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Research Article

Perturbation-Iteration Method for Solving Differential-Difference Equations Having Boundary Layer

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Abstract. In this paper, a perturbation iteration method is described for solving differential-difference equations having boundary layer. Firstly, the given differential-difference equation having boundary layer is converted into a singularly perturbed ordinary differential equation using Taylor's transformation. Then perturbation iteration method applied to solve the resulting singularly perturbed ordinary differential equation. To demonstrate the applicability of this method, three model examples are solved. It is observed that the perturbation iteration method produces very good approximation to the exact solution.

Keywords. Differential-difference equations; Boundary layer; Perturbation iteration method

MSC. 65L11; 65Q10

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1. Introduction

Ordinary differential equation which contains a delay parameter is called delay differential equation. Ordinary differential equation which contains an advance parameter is called advanced differential equation. Ordinary differential equation which contains both delay and advance parameters is called differential-difference equation. In the literature, the expressions "positive shift" and "negative shift" are also used for "advance" and "delay" terms respectively. If the highest order derivative of the differential-difference equation is multiplied by small parameter then the solution will exhibit the boundary layer phenomenon. This type of differential-difference equations arise very frequently in the mathematical modelling of various practical phenomena for example: in the modelling of the human pupil-light reflex, model of HIV infection, the study of bi-stable devices in digital electronics, variational problem in control theory; first exit time problem in modelling of activation of neuronal variability, immune response, evolutionary biology, dynamics of networks of two identical amplifier, mathematical ecology, population dynamics, the modelling of biological oscillator and in a variety of models for physiological process. Solving these problems has become most interesting and challenging task for researchers.

Lange and Miura [11, 12] have published a series of papers for solving these problems. Chakravarthy and Reddy [4] have presented an initial value approach for the solution of singularly perturbation problems. Rao and Chakravarthy [16, 17] have constructed a scheme for solving partial differential-difference equations. Salama and Al-Amery [19] have given an asymptotic method for solving differential-difference equations. Reddy *et al*. [18] have presented a new scheme for solving singularly perturbed differential-difference equations. Venkat and Palli [20] have described a simple scheme of a non-linear differential-difference equations. Adilaxmi *et al*. [1] have presented an initial value technique using exponentially fitted non-standard finite difference method for singularly perturbed differential-difference equations. Kadalbajoo *et al*. [9, 10] have described the numerical treatment of boundary value problems for second order singularly perturbed delay differential equations. Pakdemirli [15] has described application of the perturbation iteration method to boundary layer type problems. Awoke and Reddy [2] have discussed the solution of singularly perturbed differential-difference equations via fitted method. General information and theory of singular perturbation problems is available in Bellman and Cooke [3], Driver [5], El'sgol'ts and Norkin [7], Hale [8], Nayfeh [13], O'Malley [14], and van Dyke [6].

Please add here one line introduction of each section.

2. Description of the Perturbation-Iteration Method

2.1 Case I: Delay Differential Equations having Boundary Layer

Consider the delay differential equation of the form:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = 0, \quad 0 \le x \le 1, \tag{1}$$

with the boundary conditions

$$y(0) = \alpha, \quad -\delta \le x \le 0, \tag{2}$$

and

$$y(1) = \beta, \tag{3}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$ is the small delay parameter, $a(x)$ and $b(x)$ are sufficiently differentiable functions in $(0,1)$ and α, β are constants. Further, it is established that, when $a(x) \ge M > 0$ in $[0,1]$, boundary layer will be at $x = 0$ and when $a(x) \le M < 0$ in $[0,1]$, boundary layer will be at $x = 1$, where M is some positive number.

