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Research Article

Four New Sums of Second Hyper Zagreb Index Based on Cartesian Product

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Abstract. In this work, we study the second hyper Zagreb index of new operations of different subdivisions graphs related to Cartesian product of graphs.

Keywords. Topological indices; Graph operations

Mathematics Subject Classification (2020). 05C07; 05C12

1. Introduction

All graphs observed here are simple, connected and finite. Let $V(G)$, $E(G)$ and $d_G(w)$ indicate the vertex set, the edge set and the degree of a vertex of a graph G, respectively. A graph with p vertices and q edges is known as a (p,q) graph.

A topological index of a graph G is a real number which is invariant under automorphism of G and does not depend on the labeling or pictorial representation of a graph.

Gutman *et al.* [6] introduced the first and second Zagreb indices of a graph G as follows:

$$M_1(G) = \sum_{wz\in E(G)} (d_G(w)+d_G(z)) = \sum_{w\in V(G)} d_G^2(w)$$

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and

$$M_2(G) = \sum_{wz \in E(G)} d_G(w) d_G(z).$$

Shirdel *et al.* in [9] found Hyper-Zagreb index $HM(G)$ which is established as

$$HM(G) = \sum_{wz \in E(G)} [d_G(w) + d_G(z)]^2.$$

Also, they have computed the hyper-Zagreb index of the Cartesian product, composition, join and disjunction of graphs.

A forgotten topological index F-index [4] is defined for a graph G as

$$F(G) = \sum_{w \in V(G)} d_G^3(w) = \sum_{wz \in E(G)} [d_G^2(w) + d_G^2(z)].$$

Farahani *et al.* [3] defined the second hyper Zagerb as

$$HM(G) = \sum_{wz \in E(G)} [d_G(w) d_G(z)]^2.$$

Here we introduce a second forgotten topological index F_2 which is defined for a graph G as

$$F_2(G) = \sum_{w \in V(G)} d_G^4(w).$$

Kulli [6] introduced the first and second Gourava indices and defined as

$$GO_1(G) = \sum_{wz \in E(G)} (d_G(w) + d_G(z)) + (d_G(w) d_G(z))$$

and

$$GO_2(G) = \sum_{wz \in E(G)} d_G(w) d_G(z)(d_G(w) + d_G(z)).$$

The line graph $L(G)$ is the graph whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common.

The following are the four related graphs for a connected graph G.

$S(G)$ is the graph which is obtained from G by adding an extra vertex into each edge of G. In other words replaced each edge of G by a path of length 2.

The graph $R(G)$ is obtained from G by inserting an additional vertex into each edge of G and joining each additional vertex to the end vertices of the corresponding edge of G.

$Q(G)$ is a graph derived from G by adding a new vertex to each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

The total graph $T(G)$ is derived from G by inserting an new vertex to each edge of G, then joining each new vertex to the end vertices of the corresponding edge and joining with edges those pairs of new vertices on adjacent edges of G.

The Cartesian product of the graphs G_1 and G_2 is the graph $G_1 \square G_2$ with vertex set $V(G_1) \times V(G_2)$ and for which $(w_1, w_2)(z_1, z_2) \in G_1 \square G_2$ iff $w_1 = z_1$ and $w_2 z_2 \in E(G_2)$ or **(ii)** $w_2 = z_2$ and $w_1 z_1 \in E(G_1)$. It is easy to see that

$$d_{G_1 \square G_2}(w_i, z_j) = d_{G_1}(w_i) + d_{G_2}(z_j),$$

where $(w_i, z_j) \in V(G_1 \square G_2)$.

Eliasi and Taeri [2] introduced the four operations of the graphs G_1 and G_2 based on the Cartesian product of these graphs. The Zagreb indices of the our new sums of graphs are obtained by Deng *et al.* [1]. The F-index of four operations on some special graphs are computed by Ghobadi and Ghorbaninejad [5]. Eliasi and Taeri [2] have obtained the Wiener index of four new sums of graphs.

Sarala *et al.* [7] introduced the four operations of the graphs G_1 and G_2 based on the composition of these graphs.

In this sequence, we calculate the four new sums of second hyper Zagreb index based on cartesian product of graphs.

2. Main Results

In this section, we find the exact value of the second hyper Zagreb index of Cartesian product of graphs.

