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Research Article

Some Spectra of Superposition Operators Generated by an Exponential Function

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Abstract. In the present paper we consider the nonlinear superposition operator F in Banach spaces of sequences l_p $(1 \le p \le \infty)$, generated by the function $f(s,u) = d(s) + a^{ku} - 1$, with $a > 1$ and $k \in \mathbb{R} \setminus \{0\}$. We find out the Rhodius spectra $\sigma_R(F)$ and the Neuberger spectra $\sigma_N(F)$ of these operators, depending on the values of k.

Keywords. Superposition operator; Rhodius spectrum; Neuberger spectrum; Fréchet derivative

MSC. 47J10; 47H30

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1. Introduction and Preliminaries

In this paper we consider the class of nonlinear superposition operators generated by an exponential function in Banach spaces of sequences l_p $(1 \le p \le \infty)$. These spaces of sequences are equipped with the standard norm:

$$\|x\| = \begin{cases} \left(\sum\limits_{s \in \mathbb{N}} |x(s)|^p \right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \sup\limits_{s \in \mathbb{N}} |x(s)| & \text{if } p = \infty. \end{cases} \tag{1}$$

The exponential function arises in various mathematical models and dynamical systems where we have rapid growth or decay, such as model of population growth, growth of investment assets and various epidemic models (as well the models of rapidly spreading the coronavirus disease 2019 (COVID-19)). The superposition operator, also known as Nemytskii operator or composition operator, is generated by the function $f = f(s,u)$ defined on $\mathbb{N}\times\mathbb{R}$ with the values in $\mathbb{R}$. For $x = x(s) \in l_p$, by applying f we get the function $f(s,x(s))$ and the superposition operator F:

$$Fx(s) = f(s,x(s)). \tag{2}$$

There are several different ways of defining the term spectrum for nonlinear operators and corresponding nonlinear spectral theories ([1], [3], [8], [9]). We consider the Rhodius and Neuberger spectrum of nonlinear operators and they both contain the set of all eigenvalues of the operator (point spectrum). For the class $\mathfrak{C}(l_p)$ of all continuous operators F on Banach space l_p over $\mathbb{R}$, the Rhodius resolvent set ([9]) is given by:

$$\rho_R(F) = \{\lambda \in \mathbb{R} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{C}(l_p)\}$$

and the Rhodius spectrum is the set

$$\sigma_R(F) = \mathbb{R} \setminus \rho_R(F).$$

If F is Fréchet differentiable at each point $x \in X$ and the map $x \mapsto F'(x)$ is continuous, we write $F \in \mathfrak{C}^1(X,Y)$ and call F continuously Fréchet differentiable ([1], [2]).

The Neuberger resolvent set for the class of continuously Fréchet differentiable operators $F : X \to X$ is defined by

$$\rho_N(F) = \{\lambda \in \mathbb{R} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{C}^1(l_p)\}$$

and the set

$$\sigma_N(F) = \mathbb{R} \setminus \rho_N(F)$$

is called Neuberger spectrum of F ([8]).

These spectra may be useful in solvability of certain operator equations and eigenvalue problems ([8]). The Rhodius and Neuberger spectrum of some nonlinear superposition operators may be found in [1], [5], [6], [7]. The conditions of acting, continuity and differentiability of the superposition operator defined in the spaces of sequences l_p are given in the following theorems.

Theorem 1 ([4])**.** *Let $1 \le p,q < \infty$. Then the following properties are equivalent:*

- *the operator F acts from l_p to l_q;*
- *there are functions $a(s) \in l_q$ and constants $\delta > 0$, $n \in \mathbb{N}$, $b \ge 0$, for which*

$$|f(s,u)| \le a(s) + b|u|^{\frac{p}{q}} \quad (s \ge n, |u| < \delta);$$

- *for any $\varepsilon > 0$ there exists a function $a_\varepsilon \in l_q$ and constants $\delta_\varepsilon > 0$, $n_\varepsilon \in \mathbb{N}$, $b_\varepsilon \ge 0$, for which $\|a_\varepsilon(s)\|_q < \varepsilon$ and*

$$|f(s,u)| \le a_\varepsilon(s) + b_\varepsilon|u|^{\frac{p}{q}} \quad (s \ge n_\varepsilon, |u| \le \delta_\varepsilon).$$

Theorem 2 ([4])**.** *Let $1 \le p,q < \infty$ and let the superposition operator* (1)*, generated by the function $f(s,u)$, acts from l_p to l_q. Then this operator is continuous if and only if each of the functions is continuous for every $s \in \mathbb{N}$.*