We have from Taylor series expansion

$$y'(x-\delta) \approx y'(x) - \delta y''(x). \tag{4}$$

Using eq. (4) in eq. (1), we get singularly perturbed ordinary differential equation:

$$\varepsilon' y''(x) + a(x)y'(x) + b(x)y(x) = 0, \tag{5}$$

where $\varepsilon' = \varepsilon - a(x)\delta$. We re-write eq. (5) as

$$F(y'', y', y, \varepsilon') = \varepsilon' y''(x) + a(x)y'(x) + b(x)y(x) = 0. \tag{6}$$

Now, we describe the perturbation iteration method for solving the eq. (6). We start with y_o as the initial guess and define:

$$y_{n+1} = y_n + \varepsilon'(y_c)_n, \quad n = 0, 1, 2, \dots, \tag{7}$$

where y_c is the correction term. Substituting the eq. (7) in eq. (6) and expanding by the Taylor series, we get:

$$\begin{aligned} &F(y_n'', y_n', y_n, 0) + F_y(y_n'', y_n', y_n, 0)\varepsilon'(y_c)_n + F_{y'}(y_n'', y_n', y_n, 0)\varepsilon'(y_c')_n \\ &\quad + F_{y''}(y_n'', y_n', y_n, 0)\varepsilon'(y_c'')_n + F_{\varepsilon'}(y_n'', y_n', y_n, 0)\varepsilon' = 0, \end{aligned} \tag{8}$$

where $F_y = \frac{\partial F}{\partial y}$, $F_{y'} = \frac{\partial F}{\partial y'}$, $F_{y''} = \frac{\partial F}{\partial y''}$, $F_{\varepsilon'} = \frac{\partial F}{\partial \varepsilon'}$ and all derivatives are evaluated at $\varepsilon' = 0$.

Clearly, eq. (8) is a variable coefficient non-homogeneous linear second order differential equation with respect to the unknown $(y_c)_n$ in its most general form. The Iteration procedure will start with an initial guess y_0, firstly $(y_c)_0$ is calculated from eq. (8) and then its value is substituted in eq. (7) to calculate the value of y_1. This iteration process is repeated using eq. (8) and eq. (7) until a satisfactory result is obtained. To solve the boundary layer problems, for the outer solution, eq. (5) is iterated. For the inner solution, eq. (5) is expressed in terms of the boundary layer variable by using the stretching transformation and then transformed equation is iterated according to eq. (6) to eq. (8). After getting the outer solution and inner solution, we use matching principle and then obtain the composite solution which will valid throughout the domain.

Matching Principle. Van Dyke [6] has proposed the *Matching Principle*, namely: *'inner limit of outer solution = outer limit of inner solution'* in overlapping region. That is, the outer solution is written in terms of the inner variable and the inner solution is written in terms of the outer variable and then both are equated.

2.2 Case-II: Differential-Difference Equations having Boundary Layer

Consider the differential-difference equation of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x-\delta) + c(x)y(x) + d(x)y(x+\eta) = 0, \tag{9}$$

$0 < x < 1$ with the boundary conditions

$$y(x) = \varphi(x), \text{ on } -\delta \le x \le 0, \tag{10}$$

$$y(x) = \gamma(x), \text{ on } 1 \le x \le 1+\eta, \tag{11}$$

with the constant coefficients (i.e. $a(x) = a$, $b(x) = b$, $c(x) = c$, $d(x) = d$, $\varphi(x) = \varphi$, $\gamma(x) = \gamma$ are constants) where $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$ and $0 < \eta = O(\varepsilon)$ are the delay and advanced parameters, respectively.

If $b(x)+c(x)+d(x) \le 0$, $a(x)-\delta b(x)+\eta d(x) \ge M > 0$ in $[0,1]$ then eq. (9) has unique solution and a boundary layer at $x=0$ where M is a positive number.

If $b(x)+c(x)+d(x) \le 0$, $a(x)-\delta b(x)+\eta d(x) \le M < 0$ in $[0,1]$ then eq. (9) has unique solution and a boundary layer at $x=1$ where M is a positive number.