Theorem 2.1. *Let G_i, $i = 1,2$ be a (p_i, q_i) graph. Then*

$$\begin{aligned} HM_2(G_1 \square_S G_2) &= M_1(G_1)[16q_2 + 2M_2(G_2) + HM(G_2)] + 8q_1M_1(G_2) \\ &\quad + F(G_1)[4p_2 + 2M_1(G_2)] + q_2F_2(G_1) + p_1HM_2(G_2) + 4q_1GO_2(G_2). \end{aligned}$$

Proof.

$$\begin{aligned} HM_2(G_1 \square_S G_2) &= \sum_{(w,k)(z,l)\in E(G_1\square_S G_2)} [d_{G_1\square_S G_2}(w,k)d_{G_1\square_S G_2}(z,l)]^2 \\ &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d_{G_1\square_S G_2}(w,k)d_{G_1\square_S G_2}(w,l)]^2 \\ &\quad + \sum_{k\in V(G_2)}\sum_{wz\in E(S(G_1))} [d_{G_1\square_S G_2}(w,k)d_{G_1\square_S G_2}(z,k)]^2 \\ &= A_1 + A_2, \end{aligned}$$

where A_1 and A_2 are the terms of the above sums taken in order which are calculated as follows.

$$\begin{aligned} A_1 &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d_{G_1\square_S G_2}(w,k)d_{G_1\square_S G_2}(w,l)]^2 \\ &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} \big[[d_{G_1}(w)+d_{G_2}(k)][d_{G_1}(w)+d_{G_2}(l)]\big]^2 \\ &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d^2_{G_1}(w)+d_{G_1}(w)[d_{G_2}(k)+d_{G_2}(l)]+d_{G_2}(k)d_{G_2}(l)]^2 \\ &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d^4_{G_1}(w)+d^2_{G_1}(w)[d_{G_2}(w)+d_{G_2}(l)]^2+d^2_{G_2}(k)d^2_{G_2}(l) \\ &\quad + 2d^3_{G_1}(w)[d_{G_2}(k)+d_{G_2}(l)]+2d^2_{G_1}(w)d_{G_2}(k)d_{G_2}(l) \\ &\quad + 2d_{G_1}(w)[d^2_{G_2}(k)d_{G_2}(l)+d_{G_2}(k)d^2_{G_2}(l)]] \\ &= q_2F_2(G_1) + M_1(G_1)HM(G_2) + p_1HM_2(G_2) + 2F(G_1)M_1(G_2) \\ &\quad + 2M_1(G_1)M_2(G_2) + 4q_1GO_2(G_2), \\ A_2 &= \sum_{k\in V(G_2)}\sum_{wz\in E(S(G_1))} [d_{G_1\square_S G_2}(w,k)d_{G_1\square_S G_2}(z,k)]^2 \\ &= \sum_{k\in V(G_2)}\sum_{wz\in E(S(G_1))w\in V(G_1),z\in V(S(G_1))\setminus V(G_1)} [(d_{G_1}(w)+d_{G_2}(k))2]^2 \end{aligned}$$

$$= 4\sum_{k\in V(G_2)}\sum_{w\in V(G_1)} d_{G_1}(w)[d_{G_1}(w)+d_{G_2}(k)]^2$$
$$= 4\sum_{k\in V(G_2)}\sum_{w\in V(G_1)} [d^3_{G_1}(w)+d_{G_1}(w)d^2_{G_2}(k)+2d^2_{G_1}(w)d_{G_1}(k)]$$
$$= 4p_2F(G_1)+8q_1M_1(G_2)+16q_2M_1(G_1)$$

Adding A_1 and A_2 we get

$$\begin{aligned} HM_2(G_1\square_S G_2) &= M_1(G_1)[16q_2+2M_2(G_2)+HM(G_2)]+8q_1M_1(G_2)\\ &\quad +F(G_1)[4p_2+2M_1(G_2)]+q_2F_2(G_1)+p_1HM_2(G_2)\\ &\quad +4q_1GO_2(G_2). \end{aligned}$$ □

Theorem 2.2. *Let G_i, $i=1,2$ be a (p_i,q_i) graph. Then*

$$\begin{aligned} HM_2(G_1\square_R G_2) &= 8q_1GO_2(G_2)+32q_2GO_2(G_1)+p_1HM_2(G_2)+16p_2HM_2(G_1)\\ &\quad +q_1F_2(G_2)+16q_2F_2(G_1)+16p_2F(G_1)\\ &\quad +(4HM(G_1)+16F(G_1)+8M_2(G_1)+8q_1)M_1(G_2)\\ &\quad +(4HM(G_2)+4F(G_2)+8M_2(G_2)+32q_2)M_1(G_1). \end{aligned}$$