Theorem 3 ([4]). *Let $1 \le p,\ q < \infty$ and the operator F generated by the function $f(s,u)$ acts from l_p into l_q. The operator F is differentiable at $x_0 \in l_p$ if and only if $f'_u(s,\cdot)$ is continuous at x_0 for almost all $s \in \mathbb{N}$.*

Theorem 4 ([4]). *Let $f(s,u)$ be a Carathéodory function and operator F generated by the function $f(s,u)$ acts from l_p to l_q. If operator F is differentiable in $x_0 \in l_p$, then its (Fréchet) derivative in x_0 has the form*

$$F'(x_0)h(s) = a(s)h(s), \tag{3}$$

where $a \in l_q/l_p$ is given by

$$a(s) = \lim_{u\to 0} \frac{f(s,x_0(s)+u) - f(s,x_0(s))}{u}. \tag{4}$$

If superposition operator G, generated by the function

$$g(s,u) = \begin{cases} \frac{1}{u}[f(s,x(s)+u) - f(s,x(s))]; & u \neq 0, \\ a(s); & u = 0, \end{cases}$$

acts from l_p to l_q/l_p, and it is continuous in 0, then F is differentiable in x_0 and it values (3).

The space l_q/l_p is the set of all multipliers $(a(s))$ from l_p to l_q. It is a Banach space of sequences, defined by

$$l_q/l_p = \begin{cases} l_{pq(p-q)^{-1}} & \text{for } p > q, \\ l_\infty & \text{for } p \le q. \end{cases} \tag{5}$$

2. Main Results

Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = d(s) + a^{ku} - 1$, with $a > 0$, $k \in \mathbb{R} \setminus \{0\}$, where $d = (d(s))_{s\in\mathbb{N}}$ is a sequence from the space l_q $(1 \le q \le p \le \infty)$. We show that this operator acts from the space l_p to the space l_p.

(a) Case $1 \le p < \infty$

$$|f(s,u)| = |d(s) + a^{ku} - 1| \le |d(s)| + |a^{ku} - 1|. \tag{6}$$

For $|u| < 1$ it is not hard to see that

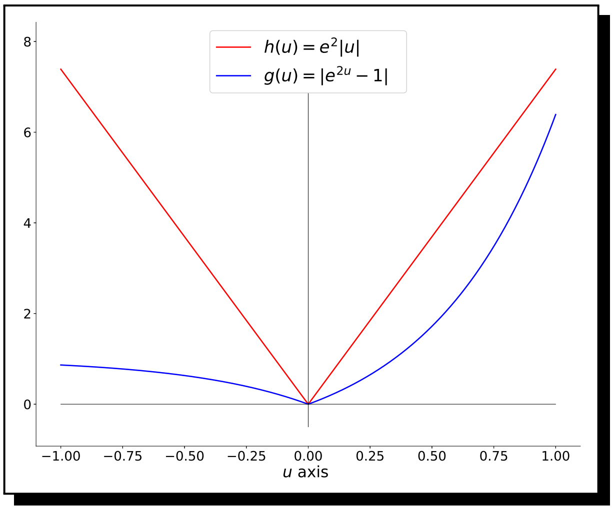
$$|a^{ku} - 1| \le a^{|k|} \cdot |u|. \tag{7}$$

Hence, from (6) and (7), we get

$$|f(s,u)| \le |d(s)| + a^{|k|} \cdot |u|. \tag{8}$$

Since $d \in l_q$ and $l_q \subseteq l_p$ we have that $(|d(s)|)_{s\in\mathbb{N}} \in l_p$. Now, we can see there exists constants $\delta = 1$, $n = 1$, $b = a^{|k|}$ from the Theorem 1 such that $\forall\ s \ge n$, $|u| < \delta$, inequality (8) holds. Hence, it follows that $F : l_p \to l_p$.

In Figure 1 we illustrate the inequality (7) with the functions $g(u) = |e^{ku} - 1|$ and $h(u) = e^{|k|} \cdot |u|$ for $k = 2 > 0$ (on the left side) and $k = -2 < 0$ (on the right side). In both cases the graph of $g(u)$ is under the graph of $h(u)$ for $|u| < 1$.