Using Taylor series expansion, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x), \tag{12}$$

$$y(x+\eta) \approx y(x) + \eta y'(x) + \frac{\eta^2}{2} y''(x). \tag{13}$$

Substituting eqs. (12)-(13) in eq. (9), we get singularly perturbed ordinary differential equation

$$\varepsilon' y''(x) + a'(x)y'(x) + b'(x)y(x) = 0, \tag{14}$$

where

$$\varepsilon' = \varepsilon + b(x)\frac{\delta^2}{2} + d(x)\frac{\eta^2}{2}, \tag{15}$$

$$a'(x) = a(x) - \delta b(x) + \eta d(x), \tag{16}$$

$$b'(x) = b(x) + c(x) + d(x). \tag{17}$$

Perturbation iteration method with mutatis mutandis is applied for solving the eq. (14).

3. Numerical Experiments

In this section, three model examples are solved and the solutions are compared with the exact/available solutions.

Example 3.1. Consider the delay differential equation having left boundary layer:

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0, \quad 0 \le x \le 1; \tag{18}$$

with the boundary conditions

$$y(0) = 1, \quad -\delta \le x \le 0, \tag{19}$$

and

$$y(1) = 1. \tag{20}$$

Using eq. (4) in eq. (18), we get singularly perturbed ordinary differential equation:

$$\varepsilon' y''(x) + y'(x) - y(x) = 0. \tag{21}$$

where$\varepsilon' = \varepsilon - \delta$.

The outer solution: We re-write eq. (21) as:

$$F(y'', y', y, \varepsilon') = \varepsilon' y''(x) + y'(x) - y(x) = 0. \tag{22}$$

For this problem, boundary layer is located at $x = 0$. Therefore, the outer solution will not satisfy the condition at $x = 0$. Substituting eq. (22) in eq. (8) and we get:

$$y_n' - y_n - \varepsilon'(y_c)_n + \varepsilon'(y_c')_n + \varepsilon' y_n'' = 0, \quad n = 0, 1, 2, \dots. \tag{23}$$

Initial guess is taken as:

$$y_0 = 1. \tag{24}$$

Substituting eq. (24) into eq. (23) for $n=0$ and solving for $(y_c)_0$ we get

$$(y_c)_0 = c_1 e^x - \frac{1}{\varepsilon'}. \tag{25}$$

Therefore, from eq. (7) we get y_1 as

$$y_1 = y_0 + \varepsilon'(y_c)_0 = c_1 \varepsilon' e^x. \tag{26}$$

Applying the right-side boundary condition (i.e. $y(1)=1$) we get the value of arbitrary constant c_1.

$$y_1 = e^{(x-1)}. \tag{27}$$

Similarly for $n=1$, we get the y_2 is

$$y_2 = e^{(x-1)} + \varepsilon' e^{(x-1)}(1-x). \tag{28}$$

This solution does not satisfy the boundary condition given at left-side i.e. $y_2(0) \neq 0$.

The inner solution: To obtain the inner solution, we use the stretching transformation

$$\xi = \frac{x}{\varepsilon'}. \tag{29}$$

Substituting eq. (29) in eq. (21), we get the equation:

$$F(Y'', Y', Y, \varepsilon') = Y'' + Y' - \varepsilon' Y = 0, \tag{30}$$

where $Y = Y(\xi)$. Substituting eq. (30) in eq. (8) and re-arranging the terms we get:

$$Y_n'' + Y_n' + \varepsilon'(Y_C')_n + \varepsilon'(Y_C'')_n - \varepsilon' Y_n = 0, \quad n = 0,1,2,\dots. \tag{31}$$

We take the initial guess as:

$$Y_0 = 1 \quad (\text{for } n=0). \tag{32}$$

Then, we get the successive two iterations as:

$$Y_1 = 1 + \varepsilon' c_3(1-e^{-\xi}) + \varepsilon'\xi, \tag{33}$$

$$Y_2 = 1 + \varepsilon' c_3(1-e^{-\xi}) + \varepsilon'\xi + \varepsilon'\left[c_6(1-e^{-\xi}) + \varepsilon'\xi\left\{c_3\left(1+\frac{e^{-\xi}}{2}\right) + \frac{\xi}{2} - 1\right\}\right]. \tag{34}$$

