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Research Article

Variational Analysis of an Electro-Elasto-Viscoplastic Contact Problem With Friction and Wear

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Abstract. We consider a dynamic contact problem with wear between two elastic-viscoplastic piezoelectric bodies. The contact is frictional and bilateral which results in the wear of contacting surface. The evolution of the wear function is described with Archard's law. We derive variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

Keywords. Electro-elasto-viscoplastic materials; Internal state variable; Normal compliance; Wear; Evolution equations; Fixed point

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1. Introduction

Scientific research in mechanics are articulated around two main components: one devoted to the laws of behavior and other boundary conditions imposed on the body. The boundary conditions reflect the binding of the body with the outside world. The piezoelectric effect is the apparition of electric charges on surfaces of particular crystals after deformation. Its reverse effect consists of the generation of stress and strain in crystals under the action of the electric field on the boundary. Materials undergoing piezoelectric effects are called piezoelectric materials; their study require techniques and results from electromagnetic theory and continuum mechanics. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore, there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. The contact between deformable bodies are very common in the industry and everyday life, contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal. Contact processes are accompanied by a number of phenomena among which the main one is the friction. Nevertheless, more is involved in contact than just friction. Indeed, during a contact process elastic or plastic deformations of the surface asperities may happen. Also, some or all of the following may take place: squeezing of oil or other fluids, breaking of the asperities' tips and production of debris, motion of the debris, formation or welding of junctions, creeping, fracture, etc. Moreover, frictional contact is associated with heat generation, material damage, wear and adhesion of contacting surfaces. As the contact process evolves, the contacting surfaces evolve too, via their wear. Wear is one of the process which reduce the lifetime of modern machine elements. It represents the untwated removal of materials from surfaces of contacting bodies in relative motion.

The aim of this paper is to make the coupling of an elastic-visco-plastic piezoelectric problem with internal state variable and a frictional contact problem with wear. Then the constitutive laws considered here are of the form:

$$\begin{aligned}
\boldsymbol{\sigma}^\kappa &= \mathcal{A}^\kappa \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\kappa) + \mathcal{G}^\kappa \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa) + (\mathcal{E}^\kappa)^* \nabla \psi^\kappa \\
&\quad + \int_0^t \mathcal{F}^\kappa \Big(\boldsymbol{\sigma}^\kappa(s) - \mathcal{A}^\kappa \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\kappa(s)) - (\mathcal{E}^\kappa)^* \nabla \psi^\kappa,\ \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa(s)), \beta^\kappa(s) \Big)\, ds, \qquad (1)\\
\dot{\beta}^\kappa &= \Theta^\kappa \big(\boldsymbol{\sigma}^\kappa - \mathcal{A}^\kappa \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\kappa) - (\mathcal{E}^\kappa)^* \nabla \psi^\kappa(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa), \beta^\kappa \big), \qquad (2)\\
\boldsymbol{D}^\kappa &= \mathcal{E}^\kappa \varepsilon(\boldsymbol{u}^\kappa) - \mathcal{B}^\kappa \nabla \psi^\kappa \qquad (3)
\end{aligned}$$

in which $\boldsymbol{u}^\kappa$, $\boldsymbol{\sigma}^\kappa$ represent, respectively, the displacement field and the stress field where the dot above denotes the derivative with respect to the time variable, $\boldsymbol{D}^\kappa$ represents the electric displacement field. Here $\mathcal{A}^\kappa$ and $\mathcal{G}^\kappa$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively. $\mathcal{F}^\kappa$ is a nonlinear constitutive function describing the viscoplastic behaviour of the material and depending on the internal state variable β^κ. Θ^κ is also a nonlinear constitutive function which depend on β^κ, and $\mathcal{G}^\kappa$ represents the elasticity operator. $E(\psi^\kappa) = -\nabla \psi^\kappa$ is the electric field, $\mathcal{E}^\kappa = (e_{ijk})$ represents the third order piezoelectric tensor, $(\mathcal{E}^\kappa)^*$ is its transpose and $\mathcal{B}^\kappa$ denotes the electric permittivity tensor. General models for elastic materials with piezoelectric effects can be found in [5, 14]. Dynamic contact problems are the topic of numerous papers, e.g. [1, 6, 9]. A quasistatic frictional contact problem with wear involving elastic-viscoplastic materials with damage and thermal effects can be found in [2]. Contact problems with friction or adhesion for electro-viscoelastic materials were studied in [11, 12]. A static frictional contact problem for electric-elastic materials was considered in [5, 7]. The normal compliance contact condition was first considered in [6] in the study of dynamic problems with linearly elastic and viscoelastic materials and then it was used in various references, see e.g. [4, 10]. This condition allows the interpenetration of the body's surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities. In this paper we consider a mathematical frictional contact between two elastic-visco-plastic piezoelectric bodies with internal state variable for rate-type materials of the form (1)–(3). The contact is frictional and bilateral which result in the wear of contacting surface.

This article is organized as follows. In Section 2 we describe the mathematical models for the frictional contact problem between two electro-elastic-viscoplastics bodies. The contact is modelled with normal compliance and wear. In Section 3 we list the assumption on the data and derive the variational formulation of the problem. In Section 4 we state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is based on arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments.

2. Problem Statement

We consider the following physical setting. Let us consider two electro-elastic-viscoplastics bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d$ $(d=2,3)$. For each domain Ω^κ, the boundary Γ^κ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^κ, Γ_2^κ and Γ_3^κ on one hand, and on two measurable parts Γ_a^κ and Γ_b^κ, on the other hand, such that $meas(\Gamma_1^\kappa)>0$, $meas(\Gamma_a^\kappa)>0$. Let $T>0$ and let $[0,T]$ be the time interval of interest. The Ω^κ body is submitted to $\boldsymbol{f}_0^\kappa$ forces and volume electric charges of density q_0^κ. The bodies are assumed to be clamped on $\Gamma_1^\kappa\times(0,T)$. The surface tractions $\boldsymbol{f}_2^\kappa$ act on $\Gamma_2^\kappa\times(0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\kappa\times(0,T)$ and a surface electric charge of density q_2^κ is prescribed on $\Gamma_b^\kappa\times(0,T)$. The two bodies can enter in bilateral contact with friction along the common part $\Gamma_3^1=\Gamma_3^2=\Gamma_3$. The bodies are in contact with friction and wear, over the contact surface Γ_3. We introduce the wear function $\omega:\Gamma_3\times(0,T)\to\mathbb{R}^+$ which measures the wear of the surface. The wear is identified as the normal depth of the material that is lost. Let g be the intial gap between the two bodies. Let p_ν and p_τ denote the normal and tangential compliance functions. We denote by $\boldsymbol{v}^*$ and $\alpha^*=\|\boldsymbol{v}^*\|$ the tangential velocity and the tangential speed at the contact surface between the two bodies. We use the modified version of Archard's law $\dot{\omega}=-\lambda_0\boldsymbol{v}^*\sigma_\nu$. To describe the evolution of wear, where $\lambda_0>0$ is a wear coefficient. We introduce the unitary vector $\delta:\Gamma_3\to\mathbb{R}^d$ defined by $\delta=\boldsymbol{v}^*/\|\boldsymbol{v}^*\|$. When the contact arises, some material of the contact surfaces worn out and immediately removed from the system. This process is measured by the wear function ω. With these assumptions above, the classical formulation of the mechanical frictional contact problem with wear between two electro-elastic-viscoplastics bodies is the following.

Problem P. For $\kappa=1,2$, find a displacement field $\boldsymbol{u}^\kappa:\Omega^\kappa\times[0,T]\to\mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\kappa:\Omega^\kappa\times[0,T]\to\mathbb{S}^d$, an electric potential field $\psi^\kappa:\Omega^\kappa\times[0,T]\to\mathbb{R}$, a wear $\omega:\Gamma_3\times[0,T]\to\mathbb{R}^+$ and a electric displacement field $\boldsymbol{D}^\kappa:\Omega^\kappa\times[0,T]\to\mathbb{R}^d$ and an internal state variable field $\beta^\kappa:\Omega^\kappa\times[0,T]\to\mathbb{R}^m$ such that

$$\boldsymbol{\sigma}^\kappa(t)=\mathcal{A}^\kappa\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\kappa(t))+\mathcal{G}^\kappa\boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa(t))+(\mathcal{E}^\kappa)^*\nabla\psi^\kappa(t)$$
$$+\int_0^t\mathcal{F}^\kappa\Big(\boldsymbol{\sigma}^\kappa(s)-\mathcal{A}^\kappa\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\kappa(s))-(\mathcal{E}^\kappa)^*\nabla\psi^\kappa(s),\boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa(s)),\beta^\kappa(s)\Big)ds,\ \text{in}\Omega^\kappa\times(0,T), \tag{4}$$

$$\dot{\beta}^\kappa(t)=\Theta^\kappa\big(\boldsymbol{\sigma}^\kappa(t)-\mathcal{A}^\kappa\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\kappa(t))-(\mathcal{E}^\kappa)^*\nabla\psi^\kappa(t),\boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa(t)),\beta^\kappa(t)\big)\quad\text{in }\Omega^\kappa\times(0,T), \tag{5}$$