Proof.

$$\begin{aligned} HM_2(G_1\square_R G_2) &= \sum_{(w,k)(z,l)\in E(G_1\square_R G_2)} [d_{G_1\square_R G_2}(w,k)d_{G_1\square_R G_2}(z,l)]^2\\ &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d_{G_1\square_R G_2}(w,k)d_{G_1\square_R G_2}(w,l)]^2\\ &\quad + \sum_{k\in V(G_2)}\sum_{wz\in E(R(G_1)),w,z\in V(G_1)} [d_{G_1\square_R G_2}(w,k)d_{G_1\square_R G_2}(z,k)]^2\\ &\quad + \sum_{k\in V(G_2)}\sum_{wz\in E(R(G_1)),w\in V(G_1),z\in V(R(G_1))\setminus V(G_1)} [d_{G_1\square_R G_2}(w,k)d_{G_1\square_R G_2}(z,k)]^2\\ &= B_1+B_2+B_3, \end{aligned}$$

where B_1, B_2 and B_3 are the terms of the above sums taken in order which are calculated as follows.

$$\begin{aligned} B_1 &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d_{G_1\square_R G_2}(w,k)d_{G_1\square_R G_2}(w,l)]^2\\ &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} \big[[d_{R(G_1)}(w)+d_{G_2}(k)][d_{R(G_1)}(w)+d_{G_2}(l)]\big]^2\\ &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} \big[[2d_{G_1}(w)+d_{G_2}(k)][2d_{G_1}(w)+d_{G_2}(l)]\big]^2\\ &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [4d^2_{G_1}(w)+2d_{G_1}(w)[d_{G_2}(k)+d_{G_2}(l)]+d_{G_2}(k)d_{G_2}(l)]^2\\ &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} \big[16d^4_{G_1}(w)+4d^2_{G_1}(w)[d_{G_2}(w)+d_{G_2}(l)]^2+d^2_{G_2}(k)d^2_{G_2}(l)\\ &\quad +16d^3_{G_1}(w)[d_{G_2}(k)+d_{G_2}(l)]+8d^2_{G_1}(w)d_{G_2}(k)d_{G_2}(l)\\ &\quad +4d_{G_1}(w)[d^2_{G_2}(k)d_{G_2}(l)+d_{G_2}(k)d^2_{G_2}(l)]\big]\\ &= 16q_2F_2(G_1)+p_1HM_2(G_2)+4M_1(G_1)HM(G_2)+8M_1(G_1)M_2(G_2) \end{aligned}$$

$$+16F(G_1)M_1(G_2)+8q_1GO_2(G_2),$$

$$B_2=\sum_{k\in V(G_2)}\sum_{wz\in E(R(G_1))w,z\in V(G_1)}[d_{G_1\square_R G_2}(w,k)d_{G_1\square_R G_2}(z,k)]^2$$

$$=\sum_{k\in V(G_2)}\sum_{wz\in E(G_1)}\left[[2d_{G_1}(w)+d_{G_2}(k)][2d_{G_1}(z)+d_{G_2}(k)]\right]^2$$

$$=\sum_{k\in V(G_2)}\sum_{wz\in E(G_1)}\left[4d_{G_1}(w)d_{G_1}(z)+2d_{G_2}(k)(2d_{G_1}(w)+d_{G_2}(k))+d^2_{G_2}(k)\right]^2$$

$$=\sum_{k\in V(G_2)}\sum_{w\in V(G_1)}[16d^2_{G_1}(w)d^2_{G_1}(z)+4d^2_{G_2}(k)(d_{G_1}(w)+d_{G_1}(z))^2+d^4_{G_2}(k)$$

$$+8d_{G_1}(w)d_{G_1}(z)d^2_{G_2}(k)+16d_{G_2}(k)d_{G_1}(w)d_{G_1}(z)(d_{G_1}(w)+d_{G_1}(z))$$