The arbitrary constants involved in eq. (33) and eq. (34) are determined by the Matching Principle.

$$(y_2)^i = e^{(\xi\varepsilon'-1)} + \varepsilon' e^{(\xi\varepsilon'-1)}(1-\xi\varepsilon'). \tag{35}$$

Taking approximation up to two terms for fixed value of ξ we have

$$(y_2)^i \cong e^{-1}(1+\xi\varepsilon') + \varepsilon' e^{-1} + \dots. \tag{36}$$

By writing eq. (36) in terms of variable x.

$$(y_2)^i \cong e^{-1}(1+x) + \varepsilon' e^{-1} + \dots. \tag{37}$$

Inner solution, in terms of the outer variable x, becomes:

$$(Y_2)^o = 1 + \varepsilon' c_3(1-e^{-x/\varepsilon'}) + x + \varepsilon'\left[c_6(1-e^{-x/\varepsilon'}) + x\left\{c_3\left(1+\frac{e^{-\frac{x}{\varepsilon'}}}{2}\right) + \frac{x}{2\varepsilon'} - 1\right\}\right]. \tag{38}$$

Taking approximation up to two terms only, as terms $e^{-x/\varepsilon'}$ and x^2 can be neglected, we get:

$$(Y_2)^o = (1+\varepsilon' c_3)(1+x) + \varepsilon' c_6. \tag{39}$$

From eq. (37) and eq. (39), we get:

$$1+\varepsilon' c_3 = e^{-1} \quad \text{and} \quad c_6 = e^{-1}. \tag{40}$$

Therefore, inner and outer solutions, in terms of original variable x can be written as

$$Y_2 = 1+(e^{-1}-1)(1-e^{-\frac{x}{\varepsilon'}})+x+\varepsilon'\left[e^{-1}(1-e^{-x/\varepsilon'})+x\left\{\frac{(e^{-1}-1)}{\varepsilon'}\left(1+\frac{e^{-\frac{x}{\varepsilon'}}}{2}\right)+\frac{x}{2\varepsilon'}-1\right\}\right], \tag{41}$$

$$y_2 = e^{(x-1)}+\varepsilon' e^{(x-1)}(1-x). \tag{42}$$

The solution which is valid throughout the domain, is given by

$$y = Y_2 + y_2 - (y_2)^i. \tag{43}$$

Substituting eq. (41), eq. (42) and eq. (37) in eq. (43), we get the composite expansion as:

$$y = e^{x-1} - e^{(-\frac{x}{\varepsilon'}-1)} + e^{-\frac{x}{\varepsilon'}} + \varepsilon' e^{-1}(1-e^{-\frac{x}{\varepsilon'}}) + x\frac{e^{(-\frac{x}{\varepsilon'}-1)}}{2} - x\frac{e^{-\frac{x}{\varepsilon'}}}{2}. \tag{44}$$

The exact solution of eq. (18) is

$$y = \frac{(1-e^{m_2})e^{m_1 x}+(e^{m_1}-1)e^{m_2 x}}{(e^{m_1}-e^{m_2})}, \tag{45}$$

where

$$m_1 = \frac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}, \tag{46}$$

$$m_2 = \frac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}. \tag{47}$$

Results are shown in Tables 1 and 2, and the layer behaviour in Figures 1 and 2.

Table 1. Results for Example 3.1 with $h = 0.01$, $\varepsilon = 0.001$ and $\delta = 0.0001$

x	Present solution	Exact solution
0.0	1.00000000	1.00000000
0.02	0.37564219	0.37564167
0.04	0.38322397	0.38322325
0.06	0.39095892	0.39095785
0.08	0.39885013	0.39884855
0.1	0.40690075	0.40689852
0.2	0.44966005	0.44965201
0.3	0.49691639	0.49689768
0.4	0.54914272	0.54910754
0.5	0.60686175	0.60680316
0.6	0.67065113	0.67056097
0.7	0.74114931	0.74101790
0.8	0.81906184	0.81887787
0.9	0.90516850	0.90491871
1.0	1.00033109	1.00000000