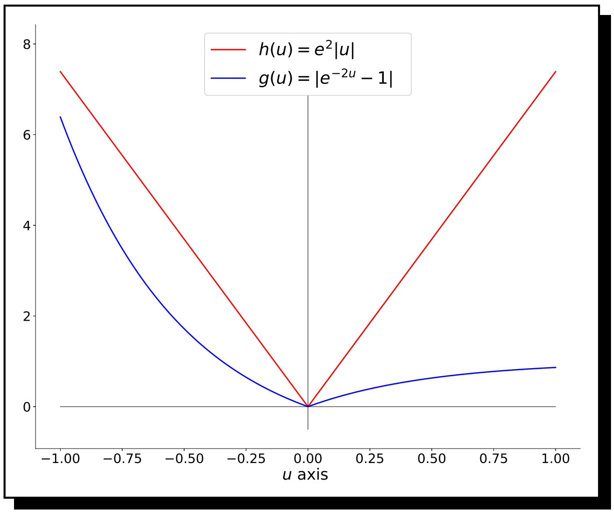


Figure 1

(b) Case $p = l_\infty$

$$d \in l_q \subseteq l_\infty \Rightarrow \exists \sup_{s\in\mathbb{N}} |d(s)| = A < \infty$$

and

$$x \in l_\infty \Rightarrow \exists \sup_{s\in\mathbb{N}} |x(s)| = B < \infty.$$

For arbitrary $x = (x(1), x(2), \cdots) \in l_\infty$ we have

$$\sup_{s\in\mathbb{N}} |Fx(s)| = \sup_{s\in\mathbb{N}} |d(s) + a^{kx(s)} - 1| \leq \sup_{s\in\mathbb{N}} |d(s)| + \sup_{s\in\mathbb{N}} a^{kx(s)} + 1.$$

Since $\sup a^{kx(s)} \leq a^{\sup|kx(s)|} = a^{|k|\sup|x(s)|}$, it follows

$$\sup_{s\in\mathbb{N}} |Fx(s)| \leq A + a^{|k|B} + 1 < \infty.$$

We see that for every $x \in l_\infty$ it holds $Fx \in l_\infty$ and hence, the operator F acts from l_∞ to l_∞.

For every $s \in \mathbb{N}$ the function $f(s,u) = d(s) + a^{ku} - 1$ is continuous, so from the Theorem 2 we have that the operator F is continuous.

Theorem 5. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = d(s) + a^{ku} - 1$, with $a > 1$, $k \in \mathbb{R} \setminus \{0\}$, where $(d(s))_s$ is a sequence from the space l_q $(1 \leq q \leq p \leq \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = [0, +\infty)$ for $k > 0$ and $\sigma_R(F) = (-\infty, 0]$ for $k < 0$.*

Proof. For given $d = (d(1), d(2), \ldots) \in l_q$ and arbitrary $x = (x(1), x(2), \ldots) \in l_p$, we have $Fx = F(x(1), x(2), \ldots) = (d(1) + a^{kx(1)} - 1, d(2) + a^{kx(2)} - 1, \ldots)$.

The operator $\lambda I - F$ for $\lambda = 0$ becomes $-F$ and

$$-Fx = (-d(1) - a^{kx(1)} + 1, -d(2) - a^{kx(2)} + 1, \ldots). \tag{9}$$

From $-Fx = -Fy$ $(x, y \in l_p)$, we have

$$(-d(1) - a^{kx(1)} + 1, -d(2) - a^{kx(2)} + 1, \ldots) = (-d(1) - a^{ky(1)} + 1, -d(2) - a^{ky(2)} + 1, \ldots)$$

$$-d(s) - a^{kx(s)} + 1 = -d(s) - a^{ky(s)} + 1, \quad \forall\, s \in \mathbb{N}$$

$$a^{kx(s)} = a^{ky(s)}, \quad \forall\ s \in \mathbb{N}. \tag{10}$$

The function $f(x) = a^{kx}$, $x \in \mathbb{R}$, $k \in \mathbb{R} \setminus \{0\}$, $a > 1$ is strictly monotonous and injective, so from (10) we get $x(s) = y(s)$, $\forall\ s \in \mathbb{N}$, i.e. $x = y$. That is why this operator (9) is injective. This is not a surjective operator since $-d(s) - a^{kx(s)} + 1 \in (-\infty, -d(s)+1)$, $(s \in \mathbb{N})$ and d is a bounded sequence. Hence, the operator (9) is not bijection and

$$0 \in \sigma_R(F). \tag{11}$$

For $\lambda \neq 0$ we have the operator

$$(\lambda I - F)x = (\lambda x(1) - d(1) - a^{kx(1)} + 1, \lambda x(2) - d(2) - a^{kx(2)} + 1, \ldots). \tag{12}$$