$$+4d^3_{G_1}(k)(d_{G_1}(w)+d_{G_1}(z))$$

$$=16p_2HM_2(G_1)+4M_1(G_2)HM(G_1)+q_1F_2(G_2)+8M_1(G_2)M_2(G_1)$$

$$+32q_2GO_2(G_1)+4F(G_2)M_1(G_1),$$

$$B_3=\sum_{k\in V(G_2)}\sum_{wz\in E(R(G_1)),w\in V(G_1),z\in V(R(G_1))\setminus V(G_1)}[d_{G_1\square_R G_2}(w,k)d_{G_1\square_R G_2}(z,k)]^2$$

$$=\sum_{k\in V(G_2)}\sum_{wz\in E(R(G_1)),w\in V(G_1),z\in V(R(G_1))\setminus V(G_1}\left[[d_{R(G_1)}(w)+d_{G_2}(k)][d_{R(G_1)}(z)]\right]^2$$

$$=\sum_{k\in V(G_2)}\sum_{wz\in E(R(G_1)),w\in V(G_1),z\in V(R(G_1))\setminus V(G_1}\left[[2d_{G_1}(w)+d_{G_2}(k)]2\right]^2$$

$$=4\sum_{k\in V(G_2)}\sum_{w\in V(G_1)}d_{G_1}(w)[2d_{G_1}(w)+d_{G_2}(k)]^2$$

$$=4\sum_{k\in V(G_2)}\sum_{w\in V(G_1)}[4d^3_{G_1}(w)+d_{G_1}(w)d^2_{G_2}(k)+4d^2_{G_1}(w)d_{G_1}(k)]$$

$$=4[4p_2F(G_1)+2q_1M_1(G_2)+8q_2M_1(G_1)].$$

Adding B_1, B_2 and B_3 we get

$$HM_2(G_1\square_R G_2)=8q_1GO_2(G_2)+32q_2GO_2(G_1)+p_1HM_2(G_2)+16p_2HM_2(G_1)$$
$$+q_1F_2(G_2)+16q_2F_2(G_1)+16p_2F(G_1)$$
$$+(4HM(G_1)+16F(G_1)+8M_2(G_1)+8q_1)M_1(G_2)$$
$$+(4HM(G_2)+4F(G_2)+8M_2(G_2)+32q_2)M_1(G_1).$$

□

Theorem 2.3. *Let G_i, $i=1,2$ be a (p_i,q_i) graph. Then*

$$HM_2(G_1\square_Q G_2)=q_2F_2(G_1)+M_1(G_1)HM(G_2)+p_1HM_2(G_2)+2F(G_1)M_1(G_2)$$
$$+2M_1(G_1)M_2(G_2)+4q_1GO_2(G_2)$$
$$+p_2\sum_{w\in V(G_1)}\sum_{z\in N_{G_1}(w)}d^2_{G_1}(w)d^2_{Q(G_1)}(z)+2HM(G_1)M_1(G_2)$$
$$+4q_2\sum_{w\in V(G_1)}\sum_{z\in N_{G_1}(w)}d_{G_1}(w)d^2_{Q(G_1)}(z)$$
$$p_2[HM_2(L(G_1))+4HM(L(G_1))+16\Big(\frac{M_I(G_1)}{2}-q_1\Big)+4GO_2(L(G_1))$$
$$+8M_2(L(G_1))+16M_1(L(G_1))].$$

Proof.

$$\begin{aligned}
HM_2(G_1\Box_Q G_2) &= \sum_{(w,k)(z,l)\in E(G_1\Box_Q G_2)} [d_{G_1\Box_Q G_2}(w,k)d_{G_1\Box_Q G_2}(z,l)]^2 \\
&= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d_{G_1\Box_Q G_2}(w,k)d_{G_1\Box_Q G_2}(w,l)]^2 \\
&\quad + \sum_{k\in V(G_2)}\sum_{wz\in E(Q(G_1)),w\in V(G_1),z\in V(Q(G_1))\setminus V(G_1)} [d_{G_1\Box_Q G_2}(w,k)d_{G_1\Box_Q G_2}(z,k)]^2 \\
&\quad + \sum_{k\in V(G_2)}\sum_{wz\in E(Q(G_1)),w,z\in V(Q(G_1))\setminus V(G_1)} [d_{G_1\Box_Q G_2}(w,k)d_{G_1\Box_Q G_2}(z,k)]^2 \\
&= C_1+C_2+C_3,
\end{aligned}$$

where C_1, C_2 and C_3 are the terms of the above sums taken in order which are calculated as follows.