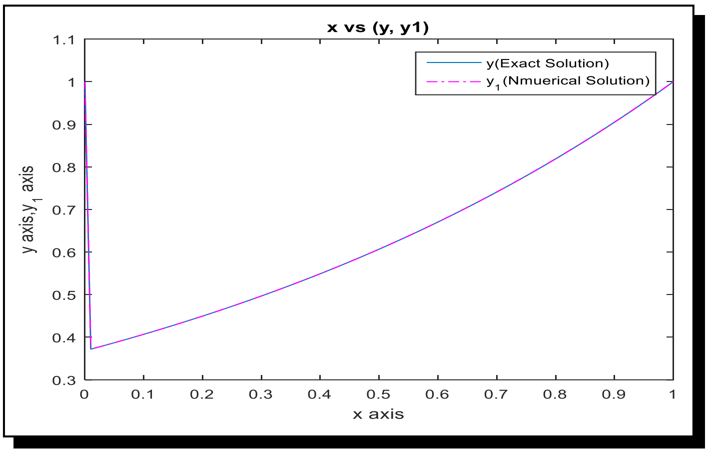

Figure 1. Example 3.1 with $h = 0.01$, $\varepsilon = 0.001$ and $\delta = 0.0001$

Table 2. Results for Example 3.1 with $h = 0.01$, $\varepsilon = 0.0001$ and $\delta = 0.00001$

x	Present solution	Exact solution
0.0	1.00000000	1.00000000
0.02	0.37534420	0.37534419
0.04	0.38292599	0.38292596
0.06	0.39066094	0.39066087
0.08	0.39855215	0.39855203
0.1	0.40660276	0.40660258
0.2	0.44936207	0.44936131
0.3	0.49661841	0.49661658
0.4	0.54884474	0.54884126
0.5	0.60656376	0.60655794
0.6	0.67035315	0.67034417
0.7	0.74085132	0.74083821
0.8	0.81876386	0.81874548
0.9	0.90487052	0.90484556
1.0	1.00003310	1.00000000

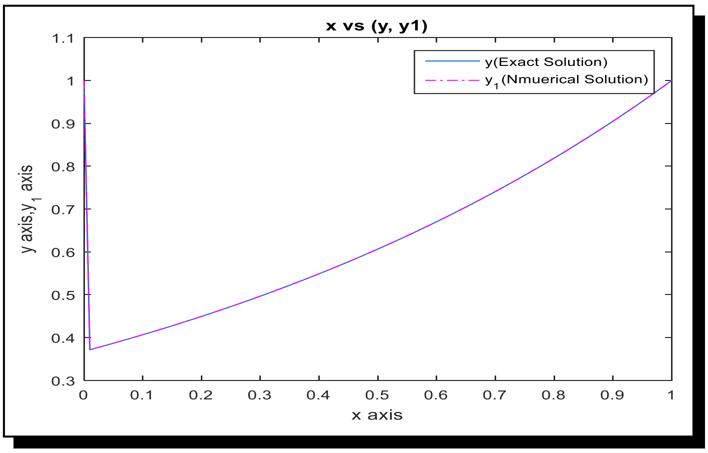

Figure 2. Example 3.1 with $h = 0.01$, $\varepsilon = 0.0001$ and $\delta = 0.00001$

Example 3.2. Now, we consider the delay differential equation having right boundary layer

$$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0, \quad 0 \le x \le 1, \tag{48}$$

with the boundary conditions

$$y(0) = 1, \quad -\delta \le x \le 0, \tag{49}$$

and

$$y(1) = -1. \tag{50}$$

Using eq. (4) into eq. (48), we get singularly perturbed ordinary differential equation

$$\varepsilon' y''(x) - y'(x) - y(x) = 0, \tag{51}$$

where $\varepsilon' = \varepsilon + \delta$.