From $(\lambda I - F)x = (\lambda I - F)y$ $(x, y \in l_p)$, we get

$$\begin{aligned} &(\lambda x(1) - d(1) - a^{kx(1)} + 1, \lambda x(2) - d(2) - a^{kx(2)} + 1, \ldots) \\ &\qquad = (\lambda y(1) - d(1) - a^{ky(1)} + 1, \lambda y(2) - d(2) - a^{ky(2)} + 1, \ldots) \\ &\lambda x(s) - d(s) - a^{kx(s)} + 1 = \lambda y(s) - d(s) - a^{ky(s)} + 1, \quad \forall / s \in \mathbb{N} \\ \Leftrightarrow \quad &\lambda x(s) - a^{kx(s)} = \lambda y(s) - a^{ky(s)}, \quad \forall / s \in \mathbb{N}. \end{aligned}$$

Now, we need to find is the function

$$f(x) = \lambda x - a^{kx} \tag{13}$$

injective or not.

(1) Let $k > 0$. From $f(x) = f(y)$ it follows:

$$\lambda x - a^{kx} = \lambda y - a^{ky} \Rightarrow \lambda(x - y) = a^{kx} - a^{ky}.$$

Then, for $x \neq y$ we have $\lambda = \frac{a^{kx} - a^{ky}}{x - y} > 0$. It means for $\lambda > 0$ the function $f(x)$ is not injection. (Because from $f(x) = f(y)$ it does not follow $x = y$ when $\lambda > 0$).

We find

$$(0, +\infty) \subset \sigma_R(F). \tag{14}$$

If $\lambda < 0$ then from $\lambda(x - y) = a^{kx} - a^{ky}$ it does follow $x = y$, so $f(x)$ is injective and operator $(\lambda I - F)$ is injective. The superposition operator $(\lambda I - F)$ is generated by the function

$$f(s, u) = \lambda u - d(s) - a^{ku} + 1. \tag{15}$$

We can consider the function (15) as the function of one variable u, where $d(s)$ is a real constant for fixed $s \in \mathbb{N}$. The first derivative of the function (15) with the respect of u is

$$f'_u(s, u) = \lambda - k a^{ku} \ln a. \tag{16}$$

For $\lambda < 0$ we see that $f'_u(s, u) < 0$, so f is a strictly decreasing function, also $u \to \pm\infty \Rightarrow f(s, u) \to \mp\infty$ holds. Hence f is bijective function for every $s \in \mathbb{N}$, so the operator $\lambda I - F$ is bijective for $\lambda < 0$.

Now, we have to find out if $(\lambda I - F)^{-1}$ is a continuous operator for $\lambda < 0$.

The function (15) is a bijective, decreasing and continuous function for $\lambda < 0$, so there exists its inverse $f^{-1}(s, u)$ which is also bijective, decreasing and continuous function (for every $s \in \mathbb{N}$) ([10]). Then, from the Theorem 2 follows that operator $(\lambda I - F)^{-1}$, generated by $f^{-1}(s, u)$ is a continuous operator and thus

$$(-\infty,0)\subset\rho_R(F). \tag{17}$$

From (11), (14) and (17) we have

$$[0,+\infty)=\sigma_R(F). \tag{18}$$

(2) Let $k<0$. Now, we have an opposite situation.

$$f(x)=f(y)\Rightarrow\lambda x-a^{kx}=\lambda y-a^{ky}\Rightarrow\lambda(x-y)=a^{kx}-a^{ky}.$$

For $x\neq y$ we have $\lambda=\frac{a^{kx}-a^{ky}}{x-y}<0$ (because a^{kx}, with $k<0$, is a decreasing function). Hence, for $\lambda<0$ the function $f(x)$ is not injection and

$$(-\infty,0)\subset\sigma_R(F). \tag{19}$$

If $\lambda>0$ then $f'(x)=\lambda-ka^x\ln a>0$, hence $f(x)$ is a strictly increasing function and it is injective, so the operator $(\lambda I-F)$ is injective.

For $\lambda>0$ we see that the first derivative of the (15) is positive, i.e. $f'_u(s,u)>0$, so f is a strictly increasing function. Further, $u\to\pm\infty\Rightarrow f(s,u)\to\pm\infty$ holds. Hence f is a bijective function for every $s\in\mathbb{N}$, so the operator $\lambda I-F$ is bijective for $\lambda>0$.

Now, we have to find out if $(\lambda I-F)^{-1}$ is a continuous operator for $\lambda>0$.