$$\boldsymbol{D}^\kappa(t)=\mathcal{E}^\kappa\varepsilon(\boldsymbol{u}^\kappa(t))-\mathcal{B}^\kappa\nabla\psi^\kappa(t)\quad\text{in }\Omega^\kappa\times(0,T), \tag{6}$$

$$\rho^\kappa \ddot{\boldsymbol{u}}^\kappa = \mathrm{Div}\,\boldsymbol{\sigma}^\kappa + \boldsymbol{f}_0^\kappa \quad \text{in } \Omega^\kappa \times (0,T), \tag{7}$$

$$\mathrm{div}\boldsymbol{D}^\kappa - q_0^\kappa = 0 \quad \text{in } \Omega^\kappa \times (0,T), \tag{8}$$

$$\boldsymbol{u}^\kappa(t) = 0 \quad \text{on } \Gamma_1^\kappa \times (0,T), \tag{9}$$

$$\boldsymbol{\sigma}^\kappa \boldsymbol{\nu}^\kappa = \boldsymbol{f}_2^\kappa \quad \text{on } \Gamma_2^\kappa \times (0,T), \tag{10}$$

$$\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \text{ where } \sigma_\nu = -p_\nu(u_\nu - \omega - g) \quad \text{on } \Gamma_3 \times (0,T), \tag{11}$$

$$\boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \text{ where } \boldsymbol{\sigma}_\tau = -p_\tau(\boldsymbol{u}_\nu - \omega - g)\frac{\boldsymbol{v}^*}{\|\boldsymbol{v}^*\|} \quad \text{on } \Gamma_3 \times (0,T), \tag{12}$$

$$u_\nu^1 + u_\nu^2 = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{13}$$

$$\dot{\omega} = -\lambda_0 \alpha^* \sigma_\nu \quad \text{on } \Gamma_3 \times (0,T), \tag{14}$$

$$\psi^\kappa(t) = 0 \quad \text{on } \Gamma_a^\kappa \times (0,T), \tag{15}$$

$$\boldsymbol{D}^\kappa.\boldsymbol{\nu}^\kappa = q_2^\kappa \quad \text{on } \Gamma_b^\kappa \times (0,T), \tag{16}$$

$$\boldsymbol{u}^\kappa(0) = \boldsymbol{u}_0^\kappa, \; \dot{\boldsymbol{u}}^\kappa(0) = \boldsymbol{v}_0^\kappa, \; \beta^\kappa(0) = \beta_0^\kappa \quad \text{in } \Omega^\kappa, \tag{17}$$

$$\omega(0) = \omega_0 \quad \text{on } \Gamma_3. \tag{18}$$

First, equations (4)-(6) represent the electro-elastic-viscoplastic constitutive law with internal state variable of the material. Equations (7) and (8) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, respectively. Next, the equations (9) and (10) represent the displacement and traction boundary condition, respectively. Conditions (11)-(13) represent the frictional bilateral contact with wear described above. The equation (14) represents the ordinary differential equation which describes the evolution of the wear function. Equations (15) and (16) represent the electric boundary conditions. Finally, the functions $\boldsymbol{u}_0^\kappa$, $\boldsymbol{v}_0^\kappa$, β_0^κ and ω_0 in (17)-(18) are the initial data.

3. Variational Formulation and Preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end, we need to introduce some notation and preliminary material. Here and below, $\mathbb{S}^d$ represent the space of second-order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\boldsymbol{u}^\kappa \cdot \boldsymbol{v}^\kappa = u_i^\kappa \cdot v_i^\kappa, \qquad |\boldsymbol{v}^\kappa| = (\boldsymbol{v}^\kappa \cdot \boldsymbol{v}^\kappa)^{\frac{1}{2}}, \quad \forall\, \boldsymbol{u}^\kappa, \boldsymbol{v}^\kappa \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma}^\kappa \cdot \boldsymbol{\tau}^\kappa = \sigma_{ij}^\kappa \cdot \tau_{ij}^\kappa, \quad |\boldsymbol{\tau}^\kappa| = (\boldsymbol{\tau}^\kappa \cdot \boldsymbol{\tau}^\kappa)^{\frac{1}{2}}, \quad \forall\, \boldsymbol{\sigma}^\kappa, \boldsymbol{\tau}^\kappa \in \mathbb{S}^d.$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted.

Now, we use standard notations for the Lebesgue and Sobolev spaces associated with Ω^α and Γ^α and, moreover, we consider the spaces

$$H^\kappa = \{\boldsymbol{v}^\kappa = (v_i^\kappa)_{1\le i\le d};\ v_i^\kappa \in L^2(\Omega^\kappa)\},$$

$$\mathcal{H}^\kappa = \{\boldsymbol{\tau}^\kappa = (\tau_{ij}^\kappa)_{1\le i,j\le d};\ \tau_{ij}^\kappa = \tau_{ji}^\kappa \in L^2(\Omega^\kappa)\},$$

$$H_1^\kappa = \{\boldsymbol{v}^\kappa = (v_i^\kappa)_{1\le i\le d};\ \varepsilon(\boldsymbol{v}^\kappa) \in \mathcal{H}^\kappa\},$$

$$\mathcal{H}_1^\kappa = \{\boldsymbol{\tau}^\kappa = (\tau_{ij}^\kappa)_{1\le i\le d};\ \boldsymbol{\tau}^\kappa \in \mathcal{H}^\kappa,\ \mathrm{Div}\boldsymbol{\tau}^\kappa \in H^\kappa\},$$

$$\boldsymbol{Y}^\kappa = \{\boldsymbol{\lambda}^\kappa = (\lambda_i^\kappa)_{1\le i\le m};\ \lambda_i^\kappa \in L^2(\Omega^\kappa)\},$$

$$\boldsymbol{V}^\kappa = \{\boldsymbol{v}^\kappa \in H^1(\Omega^\kappa)^d;\ \boldsymbol{v}^\kappa = 0 \text{ on } \Gamma_1^\kappa\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}^\kappa,\boldsymbol{v}^\kappa)_{H^\kappa} = \int_{\Omega^\kappa} \boldsymbol{u}^\kappa \cdot \boldsymbol{v}^\kappa dx,$$
$$(\boldsymbol{\sigma}^\kappa,\boldsymbol{\tau}^\kappa)_{\mathcal{H}^\kappa} = \int_{\Omega^\kappa} \boldsymbol{\sigma}^\kappa \cdot \boldsymbol{\tau}^\kappa dx,$$
$$(\boldsymbol{u}^\kappa,\boldsymbol{v}^\kappa)_{H_1^\kappa} = \int_{\Omega^\kappa} \boldsymbol{u}^\kappa \cdot \boldsymbol{v}^\kappa dx + \int_{\Omega^\kappa} \nabla\boldsymbol{u}^\kappa \cdot \nabla\boldsymbol{v}^\kappa dx,$$
$$(\boldsymbol{\sigma}^\kappa,\boldsymbol{\tau}^\kappa)_{\mathcal{H}_1^\kappa} = \int_{\Omega^\kappa} \boldsymbol{\sigma}^\kappa \cdot \boldsymbol{\tau}^\kappa dx + \int_{\Omega^\kappa} \mathrm{Div}\,\boldsymbol{\sigma}^\kappa \cdot \mathrm{Div}\,\boldsymbol{\tau}^\kappa dx,$$
$$(\boldsymbol{\lambda}^\kappa,\boldsymbol{\mu}^\kappa)_{\boldsymbol{Y}^\kappa} = \int_{\Omega^\kappa} \boldsymbol{\lambda}^\kappa \cdot \boldsymbol{\mu}^\kappa dx,$$
$$(\boldsymbol{u}^\kappa,\boldsymbol{v}^\kappa)_{\boldsymbol{V}^\kappa} = (\varepsilon(\boldsymbol{u}^\kappa),\varepsilon(\boldsymbol{v}^\kappa))_{\mathcal{H}^\kappa}$$

and the associated norms $\|\cdot\|_{H^\kappa}$, $\|\cdot\|_{\mathcal{H}^\kappa}$, $\|\cdot\|_{H_1^\kappa}$, $\|\cdot\|_{\mathcal{H}_1^\kappa}$, $\|\cdot\|_{\boldsymbol{Y}^\kappa}$ and $\|\cdot\|_{\boldsymbol{V}^\kappa}$ respectively. Here and below we use the notation

$$\nabla\boldsymbol{u}^\kappa = (u_{i,j}^\kappa),\ \varepsilon(\boldsymbol{u}^\kappa) = (\varepsilon_{ij}(\boldsymbol{u}^\kappa)),\quad \varepsilon_{ij}(\boldsymbol{u}^\kappa) = \frac{1}{2}(u_{i,j}^\kappa + u_{j,i}^\kappa),\quad \forall\ u^\kappa \in H_1^\kappa,$$
$$\mathrm{Div}\,\boldsymbol{\sigma}^\kappa = (\sigma_{ij,j}^\kappa),\quad \forall\ \boldsymbol{\sigma}^\kappa \in \mathcal{H}_1^\kappa.$$

Completeness of the space $(V^\kappa, \|\cdot\|_{V^\kappa})$ follows from the assumption $meas(\Gamma_1^\kappa > 0)$, which allows the use of Korn's inequality.