$$\begin{aligned}
C_1 &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d_{G_1\Box_Q G_2}(w,k)d_{G_1\Box_Q G_2}(w,l)]^2 \\
&= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} \big[[d_{G_1}(w)+d_{G_2}(k)][d_{G_1}(w)+d_{G_2}(l)]\big]^2 \\
&= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d^2_{G_1}(w)+d_{G_1}(w)[d_{G_2}(k)+d_{G_2}(l)]+d_{G_2}(k)d_{G_2}(l)]^2 \\
&= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d^4_{G_1}(w)+d^2_{G_1}(w)[d_{G_2}(w)+d_{G_2}(l)]^2+d^2_{G_2}(k)d^2_{G_2}(l) \\
&\quad +2d^3_{G_1}(w)[d_{G_2}(k)+d_{G_2}(l)]+2d^2_{G_1}(w)d_{G_2}(k)d_{G_2}(l) \\
&\quad +2d_{G_1}(w)[d^2_{G_2}(k)d_{G_2}(l)+d_{G_2}(k)d^2_{G_2}(l)]] \\
&= q_2F_2(G_1)+M_1(G_1)HM(G_2)+p_1HM_2(G_2)+2F(G_1)M_1(G_2) \\
&\quad +2M_1(G_1)M_2(G_2)+4q_1GO_2(G_2), \\
C_2 &= \sum_{k\in V(G_2)}\sum_{wz\in E(Q(G_1)),w\in V(G_1),z\in V(Q(G_1))\setminus V(G_1)} [d_{G_1\Box_Q G_2}(w,k)d_{G_1\Box_Q G_2}(z,k)]^2 \\
&= \sum_{k\in V(G_2)}\sum_{wz\in E(Q(G_1)),w\in V(G_1),z\in V(Q(G_1))\setminus V(G_1)} \big[[d_{G_1}(w)+d_{G_2}(k)]d_{Q(G_1)}(z)\big]^2 \\
&= \sum_{k\in V(G_2)}\sum_{wz\in E(Q(G_1)),w\in V(G_1),z\in V(Q(G_1))\setminus V(G_1)} \big[d^2_{G_1}(w)d^2_{Q(G_1)}(z)+d^2_{Q(G_1)}(z)d^2_{G_2}(k) \\
&\quad +2d_{G_1}(w)d^2_{Q(G_1)}(z)d_(G_2)(k)\big] \\
&= p_2\sum_{w\in V(G_1)}\sum_{z\in N_{G_1}(w)} d^2_{G_1}(w)d^2_{Q(G_1)}(z)+2HM(G_1)M_1(G_2) \\
&\quad +4q_2\sum_{w\in V(G_1)}\sum_{z\in N_{G_1}(w)} d_{G_1}(w)d^2_{Q(G_1)}(z), \\
C_3 &= \sum_{k\in V(G_2)}\sum_{wz\in E(Q(G_1)),w,z\in V(Q(G_1))\setminus V(G_1)} [d_{G_1\Box_Q G_2}(w,k)d_{G_1\Box_Q G_2}(z,k)]^2 \\
&= \sum_{k\in V(G_2)}\sum_{wz\in E(Q(G_1)),w,z\in V(Q(G_1))\setminus V(G_1)} [d_{Q(G_1)}(w)d_{Q(G_1)}(z)]^2 \\
&= p_2\sum_{wz\in E(Q(G_1)),w,z\in V(Q(G_1))\setminus V(G_1)} [d_{Q(G_1)}(w)d_{Q(G_1)}(z)]^2
\end{aligned}$$

$$= p_2 \sum_{t_it_j \in E(G_1), t_jt_k \in E(G_1)} \left[[d_{G_1}(t_i)+d_{G_1}(t_j)][d_{G_1}(t_j)+d_{G_1}(t_k)]\right]^2$$

$$= p_2 \sum_{Y_iY_j \in E(L(G_1))} \left[[d_{L(G_1)}(Y_i)+2][d_{L(G_1)}(Y_j)+2)]\right]^2$$

$$= p_2 \sum_{Y_iY_j \in E(L(G_1))} \left[d_{L(G_1)}(Y_i)d_{L(G_1)}(Y_j)+2(d_{L(G_1)}(Y_i)+d_{L(G_1)}(Y_j))+4)\right]^2$$