The function (15) is a bijective, increasing and continuous function for $\lambda>0$, so there exists its inverse $f^{-1}(s,u)$ which is also bijective, increasing and continuous function(for every $s\in\mathbb{N}$) ([10]). Then, from the Theorem 2 follows that operator $(\lambda I-F)^{-1}$, generated by $f^{-1}(s,u)$ is a continuous operator and thus

$$(0,+\infty)\subset\rho_R(F). \tag{20}$$

From (11), (19) and (20) we have

$$(-\infty,0]=\sigma_R(F). \tag{21}$$

□

The generating function of our superposition operator $F:l_p\to l_p$ is $f(s,u)=d(s)+a^{ku}-1$ and its first derivative with respect to u is

$$f'_u(s,u)=ka^{ku}\ln a. \tag{22}$$

This function (22) is continuous for all $s\in\mathbb{N}$ and at all $x_0\in l_p$, so according to the Theorem 3, the corresponding operator F is Fréchet differentiable at each point $x_0\in l_p$. From (3) and (4) we see that the Fréchet derivative of our operator F at $x_0=(x(1),x(2),\dots)$ along $h=(h(1),h(2),\dots)$ is a linear multiplication operator given with

$$F'(x_0)h=\left(ka^{kx(1)}\ln a\cdot h(1),ka^{kx(2)}\ln a\cdot h(2),ka^{kx(3)}\ln a\cdot h(3),\dots\right). \tag{23}$$

The multiplier is $m=\left(ka^{kx(1)}\ln a,ka^{kx(2)}\ln a,ka^{kx(3)}\ln a,\dots\right)$ and since $x\in l_p\subset l_\infty$, we have $m\in l_\infty=l_p/l_p$. The map $x\mapsto F'(x)$ is continuous, so the operator F is a continuously differentiable operator, i.e. $F\in\mathfrak{C}^1\left(l_p\right)$.

Theorem 6. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = d(s) + a^{ku} - 1$, with $a > 0$ and $k \in \mathbb{R} \setminus \{0\}$, where $(d(s))_s$ is a sequence from the space l_q $(1 \le q \le p \le \infty)$. Then the Neuberger spectrum of F is $\sigma_N(F) = [0,+\infty)$ for $k > 0$ and $\sigma_N(F) = (-\infty, 0]$ for $k < 0$.*

Proof. **(1)** Case $k > 0$.

From the previous proof of the Theorem 5 we know that the operator $\lambda I - F$ is not bijective for $\lambda \in [0,+\infty)$, so

$$[0,+\infty) \subseteq \sigma_N(F). \tag{24}$$

For $\lambda \in (-\infty, 0)$ the operator $\lambda I - F$ is bijective and we need to find out if $(\lambda I - F)^{-1}$ is a continuously differentiable operator (for $\lambda < 0$). The superposition operator $(\lambda I - F)$ is generated by the function (15) and its first derivative (with respect to u) is (16). For $\lambda < 0$ the function (16) is a continuous and negative function, so there exists $(f^{-1})'(u)$

$$(f^{-1})'(u) = \frac{1}{f'(u)} = \frac{1}{\lambda - k a^{ku} \ln a}. \tag{25}$$

The function (25) is a continuous function for every $s \in \mathbb{N}$, $\lambda < 0$ and hence the operator $(\lambda I - F)^{-1}$ is a continuously differentiable operator: $(\lambda I - F)^{-1} \in \mathfrak{C}^1(l_p)$. That is why

$$(-\infty, 0) \subseteq \rho_N(F). \tag{26}$$

Finally, from (24) and (26) it follows

$$\sigma_N(F) = [0,+\infty).$$

(2) Case $k < 0$

We have shown within the proof of the Theorem 5 that the operator $\lambda I - F$ is not bijective for $\lambda \in (-\infty, 0]$, so

$$(-\infty, 0] \subseteq \sigma_N(F). \tag{27}$$

For $\lambda \in (0,+\infty)$ the operator $\lambda I - F$ is bijective and we need to find out if $(\lambda I - F)^{-1}$ is a continuously differentiable operator.