We denote $\boldsymbol{v}^\kappa$ as the trace of an element $\boldsymbol{v}^\kappa \in H_1^\kappa$ on Γ^κ. For every element $\boldsymbol{v}^\kappa \in H_1^\kappa$, we also use the notation $\boldsymbol{v}^\kappa$ for the trace of $\boldsymbol{v}^\kappa$ on Γ^κ and we denote by v_ν^κ and $\boldsymbol{v}_\tau^\kappa$ the *normal* and the *tangential* components of $\boldsymbol{v}^\kappa$ on the boundary Γ^κ given by

$$v_\nu^\kappa = \boldsymbol{v}^\kappa.\nu^\kappa,\quad \boldsymbol{v}_\tau^\kappa = \boldsymbol{v}^\kappa - v_\nu^\kappa\boldsymbol{\nu}^\kappa.$$

Let H'_{Γ^κ} be the dual of $H_{\Gamma^\kappa} = H^{\frac{1}{2}}(\Gamma^\kappa)^d$ and let $(\cdot,\cdot)_{-\frac{1}{2},\frac{1}{2},\Gamma^\kappa}$ denote the duality pairing between H'_{Γ^κ} and H_{Γ^κ}. For every element $\boldsymbol{\sigma}^\kappa \in \mathcal{H}_1^\kappa$ let $\boldsymbol{\sigma}^\kappa\boldsymbol{\nu}^\kappa$ be the element of H'_{Γ^κ} given by

$$(\boldsymbol{\sigma}^\kappa\boldsymbol{\nu}^\kappa,\boldsymbol{v}^\kappa)_{-\frac{1}{2},\frac{1}{2},\Gamma^\kappa} = (\boldsymbol{\sigma}^\kappa,\varepsilon(\boldsymbol{v}^\kappa))_{\mathcal{H}^\kappa} + (\mathrm{Div}\,\boldsymbol{\sigma}^\kappa,\boldsymbol{v}^\kappa)_{H^\kappa}\quad \forall\ \boldsymbol{v}^\kappa \in H_1^\kappa.$$

Denote by σ_ν^κ and $\boldsymbol{\sigma}_\tau^\kappa$ the *normal* and the *tangential* traces of $\boldsymbol{\sigma}^\kappa \in \mathcal{H}_1^\kappa$, respectively. If $\boldsymbol{\sigma}^\kappa$ is continuously differentiable on $\Omega^\kappa \cup \Gamma^\kappa$, then

$$\sigma_\nu^\kappa = (\boldsymbol{\sigma}^\kappa\boldsymbol{\nu}^\kappa)\cdot\boldsymbol{\nu}^\kappa,\quad \boldsymbol{\sigma}_\tau^\kappa = \boldsymbol{\sigma}^\kappa\boldsymbol{\nu}^\kappa - \sigma_\nu^\kappa\boldsymbol{\nu}^\kappa,$$
$$(\boldsymbol{\sigma}^\kappa\boldsymbol{\nu}^\kappa,\boldsymbol{v}^\kappa)_{-\frac{1}{2},\frac{1}{2},\Gamma^\kappa} = \int_{\Gamma^\kappa} \boldsymbol{\sigma}^\kappa\boldsymbol{\nu}^\kappa.\boldsymbol{v}^\kappa da$$

fore all $\boldsymbol{v}^\kappa \in H_1^\kappa$, where da is the surface measure element. Since $meas\Gamma_1^\kappa > 0$, the following Korn's inequality holds:

$$\|\varepsilon(\boldsymbol{v}^\kappa)\|_{\mathcal{H}^\kappa} \geq c_K\|\boldsymbol{v}^\kappa\|_{H_1^\kappa}\quad \forall\ \boldsymbol{v}^\kappa \in V^\kappa, \tag{19}$$

where the constant c_K denotes a positive constant which may depends only on Ω^κ, Γ_1^κ (see [8]). Over the space V^κ we consider the inner product given by

$$(\boldsymbol{u}^\kappa,\boldsymbol{v}^\kappa)_{V^\kappa} = (\varepsilon(\boldsymbol{u}^\kappa),\varepsilon(\boldsymbol{v}^\kappa))_{\mathcal{H}^\kappa},\quad \forall\ \boldsymbol{u}^\kappa,\boldsymbol{v}^\kappa \in V^\kappa, \tag{20}$$

and let $\|\cdot\|_{V^\kappa}$ be the associated norm. It follows from Korn's inequality (19) that the norms $\|\cdot\|_{H_1^\kappa}$ and $\|\cdot\|_{V^\kappa}$ are equivalent on V^κ. Then $(V^\kappa,\|\cdot\|_{V^\kappa})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (20), there exists a constant $c_0 > 0$, depending only on Ω^κ, Γ_1^κ

and Γ_3 such that

$$\|\boldsymbol{v}^\kappa\|_{L^2(\Gamma_3)^d} \le c_0 \|\boldsymbol{v}^\kappa\|_{V^\kappa} \quad \forall\ \boldsymbol{v}^\kappa \in V^\kappa. \tag{21}$$

We also introduce the spaces

$$W^\kappa = \{\tau^\kappa \in H^1(\Omega^\kappa);\ \tau^\kappa = 0 \text{ on } \Gamma_a^\kappa\},$$
$$\mathcal{W}^\kappa = \{\boldsymbol{D}^\kappa = (D_i^\kappa);\ D_i^\kappa \in L^2(\Omega^\kappa),\ \operatorname{div}\boldsymbol{D}^\kappa \in L^2(\Omega^\kappa)\}.$$

Since $meas\Gamma_a^\kappa > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\tau^\kappa\|_{L^2(\Omega^\kappa)^d} \ge c_F \|\tau^\kappa\|_{H^1(\Omega^\kappa)} \quad \forall\ \tau^\kappa \in W^\kappa, \tag{22}$$

where $c_F > 0$ is a constant which depends only on Ω^κ, Γ_a^κ.

Over the space W^κ, we consider the inner product given by

$$(\psi^\kappa, \tau^\kappa)_{W^\kappa} = \int_{\Omega^\kappa} \nabla\psi^\kappa \cdot \nabla\tau^\kappa dx$$

and let $\|\cdot\|_{W^\kappa}$ be the associated norm. It follows from (22) that $\|\cdot\|_{H^1(\Omega^\kappa)}$ and $\|\cdot\|_{W^\kappa}$ are equivalent norms on W^κ and therefore $(W^\kappa, \|\cdot\|_{W^\kappa})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\mathbf{c}_0$, depending only on Ω^κ, Γ_a^κ and Γ_3, such that

$$\|\zeta^\kappa\|_{L^2(\Omega^\kappa)} \le \mathbf{c}_0 \|\zeta^\kappa\|_{W^\kappa} \quad \forall\ \zeta^\kappa \in W^\kappa. \tag{23}$$

The space $\mathcal{W}^\kappa$ is real Hilbert space with the inner product

$$(\boldsymbol{D}^\kappa, \boldsymbol{E}^\kappa)_{\mathcal{W}^\kappa} = \int_{\Omega^\kappa} \boldsymbol{D}^\kappa \cdot \boldsymbol{E}^\kappa dx + \int_{\Omega^\kappa} \operatorname{div}\boldsymbol{D}^\kappa \cdot \operatorname{div}\boldsymbol{E}^\kappa dx,$$

where $\operatorname{div}\boldsymbol{D}^\kappa = (\boldsymbol{D}_{i,i}^\kappa)$, and the associated norm $\|\cdot\|_{\mathcal{W}^\kappa}$.

In order to simplify the notations, we define the product spaces

$$\begin{aligned} &\boldsymbol{V} = V^1 \times V^2,\ \ H = H^1 \times H^2,\ \ H_1 = H_1^1 \times H_1^2, \\ &\mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2,\ \ \boldsymbol{Y} = \boldsymbol{Y}^1 \times \boldsymbol{Y}^2,\ \ \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \\ &W = W^1 \times W^2,\ \ \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2. \end{aligned} \tag{24}$$

The spaces $\boldsymbol{V}$, H, $\mathcal{H}$, $\boldsymbol{Y}$, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(\cdot,\cdot)_{\boldsymbol{V}}$, $(\cdot,\cdot)_H$, $(\cdot,\cdot)_{\mathcal{H}}$, $(\cdot,\cdot)_{\boldsymbol{Y}}$, $(\cdot,\cdot)_W$ and $(\cdot,\cdot)_{\mathcal{W}}$. The associate norms will be denoted by $\|\cdot\|_{\boldsymbol{V}}$, $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\boldsymbol{Y}}$, $\|\cdot\|_W$, and $\|\cdot\|_{\mathcal{W}}$, respectively.

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $1 \le p \le \infty$, $k \ge 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X,$$

$$\|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$.

In the study of the Problem **P**, we consider the following assumptions:

Assume the operators $\mathcal{A}^\kappa$, $\mathcal{G}^\kappa$, $\mathcal{F}^\kappa$, Θ^κ, $\mathcal{E}^\kappa$ and $\boldsymbol{\beta}^\kappa$ satisfy the following conditions ($L_{\mathcal{A}^\kappa}$, $L_{\mathcal{G}^\kappa}$, $m_{\mathcal{A}^\kappa}$, $L_{\mathcal{F}^\kappa}$, L_{Θ^κ}, and $m_{\boldsymbol{\beta}^\kappa}$ being positive constants), with $\kappa = 1,2$.