$$= p_2 \sum_{Y_iY_j \in E(L(G_1))} [d^2_{L(G_1)}(Y_i)d^2_{L(G_1)}(Y_j)+4(d_{L(G_1)}(Y_i)+d_{L(G_1)}(Y_j))^2+16$$
$$+4d_{L(G_1)}(Y_i)d_{L(G_1)}(Y_j)(d_{L(G_1)}(Y_i)+d_{L(G_1)}(Y_j))+8d_{L(G_1)}(Y_i)d_{L(G_1)}(Y_j)$$
$$+16(d_{L(G_1)}(Y_i)+d_{L(G_1)}(Y_j))]$$

$$= p_2[HM_2(L(G_1))+4HM(L(G_1))+16\left(\frac{M_I(G_1)}{2}-q_1\right)+4GO_2(L(G_1))$$
$$+8M_2(L(G_1))+16M_1(L(G_1))].$$

Adding C_1, C_2 and C_3 we get

$$HM_2(G_1\square_Q G_2) = q_2F_2(G_1)+M_1(G_1)HM(G_2)+p_1HM_2(G_2)+2F(G_1)M_1(G_2)$$
$$+2M_1(G_1)M_2(G_2)+4q_1GO_2(G_2)$$
$$+p_2\sum_{w\in V(G_1)}\sum_{z\in N_{G_1}(w)} d^2_{G_1}(w)d^2_{Q(G_1)}(z)+2HM(G_1)M_1(G_2)$$
$$+4q_2\sum_{w\in V(G_1)}\sum_{z\in N_{G_1}(w)} d_{G_1}(w)d^2_{Q(G_1)}(z)$$
$$= p_2[HM_2(L(G_1))+4HM(L(G_1))+16\left(\frac{M_I(G_1)}{2}-q_1\right)+4GO_2(L(G_1))$$
$$+8M_2(L(G_1))+16M_1(L(G_1))]. \qquad \square$$

Theorem 2.4. *Let G_i, $i=1,2$ be a (p_i,q_i) graph. Then*

$$HM_2(G_1\square_T G_2) = 8q_1GO_2(G_2)+32q_2GO_2(G_1)+p_1HM_2(G_2)+16p_2HM_2(G_1)$$
$$+q_1F_2(G_2)+16q_2F_2(G_1)+16p_2F(G_1)$$
$$+(4HM(G_1)+16F(G_1)+8M_2(G_1))M_1(G_2)$$
$$+(4HM(G_2)+4F(G_2)+8M_2(G_2))M_1(G_1)$$
$$= p_2\sum_{w\in V(G_1)}\sum_{z\in N_{G_1}(w)} d^2_{G_1}(w)d^2_{T(G_1)}(z)+2HM(G_1)M_1(G_2)$$
$$+4q_2\sum_{w\in V(G_1)}\sum_{z\in N_{G_1}(w)} d_{G_1}(w)d^2_{T(G_1)}(z)$$
$$= p_2[HM_2(L(G_1))+4HM(L(G_1))+16\left(\frac{M_I(G_1)}{2}-q_1\right)+4GO_2(L(G_1))$$
$$+8M_2(L(G_1))+16M_1(L(G_1))].$$

Proof.

$$HM_2(G_1\square_T G_2) = \sum_{(w,k)(z,l)\in E(G_1\square_T G_2)} [d_{G_1\square_T G_2}(w,k)d_{G_1\square_T G_2}(z,l)]^2$$

$$
\begin{aligned}
&= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d_{G_1\square_T G_2}(w,k)d_{G_1\square_T G_2}(w,l)]^2\\
&\quad + \sum_{k\in V(G_2)}\sum_{wz\in E(T(G_1)),w,z\in V(G_1)} [d_{G_1\square_T G_2}(w,k)d_{G_1\square_T G_2}(z,k)]^2\\
&\quad + \sum_{k\in V(G_2)}\sum_{wz\in E(T(G_1)),w\in V(G_1),z\in V(T(G_1))\setminus V(G_1)} [d_{G_1\square_T G_2}(w,k)d_{G_1\square_T G_2}(z,k)]^2\\
&\quad + \sum_{k\in V(G_2)}\sum_{wz\in E(Q(G_1)),w,z\in V(T(G_1))\setminus V(G_1)} [d_{G_1\square_T G_2}(w,k)d_{G_1\square_T G_2}(z,k)]^2\\
&= D_1+D_2+D_3+D_4,
\end{aligned}
$$

where D_1, D_2, D_3 and D_4 are the terms of the above sums taken in order which are calculated as follows.