For $\lambda > 0$ the function (16)(the first derivative of the generating function of operator $(\lambda I - F)$) is a continuous and positive function, so there exists $(f^{-1})'(u)$ (25). The function (25) is a continuous function for every $s \in \mathbb{N}$, $\lambda > 0$ and hence the operator $(\lambda I - F)^{-1}$ is a continuously differentiable operator: $(\lambda I - F)^{-1} \in \mathfrak{C}^1(l_p)$. That is why

$$(0,+\infty) \subseteq \rho_N(F). \tag{28}$$

Finally, from (27) and (28) it follows

$$\sigma_N(F) = (-\infty, 0].$$ □

For the superposition operators that we considered in this paper we can conclude:

Their Rhodius and Neuberger spectra are nonempty and unbounded sets and

$$\sigma_R(F) = \sigma_N(F) = \begin{cases} [0,+\infty), & \text{if } k > 0 \\ (-\infty, 0], & \text{if } k < 0. \end{cases}$$

If we have the generating function in more general form: $f(s,u) = d(s) + \phi(u)$ then we find out the following result about the spectra of its corresponding superposition operator F.

Proposition 1. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = d(s) + \phi(u)$, where $(d(s))_s$ is a sequence from the space l_q $(1 \le q \le p \le \infty)$ and ϕ is a continuous function. If the function $\phi(u)$ is not a bijection, then the Rhodius and Neuberger spectra of F contain zero ($0 \in \sigma_R(F)$ and $0 \in \sigma_N(F)$).*

Proof. The superposition operator F is continuous because ϕ is a continuous function and we may consider its Rhodius and Neuberger spectrum.

(a) If $\phi(u)$ is not an injective function, then for arbitrary $x = (x(1), x(2), \dots) \in l_p$ and $y = (y(1), y(2), \dots)$, from $-Fx = -Fy$ we get

$$\begin{aligned} & (d(1)+\phi(x(1)), d(2)+\phi(x(2)),\dots) = (d(1)+\phi(y(1)), d(2)+\phi(y(2)),\dots) \\ \Rightarrow\ & d(s)+\phi(x(s)) = d(s)+\phi(y(s)), \forall s \in \mathbb{N} \\ \Rightarrow\ & \phi(x(s)) = \phi(y(s)), \forall\ s \in \mathbb{N}. \end{aligned} \tag{29}$$

Since ϕ is not an injective function, from (29) it does not follow $x(s) = y(s)$, $\forall\ s \in \mathbb{N}$ and it means that $-F$ is not an injective operator. Hence $-F$ is not a bijective operator and that is why $0 \in \sigma_R(F)$ and $0 \in \sigma_N(F)$.

(b) If $\phi(u)$ is not a surjective function, then for arbitrary $x = (x(1), x(2), \dots) \in l_p$ we have $-Fx = (-d(1)-\phi(x(1)), -d(2)-\phi(x(2)), \dots)$. The sequence $(d(s))_s$ is bounded (because $d \in l_q \subset l_\infty$), so there exists $A < \infty$ such that $\forall\ s \in \mathbb{N}$, $|d(s)| < A < \infty$ and $\{-d(s) : s \in \mathbb{N}\} \subset (-A, A)$. Since ϕ is a continuous and not surjective function, the set $\{-\phi(u) : u \in \mathbb{R}\}$ has to be bounded at least from the one side. If it is bounded from above, then $(\exists\ B \in \mathbb{R})\{-\phi(u) : u \in \mathbb{R}\} \subset (-\infty, B)$. Now, we get $\{-d(s) - \phi(x(s) : s \in \mathbb{N}\} \subset (-\infty, B + A)$. If $\{-\phi(u) : u \in \mathbb{R}\}$ is bounded from below, then $(\exists C \in \mathbb{R})\{-\phi(u) : u \in \mathbb{R}\} \subset (C, +\infty)$ and $\{-d(s) - \phi(x(s) : s \in \mathbb{N}\} \subset (C - A, +\infty,)$. Therefore, we show that the operator $-F$ is not surjective and because of that, it follows that 0 belongs to these spectra ($0 \in \sigma_R(F)$ and $0 \in \sigma_N(F)$). □

3. Conclusion

In this paper we investigate some properties of the Rhodius and Neuberger spectra of the superposition operator F generated by the function $f(s,u) = d(s) + \phi(u)$, where ϕ is a continuous function. We find that if the function ϕ is not bijective, then these spectra contain zero. In case when ϕ is an exponential function, i.e. $\phi(u) = a^{ku} - 1$, $a > 1$, we find that these spectra are unbounded above for $k > 0$ or unbounded below for $k < 0$. This exponential function is not surjective, so it is not bijective for any real constant k, and we see that corresponding spectra contain zero. It means that Proposition 1 is verified on this example. In further work the scope is to investigate some other properties and find the relations between the generating function of the superposition operator F and its Rhodius and Neuberger spectra.

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Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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