H(1): (a) $\mathcal{A}^\kappa : \Omega^\kappa \times \mathbb{S}^d \to \mathbb{S}^d$

(b) $|\mathcal{A}^\kappa(\boldsymbol{x},\boldsymbol{\psi}_1) - \mathcal{A}^\kappa(\boldsymbol{x},\boldsymbol{\psi}_2)| \le L_{\mathcal{A}^\kappa}|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2|$, for any $\boldsymbol{\psi}_1,\boldsymbol{\psi}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\kappa$.

(c) $(\mathcal{A}^\kappa(\boldsymbol{x},\boldsymbol{\psi}_1) - \mathcal{A}^\kappa(\boldsymbol{x},\boldsymbol{\psi}_2)) \cdot (\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2) \ge m_{\mathcal{A}^\kappa}|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2|^2$, for any $\boldsymbol{\psi}_1,\boldsymbol{\psi}_2 \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\kappa$.

(d) $\mathcal{A}^\kappa(\cdot,\boldsymbol{\psi})$ is measurable on Ω^κ, for any $\boldsymbol{\psi} \in \mathbb{S}^d$.

(e) $\mathcal{A}^\kappa(\boldsymbol{x},.)$ is continuous on $\mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\kappa$.

H(2): (a) $\mathcal{G}^\kappa : \Omega^\kappa \times \mathbb{S}^d \to \mathbb{S}^d$

(b) $|\mathcal{G}^\kappa(\boldsymbol{x},\boldsymbol{\psi}_1) - \mathcal{G}^\kappa(\boldsymbol{x},\boldsymbol{\psi}_2)| \le L_{\mathcal{G}^\kappa}|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2|$ for any $\boldsymbol{\psi}_1,\boldsymbol{\psi}_2 \in \mathbb{S}^d$, a.e.$\boldsymbol{x} \in \Omega^\kappa$.

(c) $\mathcal{G}^\kappa(\cdot,\boldsymbol{\psi})$ is measurable on Ω^κ, for any $\boldsymbol{\psi} \in \mathbb{S}^d$.

(d) $\mathcal{G}^\kappa(\cdot,\mathbf{0})$ belongs to $\mathcal{H}^\kappa$.

H(3): (a) $\mathcal{F}^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \to \mathbb{S}^d$

(b) $|\mathcal{F}^\kappa(\boldsymbol{x},\boldsymbol{\eta}_1,\boldsymbol{\psi}_1,\beta_1) - \mathcal{F}^\kappa(\boldsymbol{x},\boldsymbol{\eta}_2,\boldsymbol{\psi}_2,\beta_2)| \le L_{\mathcal{F}^\kappa}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2| + |\beta_1 - \beta_2|\big)$
for any $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\psi}_1,\boldsymbol{\psi}_2 \in \mathbb{S}^d, \beta_1,\beta_2 \in \mathbb{R}^m$, a.e. $\boldsymbol{x} \in \Omega^\kappa$.

(c) $\mathcal{F}^\kappa(\cdot,\boldsymbol{\eta},\boldsymbol{\psi},\beta)$ is measurable in Ω^κ, for any $\boldsymbol{\eta},\boldsymbol{\psi} \in \mathbb{S}^d$, $\beta \in \mathbb{R}^m$.

(d) $\mathcal{F}^\kappa(\cdot,\mathbf{0},\mathbf{0},\mathbf{0})$ belongs to $\mathcal{H}^\kappa$.

H(4): (a) $\Theta^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \to \mathbb{S}^d$

(b) $|\Theta^\kappa(\boldsymbol{x},\boldsymbol{\eta}_1,\boldsymbol{\psi}_1,\beta_1) - \Theta^\kappa(\boldsymbol{x},\boldsymbol{\eta}_2,\boldsymbol{\psi}_2,\beta_2)| \le L_{\Theta^\kappa}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2| + |\beta_1 - \beta_2|\big)$,
for any $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\psi}_1,\boldsymbol{\psi}_2 \in \mathbb{S}^d, \beta_1,\beta_2 \in \mathbb{R}^m$, $a.e.$ $\boldsymbol{x} \in \Omega^\kappa$.

(c) $\Theta^\kappa(\cdot,\boldsymbol{\eta},\boldsymbol{\psi},\beta)$ is measurable on Ω^κ, $\forall$ $\boldsymbol{\eta},\boldsymbol{\omega} \in \mathbb{S}^d, \beta \in \mathbb{R}^m$

(d) $\Theta^\kappa(\cdot,\mathbf{0},\mathbf{0},\mathbf{0})$ belongs to $L^2(\Omega^\kappa)$.

H(5): (a) $\mathcal{E}^\kappa : \Omega^\kappa \times \mathbb{S}^d \to \mathbb{R}^d$

(b) $\mathcal{E}^\kappa = (e^\kappa_{ijk})$, $e^\kappa_{ijk} = e^\kappa_{ikj} \in L^\infty(\Omega^\kappa)$, $1 \le i,j,k \le d$.

(c) $\mathcal{E}^\kappa \sigma.\boldsymbol{v} = \sigma.(\mathcal{E}^\kappa)^* \boldsymbol{v}$, for any $\sigma \in \mathbb{S}^d$, for any $\boldsymbol{v} \in \mathbb{R}^d$.

H(6): (a) $\mathcal{B}^\kappa : \Omega^\kappa \times \mathbb{R}^d \to \mathbb{R}^d$

(b) $\mathcal{B}^\kappa = (b^\kappa_{ij})$, $b^\kappa_{ij} = b^\kappa_{ji} \in L^\infty(\Omega^\kappa)$, $1 \le i,j \le d$.

(c) $\mathcal{B}^\kappa \mathbf{E}.\mathbf{E} \ge m_{\mathcal{B}^\kappa}|\mathbf{E}|^2$, for any $\mathbf{E} = (E_i) \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Omega^\kappa$.

The normal compliance function p_ν and the tangential function p_τ satisfy the assumptions $\big(L_\nu$, L_τ and M_τ being positive constants$\big)$

H(7): (a) $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$

(b) $|p_\nu(\boldsymbol{x},r_1) - p_\nu(\boldsymbol{x},r_2)| \le L_\nu|r_1 - r_2|$, $\forall$ $r_1,r_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

(c) $p_\nu(\cdot,r)$ is measurable on Γ_3, for any $r \in \mathbb{R}$.

(d) $p_\nu(\boldsymbol{x},r) = 0$, for any $r \le 0$, a.e. $\boldsymbol{x} \in \Gamma_3$.

H(8): (a) $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$

(b) $|p_\tau(\boldsymbol{x},d_1) - p_\tau(\boldsymbol{x},d_2)| \le L_\tau|d_1 - d_2|$, for any $d_1,d_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

(c) $|p_\tau(\boldsymbol{x},d)| \le M_\tau$ for any $d \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

(d) $p_\tau(\cdot,d)$ is measurable on Γ_3, for any $d \in \mathbb{R}$.

(e) $p_\tau(\cdot,0) \in L^2(\Gamma_3)$.

We suppose that the mass density, the forces and the traction densities satisfy

H(9): (a) $\rho^\kappa \in L^\infty(\Omega^\kappa)$, $\exists \rho_0 > 0$; $\rho^\kappa(x) \ge \rho_0$ a.e. $x \in \Omega^\kappa$.
(b) $g \in L^2(\Gamma_3)$, $g \ge 0$ a.e. on Γ_3.
(c) $\mathbf{f}_0^\kappa \in L^2(0,T;L^2(\Omega^\kappa)^d)$, $\mathbf{f}_2^\kappa \in L^2(0,T;L^2(\Gamma_2^\kappa)^d)$.
(d) $q_0^\kappa \in C(0,T;L^2(\Omega^\kappa))$, $q_2^\kappa \in C(0,T;L^2(\Gamma_b^\kappa))$.

Also, we assume that the initial values satisfy

H(10): (a) $\beta_0^\kappa \in \boldsymbol{Y}^\kappa$, $\boldsymbol{u}_0^\kappa \in \boldsymbol{V}^\kappa$, $\boldsymbol{v}_0^\kappa \in H^\kappa$.
(b) $\omega_0 \in L^2(\Gamma_3)$.

We will use a modified inner product on H, given by

$$((\boldsymbol{u},\boldsymbol{v}))_H = \sum_{\kappa=1}^{2} (\rho^\kappa \boldsymbol{u}^\kappa, \boldsymbol{v}^\kappa)_{H^\kappa}, \quad \forall\ \boldsymbol{u},\boldsymbol{v} \in H,$$

and let $|||\cdot|||_H$ be the associated norm. It follows from assumption H(9)(a), that $|||\cdot|||_H$ and $\|\cdot\|_H$ are equivalent norms on H, and the inclusion mapping of $(\boldsymbol{V},\|\cdot\|_{\boldsymbol{V}})$ into $(H,|||\cdot|||_H)$ is continuous and dense. We denote by $\boldsymbol{V}'$ the dual of $\boldsymbol{V}$. Identifying H with its own dual. Then $(\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = ((\boldsymbol{u},\boldsymbol{v}))_H$, $\forall\ \boldsymbol{u} \in H, \forall\ \boldsymbol{v} \in \boldsymbol{V}$.