$$
\begin{aligned}
D_1 &= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [d_{G_1\square_T G_2}(w,k)d_{G_1\square_T G_2}(w,l)]^2\\
&= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} \left[[d_{T(G_1)}(w)+d_{G_2}(k)][d_{T(G_1)}(w)+d_{G_2}(l)]\right]^2\\
&= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} \left[[2d_{G_1}(w)+d_{G_2}(k)][2d_{G_1}(w)+d_{G_2}(l)]\right]^2\\
&= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} [4d^2_{G_1}(w)+2d_{G_1}(w)[d_{G_2}(k)+d_{G_2}(l)]+d_{G_2}(k)d_{G_2}(l)]^2\\
&= \sum_{w\in V(G_1)}\sum_{kl\in E(G_2)} \big[16d^4_{G_1}(w)+4d^2_{G_1}(w)[d_{G_2}(w)+d_{G_2}(l)]^2+d^2_{G_2}(k)d^2_{G_2}(l)\\
&\quad +16d^3_{G_1}(w)[d_{G_2}(k)+d_{G_2}(l)]+8d^2_{G_1}(w)d_{G_2}(k)d_{G_2}(l)\\
&\quad +4d_{G_1}(w)[d^2_{G_2}(k)d_{G_2}(l)+d_{G_2}(k)d^2_{G_2}(l)]\big]\\
&= 16q_2F_2(G_1)+p_1HM_2(G_2)+4M_1(G_1)HM(G_2)+8M_1(G_1)M_2(G_2)\\
&\quad +16F(G_1)M_1(G_2)+8q_1GO_2(G_2),\\
D_2 &= \sum_{k\in V(G_2)}\sum_{wz\in E(T(G_1))w,z\in V(G_1)} [d_{G_1\square_T G_2}(w,k)d_{G_1\square_T G_2}(z,k)]^2\\
&= \sum_{k\in V(G_2)}\sum_{wz\in E(G_1)} \left[[2d_{G_1}(w)+d_{G_2}(k)][2d_{G_1}(z)+d_{G_2}(k)]\right]^2\\
&= \sum_{k\in V(G_2)}\sum_{wz\in E(G_1)} \left[4d_{G_1}(w)d_{G_1}(z)+2d_{G_2}(k)(2d_{G_1}(w)+d_{G_2}(k))+d^2_{G_2}(k)\right]^2\\
&= \sum_{k\in V(G_2)}\sum_{w\in V(G_1)} [16d^2_{G_1}(w)d^2_{G_1}(z)+4d^2_{G_2}(k)(d_{G_1}(w)+d_{G_1}(z))^2+d^4_{G_2}(k)\\
&\quad +8d_{G_1}(w)d_{G_1}(z)d^2_{G_2}(k)+16d_{G_2}(k)d_{G_1}(w)d_{G_1}(z)(d_{G_1}(w)+d_{G_1}(z))\\
&\quad +4d^3_{G_1}(k)(d_{G_1}(w)+d_{G_1}(z))\\
&= 16p_2HM_2(G_1)+4M_1(G_2)HM(G_1)+q_1F_2(G_2)+8M_1(G_2)M_2(G_1)\\
&\quad +32q_2GO_2(G_1)+4F(G_2)M_1(G_1),\\
D_3 &= \sum_{k\in V(G_2)}\sum_{wz\in E(T(G_1)),w\in V(G_1),z\in V(T(G_1))\setminus V(G_1)} [d_{G_1\square_T G_2}(w,k)d_{G_1\square_T G_2}(z,k)]^2\\
&= \sum_{k\in V(G_2)}\sum_{wz\in E(T(G_1)),w\in V(G_1),z\in V(T(G_1))\setminus V(G_1)} \left[[d_{G_1}(w)+d_{G_2}(k)]d_{Q(G_1)}(z)\right]^2
\end{aligned}
$$

$$= \sum_{k\in V(G_2)} \sum_{wz\in E(T(G_1)), w\in V(G_1), z\in V(T(G_1))\setminus V(G_1)} [d^2_{G_1}(w)d^2_{T(G_1)}(z) + d^2_{T(G_1)}(z)d^2_{G_2}(k)$$

