

The latter is because the equivalent characteristic identities, so called φM-*identity*:

$$L_a \circ L_x \circ L_{\varphi a}^{-1} = L_{a\cdot(x\cdot\varphi a^{-1})}\,, \tag{2.9}$$

where $\varphi : Q \to Q$ is an endomorphism and $f : x \mapsto \varphi x \cdot x^{-1}$ is bijective, and

$$x^{-1}\cdot(x\cdot y) = y\,,$$

imply

$$L_a \circ L_x \circ L_{(\varphi a)^{-1}} = L_{a\cdot(x\cdot(\varphi a)^{-1})}\,, \tag{2.10}$$

being the left Iq-half Bol identity ([9, Ap. 5]) of the form

$$L_a \circ L_x \circ L_{\mathrm{Iq}\,a} = L_{a\cdot(x\cdot\mathrm{Iq}\,a)}, \quad \mathrm{Iw} = w^{-1}\,. \tag{2.11}$$

We should restrict our consideration for the case when $\varphi = s \in \mathrm{Aut}(Q)$, thus we have (2.6) (see [9, p. 173]).

Let us note, that if $\mathrm{Jq} = \mathrm{id}$ we get the left Bol identity.

2.2. *Infinitesimal characteristics*

The construction of the infinitesimal object which describes the structure of smooth local loops is one of the principal problems in the study of different classes of loops.

For some classes of smooth loops (local analytic Moufang loops, Bol loops and homogeneous Lie loops) the adequate infinitesimal object is a binary-ternary algebra.

In a general case, to any local smooth loop there corresponds only one algebra, endowed with a set of multilinear operations [9, 0.17, p.70].

For the case of a loop in Theorem 2.3 we have the reductive decomposition, namely,

$$\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}, \qquad (\operatorname{ad}\mathfrak{h})\mathfrak{m} \subset \mathfrak{m},$$

where $\mathfrak{h}$ is the subalgebra in $\mathfrak{g}$ corresponding to the subgroup $H \subset G$, $\mathfrak{m}$ is a tangent subspace to Q at $\varepsilon \in G$ and $Q \bigcap H = \{\varepsilon\}$, locally.

The tangent space at the point ε of the group G will be identified with its Lie algebra $\mathfrak{g}$ as usual.

The condition $(\operatorname{ad}\mathfrak{h})\mathfrak{m} \subset \mathfrak{m}$ is satisfied because of the definition of the reductant $hQh^{-1} \subset Q$, $\forall h \in H$ (see reductivity condition (1.1)).

Thus we have the definition of the reductive homogeneous space in the Lie algebras language.

Furthermore, some additional conditions on the reductive decomposition have to hold true [13].

$$[\xi, (\tilde{s} + \mathrm{id})\eta] \in \mathfrak{m}, \qquad [\xi, [\eta, \zeta]] \in \mathfrak{m}, \quad \forall\ \xi, \eta, \zeta \in \mathfrak{m},$$

where $\tilde{s} = s_{*,\varepsilon}$, the tangent map of the automorphism s of the group G, is an automorphism of the Lie algebra $\mathfrak{g}$.

The latter inclusion represents the necessary and sufficient infinitesimal condition for a loop to satisfy the Bol identity [6, p. 64].

These conditions and the Theorem 2.3 allow us to construct an example of a Bol loop with the generalized Bruck identity of the Definition 2.2.

3. **Example of a generalized Bruck loop included into a Lie group**

Let G be the group of matrices

$$\begin{pmatrix} \cosh t & \sinh t & 0 & 0 & a \\ \sinh t & \cosh t & 0 & 0 & b \\ 0 & 0 & \cosh t & -\sinh t & c \\ 0 & 0 & -\sinh t & \cosh t & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $a, b, c, d, t \in R$, and let H be a subgroup of G whose elements are of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The map s given by $s : t \to -t,\ a \to -b,\ b \to a,\ c \to -c,\ d \to d$ is an automorphism of the group G, and $s : H \to H$. Moreover, $sA = A$ if and only if $A \in H$.

Let us consider the set Q of matrices of the form

$$\begin{pmatrix} \cosh t & \sinh t & 0 & 0 & a \\ \sinh t & \cosh t & 0 & 0 & b \\ 0 & 0 & \cosh t & -\sinh t & c \\ 0 & 0 & -\sinh t & \cosh t & [-\tanh(t/2)]c \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is obvious that $sQ \subset Q$.

It is easy to verify that Q is a reductant, i.e., $h^{-1}Qh \subset Q$, $\forall\ h \in H$. In fact, it is possible to represent the matrices from Q by means of a new parametrization:

$$\begin{pmatrix} \cosh t & \sinh t & 0 & 0 & [(\sinh(t/2))/(t/2)]([\cosh(t/2)]\alpha + [\sinh(t/2)]\beta) \\ \sinh t & \cosh t & 0 & 0 & [(\sinh(t/2))/(t/2)]([\sinh(t/2)]\alpha + [\cosh(t/2)]\beta) \\ 0 & 0 & \cosh t & -\sinh t & [(\sinh(t/2))/(t/2)](\cosh(t/2))\gamma \\ 0 & 0 & -\sinh t & \cosh t & [(\sinh(t/2))/(t/2)](-\sinh(t/2))\gamma \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \exp\begin{pmatrix} 0 & t & 0 & 0 & \alpha \\ t & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & -t & \gamma \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{e}^{q}.$$

It follows, that the one parametric subgroups of the form $\mathrm{e}^{\lambda q}$ $(\lambda \in \mathbb{R})$ belong to Q if $q \in T_e(Q) = \mathfrak{m}$, where the elements of $\mathfrak{m}$ are of the form

$$\begin{pmatrix} 0 & t & 0 & 0 & \alpha \\ t & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & -t & \gamma \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us note, that in our case $\mathfrak{h}$ consists of matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$ is a reductive decomposition, i.e., $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

It is a routine calculation to verify that $[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] \subset \mathfrak{m}$.

Let us find now the set $(S + \mathrm{id})\mathfrak{m}$, where $S = (s)_{*,e}$.

$$S\begin{pmatrix} 0 & t & 0 & 0 & \alpha \\ t & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & -t & \gamma \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 & 0 & \alpha \\ t & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & -t & \gamma \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -t & 0 & 0 & -\beta \\ -t & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & t & -\gamma \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 & 0 & \alpha \\ t & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & -t & \gamma \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha-\beta \\ 0 & 0 & 0 & 0 & \alpha+\beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally we verify that $[\mathfrak{m}, (S+\mathrm{id})\mathfrak{m}] \subset \mathfrak{m}$.

Thus we have constructed an example of the generalized Bruck loop realized as a section on the homogeneous space G/H.

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