We define three mappings $\mathbf{f} : [0,T] \to \boldsymbol{V}'$, $q : [0,T] \to W$, $j : \boldsymbol{V} \times \boldsymbol{V} \times L^2(\Gamma_3) \to \mathbb{R}$ respectively, by

$$(\mathbf{f}(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\kappa=1}^{2} \int_{\Omega^\kappa} \mathbf{f}_0^\kappa(t)\cdot \boldsymbol{v}^\kappa\, dx + \sum_{\kappa=1}^{2} \int_{\Gamma_2^\kappa} \mathbf{f}_2^\kappa(t)\cdot \boldsymbol{v}^\kappa\, da\ \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{25}$$

$$(q(t),\varsigma)_W = \sum_{\kappa=1}^{2} \int_{\Omega^\kappa} q_0^\kappa(t)\varsigma^\kappa\, dx - \sum_{\kappa=1}^{2} \int_{\Gamma_b^\kappa} q_2^\kappa(t)\varsigma^\kappa\, da \quad \forall\ \varsigma \in W, \tag{26}$$

$$j(\boldsymbol{u},\boldsymbol{v},\omega) = \int_{\Gamma_3} \Big(p_\nu(\boldsymbol{u}_\nu - \omega - g)\boldsymbol{v}_\nu\Big)\, da + \int_{\Gamma_3} \Big(p_\tau(\boldsymbol{u}_\nu - \omega - g)\Big)\|\boldsymbol{v}_\tau - \boldsymbol{v}^*\|\, da. \tag{27}$$

We note that conditions H(9)(b) and H(9)(c) imply

$$\mathbf{f} \in L^2(0,T;\boldsymbol{V}'), q \in C(0,T;W). \tag{28}$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical (4)-(18).

Problem PV. Find a displacement field $\boldsymbol{u} : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$, an electric potential field $\psi : [0,T] \to W$, a wear $\omega : [0,T] \to L^2(\Gamma_3)$ a electric displacement field $\boldsymbol{D} : [0,T] \to \mathcal{W}$ and an internal state variable field $\beta : [0,T] \to \boldsymbol{Y}$ such that

$$\boldsymbol{\sigma}^\kappa = \mathcal{A}^\kappa \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\kappa) + \mathcal{G}^\kappa \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa) + (\mathcal{E}^\kappa)^* \nabla\psi^\kappa + \int_0^t \mathcal{F}^\kappa\Big(\boldsymbol{\sigma}^\kappa(s) - \mathcal{A}^\kappa \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\kappa(s)) - (\mathcal{E}^\kappa)^*\nabla\psi^\kappa, \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa(s)), \beta^\kappa(s)\Big)\, ds$$
$$\text{in } \Omega^\kappa \times (0,T), \tag{29}$$

$$\dot{\beta}^\kappa = \Theta^\kappa\big(\boldsymbol{\sigma}^\kappa - \mathcal{A}^\kappa \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\kappa) - (\mathcal{E}^\kappa)^*\nabla\psi^\kappa, \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa), \beta^\kappa\big) \quad \text{in } \Omega^\kappa \times (0,T), \tag{30}$$

$$\boldsymbol{D}^\kappa = \mathcal{E}^\kappa \varepsilon(\boldsymbol{u}^\kappa) - \mathcal{B}^\kappa \nabla\psi^\kappa \quad \text{in } \Omega^\kappa \times (0,T), \tag{31}$$

$$(\ddot{u},v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\kappa=1}^{2} (\boldsymbol{\sigma}^\kappa,\ \varepsilon(\boldsymbol{v}^\kappa))_{\mathcal{H}^\kappa} + j(\boldsymbol{u}(t),\boldsymbol{v},\omega) = (\mathbf{f}(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}, t \in (0,T), \tag{32}$$

$$\sum_{\kappa=1}^{2} (\mathcal{B}^\kappa \nabla\psi^\kappa(t) - \mathcal{E}^\kappa \varepsilon(\boldsymbol{u}^\kappa(t)),\ \nabla\phi^\kappa)_{H^\kappa} = (q(t),\phi)_W,\ \forall\ \phi \in W,\ t \in (0,T), \tag{33}$$

$$\dot{\omega} = \lambda_0 \alpha^* p_\nu(\boldsymbol{u}_\nu - \omega - g) \quad \text{on } \Gamma_3 \times (0,T), \tag{34}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad \dot{\boldsymbol{u}}(0)=\boldsymbol{v}_0,\quad \beta(0)=\beta_0,\quad \omega(0)=\omega_0. \tag{35}$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field, a electric displacement field and a wear. The existence of the unique solution to Problem **PV** is stated and proved in the next section.

4. Existence and Uniqueness Result

Now, we propose our existence and uniqueness result

Theorem 4.1. *Assume that* H(1)-H(10) *hold. Then there exists a unique solution* $\{\boldsymbol{u},\beta,\boldsymbol{\sigma},\psi,\omega,\boldsymbol{D}\}$ *to* Problem **PV**. *Moreover, the solution satisfies*

$$\boldsymbol{u}\in W^{1,2}(0,T;\boldsymbol{V})\cap C^1(0,T;H),\ \ \ddot{\boldsymbol{u}}\in L^2(0,T;\boldsymbol{V}'), \tag{36}$$

$$\beta\in W^{1,2}(0,T;\boldsymbol{Y}), \tag{37}$$

$$\psi\in C(0,T;W), \tag{38}$$

$$\boldsymbol{\sigma}\in L^2(0,T;\mathcal{H}),\ \ \mathrm{Div}\,\boldsymbol{\sigma}\in L^2(0,T;\boldsymbol{V}') \tag{39}$$

$$\boldsymbol{D}\in C(0,T;\mathcal{W}). \tag{40}$$

$$\omega\in C^1(0,T;L^2(\Gamma_3)). \tag{41}$$

The functions $\boldsymbol{u},\beta,\psi,\boldsymbol{\sigma},\boldsymbol{D}$ and ω which satisfy (29)-(35) are called a weak solution to the contact Problem **P**. We conclude that, under the assumptions H(1)-H(10), the mechanical problem (4)-(18) has a unique weak solution satisfying (36)-(41). We turn now to the proof of Theorem 4.1 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed point arguments. We assume in what follows that assumptions of Theorem 4.1 hold, and we consider that C is a generic positive constant which depends on Ω^κ, $\mathcal{A}^\kappa$, $\mathcal{G}^\kappa$, $\mathcal{E}^\kappa$, $\mathcal{F}^\kappa$, Γ_1^κ, Γ_2^κ, Γ_3, p_ν,p_τ, κ and T and may change from place to place.
Let $\eta\in L^2(0,T;\boldsymbol{V}')$ be given. In the first step we consider the following variational problem.

Problem $\mathbf{PV}_\eta^{u\psi}$. Find $(\boldsymbol{u}_\eta,\psi_\eta):[0,T]\to\boldsymbol{V}\times W$ such that

$$(\ddot{u}_\eta(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\kappa=1}^{2}(\mathcal{A}^\kappa\varepsilon(\dot{\boldsymbol{u}}^\kappa(t)),\ \varepsilon(\boldsymbol{v}^\kappa))_{\mathcal{H}^\kappa}=(\mathbf{f}(t)-\eta(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}},\ \ \forall\ \boldsymbol{v}\in\boldsymbol{V},\text{a.e. } t\in(0,T), \tag{42}$$

$$\sum_{\kappa=1}^{2}(\mathcal{B}^\kappa\nabla\psi_\eta^\kappa(t)-\mathcal{E}^\kappa\varepsilon(\boldsymbol{u}_\eta^\kappa(t)),\nabla\phi^\kappa)_{H^\kappa}=(q(t),\phi)_W,\ \forall\ \phi\in W \text{ a.e. } t\in(0,T), \tag{43}$$

$$\boldsymbol{u}_\eta^\kappa(0)=\boldsymbol{u}_0^\kappa,\ \ \dot{\boldsymbol{u}}_\eta^\kappa(0)=\boldsymbol{v}_0^\kappa\quad \text{in } \Omega^\kappa. \tag{44}$$

We have the following result for the problem.