$$+ 2d_{G_1}(w)d^2_{T(G_1)}(z)d_(G_2)(k)]$$

$$= p_2 \sum_{w\in V(G_1)} \sum_{z\in N_{G_1}(w)} d^2_{G_1}(w)d^2_{T(G_1)}(z) + 2HM_(G_1)M_1(G_2)$$

$$+ 4q_2 \sum_{w\in V(G_1)} \sum_{z\in N_{G_1}(w)} d_{G_1}(w)d^2_{T(G_1)}(z),$$

$$D_4 = \sum_{k\in V(G_2)} \sum_{wz\in E(T(G_1)), w,z\in V(T(G_1))\setminus V(G_1)} [d_{G_1\Box_T G_2}(w,k)d_{G_1\Box_T G_2}(z,k)]^2$$

$$= \sum_{k\in V(G_2)} \sum_{wz\in E(T(G_1)), w,z\in V(T(G_1))\setminus V(G_1)} [d_{T(G_1)}(w)d_{T(G_1)}(z)]^2$$

$$= p_2 \sum_{wz\in E(T(G_1)), w,z\in V(T(G_1))\setminus V(G_1)} [d_{T(G_1)}(w)d_{T(G_1)}(z)]^2$$

$$= p_2 \sum_{t_it_j\in E(G_1), t_jt_k\in E(G_1)} \left[[d_{G_1}(t_i)+d_{G_1}(t_j)][d_{G_1}(t_j)+d_{G_1}(t_k)]\right]^2$$

$$= p_2 \sum_{Y_iY_j\in E(L(G_1))} \left[[d_{L(G_1)}(Y_i)+2][d_{L(G_1)}(Y_j)+2)]\right]^2$$

$$= p_2 \sum_{Y_iY_j\in E(L(G_1))} \left[d_{L(G_1)}(Y_i)d_{L(G_1)}(Y_j) + 2(d_{L(G_1)}(Y_i)+d_{L(G_1)}(Y_j))+4)\right]^2$$

$$= p_2 \sum_{Y_iY_j\in E(L(G_1))} [d^2_{L(G_1)}(Y_i)d^2_{L(G_1)}(Y_j) + 4(d_{L(G_1)}(Y_i)+d_{L(G_1)}(Y_j))^2 + 16$$

$$+ 4d_{L(G_1)}(Y_i)d_{L(G_1)}(Y_j)(d_{L(G_1)}(Y_i)+d_{L(G_1)}(Y_j)) + 8d_{L(G_1)}(Y_i)d_{L(G_1)}(Y_j)$$

$$+ 16(d_{L(G_1)}(Y_i)+d_{L(G_1)}(Y_j))]$$

$$= p_2[HM_2(L(G_1)) + 4HM(L(G_1)) + 16\Big(\frac{M_I(G_1)}{2} - q_1\Big) + 4GO_2(L(G_1))$$

$$+ 8M_2(L(G_1)) + 16M_1(L(G_1))].$$

Adding D_1, D_2, D_3 and D_4 we get

$$HM_2(G_1\Box_T G_2) = 8q_1GO_2(G_2) + 32q_2GO_2(G_1) + p_1HM_2(G_2) + 16p_2HM_2(G_1)$$

$$+ q_1F_2(G_2) + 16q_2F_2(G_1) + 16p_2F(G_1)$$

$$+ (4HM(G_1) + 16F(G_1) + 8M_2(G_1))M_1(G_2)$$

$$+ (4HM(G_2) + 4F(G_2) + 8M_2(G_2))M_1(G_1)$$

$$= p_2 \sum_{w\in V(G_1)} \sum_{z\in N_{G_1}(w)} d^2_{G_1}(w)d^2_{T(G_1)}(z) + 2HM_(G_1)M_1(G_2)$$

$$+ 4q_2 \sum_{w\in V(G_1)} \sum_{z\in N_{G_1}(w)} d_{G_1}(w)d^2_{T(G_1)}(z)$$

$$= p_2[HM_2(L(G_1)) + 4HM(L(G_1)) + 16\Big(\frac{M_I(G_1)}{2} - q_1\Big) + 4GO_2(L(G_1))$$

$$+ 8M_2(L(G_1)) + 16M_1(L(G_1))]. \qquad \Box$$

3. Conclusion

In this paper, we have studied the second hyper Zagreb index of new four sums of Cartesian product of graphs. For further research, one can study the other topological indices of these new operations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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