Lemma 4.1. *There exists a unique solution* $(\boldsymbol{u}_\eta,\psi_\eta)$ *of Problem* $\boldsymbol{PV}_\eta^{u\psi}$ *and it satisfies*

$$\boldsymbol{u}_\eta\in W^{1,2}(0,T;\boldsymbol{V})\cap C^1(0,T;H),\ \ddot{\boldsymbol{u}}_\eta\in L^2(0,T;\boldsymbol{V}'), \tag{45}$$

$$\psi_\eta\in C(0,T;W). \tag{46}$$

Proof. We define the operator $A:\boldsymbol{V}\to\boldsymbol{V}'$ by

$$(A\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}}=\sum_{\kappa=1}^{2}(\mathcal{A}^\kappa\varepsilon(\boldsymbol{u}^\kappa),\ \varepsilon(\boldsymbol{v}^\kappa))_{\mathcal{H}^\kappa}\quad \forall\ \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}. \tag{47}$$

We use (47) and H(1) to find that

$$\|A\boldsymbol{u}-A\boldsymbol{v}\|_{\boldsymbol{V}'}^2 \le \sum_{\kappa=1}^{2} \|\mathcal{A}^\kappa\varepsilon(\boldsymbol{u}^\kappa)-\mathcal{A}^\kappa\varepsilon(\boldsymbol{v}^\kappa)\|_{\mathcal{H}^\kappa}^2 \quad \forall\ \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}.$$

Keeping in mind H(1) and Krasnoselski Theorem (see, e.g. [3, p.60]), we deduce that $A:\boldsymbol{V}\to\boldsymbol{V}'$ is a continuous, and so hemicontinuous. Now, by H(1)(c) and (47), it follows that

$$(A\boldsymbol{u}-A\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \ge m\|\boldsymbol{u}-\boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall\ \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}, \tag{48}$$

where the positive constant $m=\min\{m_{\mathcal{A}^1},m_{\mathcal{A}^2}\}$. Choosing $\boldsymbol{v}=0$ in (48) we obtain

$$\begin{aligned}(A\boldsymbol{u},\boldsymbol{u})_{\boldsymbol{V}'\times\boldsymbol{V}} &\ge m\|\boldsymbol{u}\|_{\boldsymbol{V}}^2-\|Ao\|_{\boldsymbol{V}'}^2\|\boldsymbol{u}\|_{\boldsymbol{V}}\\ &\ge \frac{1}{2}m\|\boldsymbol{u}\|_{\boldsymbol{V}}^2-\frac{1}{2m}\|Ao\|_{\boldsymbol{V}'}^2 \quad \forall\ \boldsymbol{u}\in\boldsymbol{V}.\end{aligned} \tag{49}$$

Moreover, by (47) and H(1)(b) we find

$$\|A\boldsymbol{u}\|_{\boldsymbol{V}'} \le C^1\|u\|_{\boldsymbol{V}}+C^2 \quad \forall\ \boldsymbol{u}\in\boldsymbol{V},$$

where $C^1=\max\{C^1_{\mathcal{A}^1},C^1_{\mathcal{A}^2}\}$ and $C^2=\max\{C^2_{\mathcal{A}^1},C^2_{\mathcal{A}^2}\}$. Finally, we recall that by (28) we have $\mathbf{f}-\eta\in L^2(0,T;\boldsymbol{V}')$ and $\boldsymbol{v}_0\in H$. Therefore, using a standard for ordinary differential equations in abstract spaces (see, e.g. [13, Theorem 2.29]), we know there exists a unique function ϑ_η such that

$$\vartheta_\eta\in L^2(0,T;\boldsymbol{V})\cap C(0,T;H),\quad \dot{\vartheta}_\eta\in L^2(0,T;\boldsymbol{V}'), \tag{50}$$

$$\dot{\vartheta}_\eta(t)+A\vartheta_\eta(t)=\mathbf{f}(t)-\eta(t),\quad a.e.\ t\in[0,T] \tag{51}$$

$$\vartheta_\eta(0)=\boldsymbol{v}_0. \tag{52}$$

Let $\boldsymbol{u}_\eta:[0,T]\to\boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}_\eta(t)=\int_0^t\vartheta_\eta(s)ds+\boldsymbol{u}_0 \quad \forall\ t\in[0,T]. \tag{53}$$

It follows from (47) and (50)-(53), that $\boldsymbol{u}_\eta$ is a solution to (42), (44), with the regularity (45).

Next, we define a bilinear form: $b(\cdot,\cdot):W\times W\to\mathbb{R}$ such that

$$b(\psi,\phi)=\sum_{\kappa=1}^{2}(\boldsymbol{\beta}^\kappa\nabla\psi^\kappa,\nabla\phi^\kappa)_{H^\kappa} \quad \forall\ \psi,\phi\in W. \tag{54}$$

We use H(9) and (54) to show that the bilinear form $b(\cdot,\cdot)$ is continuous, symmetric and coercive on W. Moreover, using (33) and the Riesz Representation Theorem we may define an element $q_\eta:[0,T]\to W$ such that

$$(q_\eta(t),\phi)_W=(q(t),\phi)_W+\sum_{\kappa=1}^{2}(\mathcal{E}^\kappa\varepsilon(\boldsymbol{u}_\eta^\kappa(t)),\nabla\phi^\kappa)_{H^\kappa} \quad \forall\ \phi\in W,\ \ t\in[0,T].$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\psi_\eta(t)\in W$ such that

$$b(\psi_\eta(t),\phi)=(q_\eta(t),\phi)_W \quad \forall\ \phi\in W. \tag{55}$$

It follows from (55), that the pair $(\boldsymbol{u}_\eta,\psi_\eta)$ is the solution to the nonlinear variational equation (43). Let now $t_1,t_2\in[0,T]$, it follows from (43) that

$$\|\psi_\eta(t_1)-\psi_\eta(t_2)\|_W \le C\big(\|\boldsymbol{u}_\eta(t_1)-\boldsymbol{u}_\eta(t_2)\|_{\boldsymbol{V}}+\|q(t_1)-q(t_2)\|_W\big). \tag{56}$$

Since $\boldsymbol{u}_\eta\in C^1(0,T;H)$ and $q\in C(0,T;W)$, inequality (56) implies that $\psi_\eta\in C(0,T;W)$. This completes the proof. □

In the second step, we let $\mu \in L^2(0,T,\boldsymbol{Y})$ be given, and define $\beta_\mu \in W^{1,2}(0,T,\boldsymbol{Y})$ by

$$\beta_\mu(t) = \beta_0 + \int_0^t \mu(s)ds. \tag{57}$$

We use $(\boldsymbol{u}_\eta, \psi_\eta)$ obtained in Lemma 4.1, and β_μ defined in (57) to construct the following Cauchy problem for the stress field.

Problem $\mathbf{PV}^{\sigma}_{\eta\mu}$. Find $\boldsymbol{\sigma}_{\eta\mu} = (\boldsymbol{\sigma}^1_{\eta\mu}, \boldsymbol{\sigma}^2_{\eta\mu}) : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}^\kappa_{\eta\mu}(t) = \mathcal{G}^\kappa \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa_\eta(t)) + \int_0^t \mathcal{F}^\kappa(\boldsymbol{\sigma}^\kappa_{\eta\mu}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa_\eta(s)), \beta^\kappa_\mu(s))ds, \qquad \text{a.e. } t \in (0,T),\ \kappa = 1,2. \tag{58}$$

In the study of Problem $\mathbf{PV}^{\sigma}_{\eta\mu}$ we have the following result.

Lemma 4.2. *There exists a unique solution of Problem $\boldsymbol{PV}^{\sigma}_{\eta\mu}$ and it satisfies $\boldsymbol{\sigma}_{\eta\mu} \in L^2(0,T;\mathcal{H})$.*

Proof. We introduce the operator $\Lambda_{\eta\mu} = (\Lambda^1_{\eta\mu}, \Lambda^2_{\eta\mu}) : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ defined by

$$\Lambda^\kappa_{\eta\mu}\boldsymbol{\sigma}(t) = \mathcal{G}^\kappa \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa_\eta(t)) + \int_0^t \mathcal{F}^\kappa\big(\boldsymbol{\sigma}^\kappa(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\kappa_\eta(s)), \beta^\kappa_\mu(s)\big)ds, \tag{59}$$

for all $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \in L^2(0,T;\mathcal{H})$, $t \in [0,T]$ and $\kappa = 1,2$. For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in L^2(0,T;\mathcal{H})$ we use (59) and H(3), to obtain

$$\|\Lambda_{\eta\mu}\boldsymbol{\sigma}_1(t) - \Lambda_{\eta\mu}\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le \max(L_{\mathcal{F}^1}, L_{\mathcal{F}^2}) \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\, ds$$

for all $t \in [0,T]$. It follows from this inequality that for p large enough, a power $\Lambda^p_{\eta\mu}$ of the operator $\Lambda_{\eta\mu}$ is a contraction on the Banach space $L^2(0,T;\mathcal{H})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_{\eta\mu} \in L^2(0,T;\mathcal{H})$ such that $\Lambda_{\eta\mu}\boldsymbol{\sigma}_{\eta\mu} = \boldsymbol{\sigma}_{\eta\mu}$. Moreover, $\boldsymbol{\sigma}_{\eta\mu}$ is the unique solution of Problem $\mathbf{PV}^{\sigma}_{\eta\mu}$, which concludes the proof. □

Lemma 4.3. *Let $(\eta_1,\mu_1),(\eta_2,\mu_2) \in L^2(0,T;\boldsymbol{V}' \times \boldsymbol{Y})$ and let $\boldsymbol{\sigma}_i$ denote the functions obtained in Lemma 4.2, for $i = 1,2$. Then, the following inequalities hold:*

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}} \le C\Big(&\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|^2_{\mathbf{V}} + \int_0^t \|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\|^2_{\mathbf{V}}\, ds \\ &+ \int_0^t \|\beta_{\mu_1}(s) - \beta_{\mu_2}(s)\|^2_{Y} ds\Big), \qquad \text{a.e. } t \in (0,T).\end{aligned} \tag{60}$$

Proof. Let $t \in [0,T]$. Using (58) and the properties H(2)-H(3) of $\mathcal{G}^\kappa$ and $\mathcal{F}^\kappa$, we find

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}} \le C\Big(&\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|^2_{\mathbf{V}} + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\, ds \\ &+ \int_0^t \|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\|^2_{V}\, ds + \int_0^t \|\beta_{\mu_1}(s) - \beta_{\mu_2}(s)\|^2_{\mathbf{Y}}\, ds\Big).\end{aligned}$$

Using the Gronwall's inequality in the previous inequality we deduce the estimate (60), which concludes the proof of Lemma 4.3. □

In the third step, we use the displacement field u_η obtained in Lemma 4.1. We consider the following intial-value problem.

Problem $\mathbf{PV}^{\omega}_{\eta}$. Find $\omega_\eta \in C^1(0,T,L^2(\Gamma_3))$ such that

$$\dot{\omega}_\eta = \lambda_0 \alpha^* p_\nu(\boldsymbol{u}_{\eta\nu} - \omega_\eta - g), \tag{61}$$

$$\omega_\eta(0)=\omega_0. \tag{62}$$

Let us now we consider the operator $\mathfrak{L}_\eta : C(0,T,L^2(\Gamma_3)) \to C(0,T,L^2(\Gamma_3))$ defined by

$$\mathfrak{L}_\eta\omega(t)=\lambda_0\alpha^*\int_0^t p_\nu(\boldsymbol{u}_{\eta\nu}(s)-\omega(s)-g)ds+\omega_0, \quad \forall\ t\in[0,T]. \tag{63}$$

Lemma 4.4. *The operator $\mathfrak{L}_\eta$ has a unique fixed point ω_η and it satisfies*

$$\omega_\eta\in C^1(0,T,L^2(\Gamma_3)). \tag{64}$$

Proof. Let $\omega_1,\omega_2\in C(0,T,L^2(\Gamma_3))$ and $t\in[0,T]$. From (63) and H(7)(b), we deduce that

$$\|\mathfrak{L}_\eta\omega_1(t)-\mathfrak{L}_\eta\omega_2(t)\|^2_{L^2(\Gamma_3)}ds\le C\int_0^t\|\omega_1(s)-\omega_2(s)\|^2_{L^2(\Gamma_3)}ds.$$

By reiterating m times the previous inequality, we obtain

$$\|\mathfrak{L}^m_\eta\omega_1-\mathfrak{L}^m_\eta\omega_2\|^2_{C(0,T;L^2(\Gamma_3))}\le\frac{(CT)^m}{m!}\|\omega_1-\omega_2\|^2_{C(0,T;L^2(\Gamma_3))}.$$

For m sufficiently large, $\mathfrak{L}^m_\eta$ is a contractive operator on the Banach space $C(0,T;L^2(\Gamma_3))$. Thus, from Banach's fixed point theorem the operator $\mathfrak{L}_\eta$ has a unique fixed point $\omega_\eta\in C(0,T;L^2(\Gamma_3))$, and from H(7)(b), (45), we deduce that (64). □

We now pass to the final step of the proof of Theorem 4.1 in which we use a fixed point argument. To this end, we consider the operator:

$$\Pi : L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y})\to L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y})$$

defined by

$$\Pi(\eta,\mu)=\big(\Pi^1(\eta,\mu),\Pi^2(\eta,\mu)\big) \tag{65}$$

with

$$\begin{aligned}(\Pi^1(\eta,\mu)(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}}&=\sum_{\kappa=1}^2\big(\mathcal{G}^\kappa\varepsilon(\boldsymbol{u}^\kappa_\eta(t))+(\mathcal{E}^\kappa)^*\nabla\psi^\kappa_\eta\ ,\ \varepsilon(\boldsymbol{v}^\kappa)\big)_{\mathcal{H}^\kappa}\\&\quad+\sum_{\kappa=1}^2\Big(\int_0^t\mathcal{F}^\kappa\big(\boldsymbol{\sigma}^\kappa_{\eta\mu},\varepsilon(\boldsymbol{u}^\kappa_\eta(s)),\beta^\kappa_\mu(s)\big)ds\ ,\ \varepsilon(\boldsymbol{v}^\kappa)\Big)_{\mathcal{H}^\kappa}+j(\boldsymbol{u}_\eta(t),\boldsymbol{v},\omega_\eta),\end{aligned} \tag{66}$$

$$\Pi^2(\eta,\mu)(t)=\Big(\Theta^1\big(\boldsymbol{\sigma}^1_{\eta\mu}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}^1_\eta(t)),\beta^1_\mu(t)\big),\Theta^2\big(\boldsymbol{\sigma}^2_{\eta\mu}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}^2_\eta(t)),\beta^2_\mu(t)\big)\Big) \tag{67}$$

for all $\boldsymbol{v}\in\boldsymbol{V}$ and $t\in[0,T]$. We have the following result.

Lemma 4.5. *The operator Π has a unique fixed point $(\eta^*,\mu^*)\in L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y})$.*

Proof. Let $(\eta_1,\mu_1),(\eta_2,\mu_2)$ in $L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y})$ and let $t\in[0,T]$. We use the notation $\boldsymbol{u}_i=\boldsymbol{u}_{\eta_i}$, $\boldsymbol{v}_i=\dot{\boldsymbol{u}}_{\eta_i}$, $\boldsymbol{\sigma}_i=\boldsymbol{\sigma}_{\eta_i\mu_i}$, and $\beta_i=\beta_{\mu_i}$ for $i=1,2$. We use H(2), H(3), H(7), H(8) to obtain

$$\begin{aligned}&\|\Pi^1(\eta_1,\mu_1)(t)-\Pi^1(\eta_2,\mu_2)(t)\|^2_{\boldsymbol{V}'}\\&\quad\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}ds+\|\psi_1(t)-\psi_2(t)\|^2_W+\int_0^t\|\beta_1(s)-\beta_2(s))\|^2_{\boldsymbol{Y}}ds\Big).\end{aligned}$$

By similar arguments, from (60), (67) and H(4) it follows that

$$\begin{aligned}&\|\Pi^2(\eta_1,\mu_1)(t)-\Pi^2(\eta_2,\mu_2)(t)\|^2_{\boldsymbol{Y}}\\&\quad\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}ds+\int_0^t\|\beta_1(s)-\beta_2(s))\|^2_{\boldsymbol{Y}}\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}\|\Pi(\eta_1,\mu_1)(t)-\Pi(\eta_2,\mu_2)(t)\|^2_{\boldsymbol{V}'\times\boldsymbol{Y}} \le C\Big(&\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\|\psi_1(t)-\psi_2(t)\|^2_W\\ &+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}ds\\ &+\int_0^t\|\beta_1(s)-\beta_2(s))\|^2_{\boldsymbol{Y}}ds+\|\beta_1(t)-\beta_2(t))\|^2_{\boldsymbol{Y}}\Big).\end{aligned} \tag{68}$$

Moreover, from (42) we obtain

$$(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\kappa=1}^{2}(\mathcal{A}^\kappa\varepsilon(\boldsymbol{v}_1^\kappa)-\mathcal{A}^\kappa\varepsilon(\boldsymbol{v}_2^\kappa),\varepsilon(\boldsymbol{v}_1^\kappa-\boldsymbol{v}_2^\kappa))_{\mathcal{H}^\kappa}=-(\eta_1-\eta_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}.$$

We integrate this equality with respect to time, use the initial conditions $\boldsymbol{v}_1(0)=\boldsymbol{v}_2(0)=\boldsymbol{v}_0$ and condition H(1)(c) to find

$$m\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}ds\le-\int_0^t(\eta_1(s)-\eta_2(s),\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))_{\boldsymbol{V}'\times\boldsymbol{V}}ds$$

where $m=\min(m_{\mathcal{A}^1},m_{\mathcal{A}^2})$. Then, using $2ab\le\frac{a^2}{\delta}+\delta b^2$ we obtain

$$\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}ds\le C\int_0^t\|\eta_1(s)-\eta_2(s)\|^2_{\boldsymbol{V}'}ds. \tag{69}$$

The definition (57) yields

$$\|\beta_1(t)-\beta_2(t)\|^2_Y\le C\Big(\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_Y ds\Big). \tag{70}$$

Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ have the same initial value we get

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}\le\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}ds. \tag{71}$$

We substitute (69)-(71) in (68) to obtain

$$\big\|\Pi(\eta_1,\mu_1)(t)-\Pi(\eta_2,\mu_2)(t)\big\|^2_{\boldsymbol{V}'\times\boldsymbol{Y}}\le C\int_0^t\big\|(\eta_1,\mu_1)(s)-(\eta_2,\mu_2)(s)\big\|^2_{\boldsymbol{V}'\times\boldsymbol{Y}}ds.$$

Reiterating this inequality n times we obtain

$$\big\|\Pi^n(\eta_1,\mu_1)-\Pi^n(\eta_2,\mu_2)\big\|^2_{L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y})}\le\frac{C^nT^n}{n!}\big\|(\eta_1,\mu_1)-(\eta_2,\mu_2)\big\|^2_{L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y})}.$$

Thus, for n sufficiently large, Π^n is a contraction on the Banach space $L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y})$, and so Π has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof. Existence. Let $(\eta^*,\mu^*)\in L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y})$ be the fixed point of Π defined by (66)-(67) and denote

$$\boldsymbol{u}_*=\boldsymbol{u}_{\eta^*},\quad \psi_*=\psi_{\eta^*},\quad \beta_*=\beta_{\mu^*},\quad \omega_*=\omega_{\eta^*}, \tag{72}$$

$$\boldsymbol{\sigma}_*^\kappa=\mathcal{A}^\kappa\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\eta^*}^\kappa)+(\mathcal{E}^\kappa)^*\nabla\psi_{\eta^*}^\kappa+\boldsymbol{\sigma}_{\eta^*\mu^*}^\kappa,\quad \kappa=1,2, \tag{73}$$

$$\boldsymbol{D}_*^\kappa=\mathcal{E}^\kappa\varepsilon(\boldsymbol{u}_*^\kappa)-\mathcal{B}^\kappa\nabla\psi_*^\kappa,\quad \kappa=1,2. \tag{74}$$

We prove $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\boldsymbol{D}_*,\psi_*,\beta_*,\omega_*\}$ satisfies (29)-(35) and the regularities (36)-(41). Indeed, we write (42) for $\eta=\eta^*$ and use (72) to find

$$(\ddot{u}_*(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\kappa=1}^{2}(\mathcal{A}^\kappa\varepsilon(\dot{\boldsymbol{u}}_*^\kappa(t)),\ \varepsilon(\boldsymbol{v}^\kappa))_{\mathcal{H}^\kappa}+(\eta^*(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}}$$

$$= (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall\, \boldsymbol{v}\in\boldsymbol{V}, a.e.\ t\in(0,T). \tag{75}$$

Equation $\Pi^1(\eta^*,\mu^*) = \eta^*$ combined with (66) and (73) show that

$$\begin{aligned}(\eta^*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} &= \sum_{\kappa=1}^{2}\left(\mathcal{B}^\kappa\varepsilon(\boldsymbol{u}_*^\kappa(t)),\ \varepsilon(\boldsymbol{v}^\kappa)\right)_{\mathcal{H}^\kappa} + \sum_{\kappa=1}^{2}\left((\mathcal{E}^\kappa)^*\nabla\psi_*^\kappa,\ \varepsilon(\boldsymbol{v}^\kappa)\right)_{\mathcal{H}^\kappa}\\ &\quad + \sum_{\kappa=1}^{2}\left(\int_0^t \mathcal{F}^\kappa\big(\boldsymbol{\sigma}_*^\kappa - \mathcal{A}^\kappa\varepsilon(\dot{\boldsymbol{u}}_*^\kappa) - (\mathcal{E}^\kappa)^*\nabla\psi_*^\kappa, \varepsilon(\boldsymbol{u}_*^\kappa), \beta_*^\kappa\big)(s)ds\ ,\ \varepsilon(\boldsymbol{v}^\kappa)\right)_{\mathcal{H}^\kappa}\\ &\quad + j(\omega, \boldsymbol{u}_*(t), \boldsymbol{v}),\quad \forall\, \boldsymbol{v}\in\boldsymbol{V}.\end{aligned} \tag{76}$$

We substitute (76) in (75) and use (72)-(73) to see that (32) is satisfied. From $\Pi^2(\eta^*,\mu^*) = \mu^*$ and (57) we see that (30) is satisfied. We write now (43) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use (72) to find (33). Next, (35) and the regularities (36), (37), (38) follow from Lemma 4.1, and the relation (57). The regularity $\sigma\in L^2(0,T;\mathcal{H})$ follows from Lemmas 4.2, 4.3, assumptions H(1), H(3) and (73). Finally, (32) implies that

$$\rho^\kappa\ddot{\boldsymbol{u}}_*^\kappa = \mathrm{Div}\,\boldsymbol{\sigma}_*^\kappa + \boldsymbol{f}_0^\kappa \quad \text{a.e. } t\in[0,T],\ \kappa = 1,2$$

and from H(9)(a), H(9)(b) and (36) we find that $(\mathrm{Div}\,\boldsymbol{\sigma}_*^1, \mathrm{Div}\,\boldsymbol{\sigma}_*^2)\in L^2(0,T;\boldsymbol{V}')$. We deduce that the regularity (39) holds. Let now $t_1, t_2\in[0,T]$, by H(5), H(6), (22) and (74), we deduce that

$$\|\boldsymbol{D}_*(t_1) - \boldsymbol{D}_*(t_2)\|_H \le C\left(\|\varphi_*(t_1) - \varphi_*(t_2)\|_W + \|\boldsymbol{u}_*(t_1) - \boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\right).$$

The regularity of $\boldsymbol{u}_*$ and φ_* given by (36) and (38) implies

$$\boldsymbol{D}_*\in C(0,T;H). \tag{77}$$

For $\kappa = 1,2$, we choose $\phi = (\phi^1,\phi^2)$ with $\phi^\kappa\in D(\Omega^\kappa)^d$ and $\phi^{3-\kappa} = 0$ in (43) and using (74) we find

$$\mathrm{div}\,\boldsymbol{D}_*^\kappa(t) = q_0^\kappa(t) \quad \forall\, t\in[0,T],\ \ \kappa = 1,2. \tag{78}$$

Property (40) follows from H(9)(d), (77) and (78).

Uniqueness. The unique solution part of Theorem 4.1 is a consequence of the uniqueness of the fixed point of the operator Π defined by (65)-(67) and the unique solvability of the Problems $\mathbf{PV}_\eta^{u\psi}$, $\mathbf{PV}_{\eta\mu}^{\sigma}$ and $\mathbf{PV}_\eta^{\omega}$, which completes the proof. □

Conclusion

We presented a mathematical models for the frictional contact problem between two elastic-viscoplastic piezoelectric bodies with internal state variables. The contact is modelled with normal compliance and wear. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic equalities, differential equations and fixed point argument.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] O. Chau, J. R. Fernández, H. Han and M. Sofonea, Variational and numerical analysis of a dynamic frictionless contact problem with adhesion, *Journal of Computational and Applied Mathematics* **156** (2003), 127 – 157, DOI: 10.1016/s0377-0427(02)00909-3.

[2] A. Djabi, A. Merouani and A. Aissaoui, A frictional contact problem with wear involving elastic-viscoplastic materials with damage and thermal effects, *Electronic Journal of Qualitative Theory of Differential Equations* **27** (2015), 401 – 408, DOI: 10.14232/ejqtde.2015.1.27.

[3] O. Kavian, *Introduction à la théorie des points critiques et Applications aux équations elliptiques*, Springer-Verlag (1993).

[4] N. Kikuchi and J. T. Oden, *Contact Problems in Elasticity, A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia (1988), DOI: 10.1137/1.9781611970845.

[5] F. Maceri and P. Bisegna, The unilateral frictionless contact of a piezoelectric body with a rigid support, *Mathematical and Computer Modelling* **28** (1998), 19 – 28, DOI: 10.1016/s0895-7177(98)00105-8.

[6] J. A. C. Martins and J. T. Oden, Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws, *Nonlinear Analysis: Theory, Methods & Applications* **11** (1987), 407 – 428, DOI: 10.1016/0362-546x(87)90055-1.

[7] S. Migórski, Hemivariational inequality for a frictional contact problem in elasto-piezoelectricity, *Discrete & Continuous Dynamical Systems – B* **6** (2006), 1339 – 1356, DOI: 10.3934/dcdsb.2006.6.1339.

[8] J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction*, Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York (1981).

[9] J. T. Oden and J. A. C. Martins, Models and computational methods for dynamic friction phenomena, *Computer Methods in Applied Mechanics and Engineering* **52** (1985), 527 – 634, DOI: 10.1016/0045-7825(85)90009-x.

[10] M. Rochdi, M. Shillor and M. Sofonea, A quasistatic viscoelastic contact problem with normal compliance and friction, *Journal of Elasticity* **51** (1998), 105 – 126, DOI: 10.1023/A:1007413119583.

[11] M. Sofonea and R. Arhab, An electro-viscoelastic contact problem with adhesion, *Dynamics of Continuous, Discrete and Impulsive Systems* **14** (2007), 577 – 991.

[12] M. Sofonea and El-H. Essoufi, A piezoelectric contact problem with slip dépendent coefficient of friction, *Mathematical Modelling and Analysis* **9** (2004), 229 – 242, DOI: 10.3846/13926292.2004.9637256.

[13] M. Sofonea, W. Han and M. Shillor, *Analysis and Approximation of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics, **276**, Chapman-Hall/CRC Press, New York (2005), DOI: 10.1201/9781420034837.

[14] L. Zhor, Z. Zellagui, H. Benseridi and S. Drabla, Variational analysis of an electro viscoelastic contact problem with friction, *Journal of the Association of Arab Universities for Basic and Applied Sciences* **14** (2013), 93 – 100, DOI: 10.1016/j.jaubas.2012.10.001.