The outer solution: To obtain the outer solution, we write eq. (51) as:

$$F(y'', y', y, \varepsilon') = \varepsilon' y''(x) - y'(x) - y(x) = 0. \tag{52}$$

In this problem, boundary layer is at $x = 1$. Therefore, the outer solution will not satisfy the boundary condition at $x = 1$. Substituting eq. (52) in eq. (8) and get

$$-y'_n - y_n - \varepsilon'(y_c)_n - \varepsilon'(y'_c)_n + \varepsilon' y''_n = 0, \quad n = 0, 1, 2, \ldots \tag{53}$$

Initial guess is taken as:

$$y_0 = 1. \tag{54}$$

Substituting eq. (54) in eq. (53) for $n = 0$ and solving we get:

The first iteration solution as

$$y_1 = e^{-x}. \tag{55}$$

Similarly, the second iteration result for $n = 1$, we get y_2 as

$$y_2 = e^{-x} + \varepsilon' x e^{-x}. \tag{56}$$

This solution does not satisfy the boundary condition at right side i.e. $y_2(1) \ne -1$.

The inner solution: To obtain the inner solution, we use the stretching variable as

$$\xi = \frac{1-x}{\varepsilon'}. \tag{57}$$

Substituting eq. (57) in eq. (51), we get

$$F(Y'', Y', Y, \varepsilon') = Y'' + Y' - \varepsilon' Y = 0, \tag{58}$$

where $Y = Y(\xi)$. Substituting eq. (58) in eq. (8) and re-arranging terms, we get:

$$Y''_n + Y'_n + \varepsilon'(Y'_C)_n + \varepsilon'(Y''_C)_n - \varepsilon' Y_n = 0, \quad n = 0, 1, 2, \ldots. \tag{59}$$

Initial guess is taken as:

$$Y_0 = -1 \quad (\text{for } n = 0). \tag{60}$$

Then, we get the successive two iterations as:

$$Y_1 = -1 + \varepsilon' c_3(1 - e^{-\xi}) - \varepsilon'\xi, \tag{61}$$

$$Y_2 = -1 + \varepsilon' c_3(1 - e^{-\xi}) - \varepsilon'\xi + \varepsilon'\left[c_6(1 - e^{-\xi}) + \varepsilon'\xi\left\{c_3\left(1 + \frac{e^{-\xi}}{2}\right) - \frac{\xi}{2} + 1\right\}\right]. \tag{62}$$

The arbitrary constants involved in eq. (61) and eq. (62) are determined by the Matching Principle.

Matching Principle. Taking approximation up to two terms only for fixed value of ξ we have:

$$(Y_2)^i \cong e^{-1}(2 - x) + \varepsilon' e^{-1} + \dots. \tag{63}$$

From eq. (62), inner solution, in terms of the outer variable x becomes:

$$(Y_2)^o = -1 + \varepsilon' c_3\left(1 - e^{-\frac{(1-x)}{\varepsilon'}}\right) - (1-x) + \varepsilon'\left[c_6\left(1 - e^{-\frac{(1-x)}{\varepsilon'}}\right) + (1-x)\left\{c_3\left(1 + \frac{e^{-\frac{(1-x)}{\varepsilon'}}}{2}\right) - \frac{(1-x)}{2\varepsilon'} + 1\right\}\right]. \tag{64}$$

Taking approximation up to two terms only as the terms $e^{-\frac{(1-x)}{\varepsilon'}}$ and $(1-x)^2$ can be neglected, we get:

$$(Y_2)^o = (\varepsilon' c_3 - 1)(2 - x) + \varepsilon' c_6. \tag{65}$$

Equating eq. (63) and eq. (65)

$$\varepsilon' c_3 - 1 = e^{-1} \quad \text{and} \quad c_6 = e^{-1}. \tag{66}$$

Therefore, inner and outer solutions, in terms of original variable x can be written as

$$Y_2 = -1 + (e^{-1} + 1)\left(1 - e^{-\frac{(1-x)}{\varepsilon'}}\right) - (1-x)$$
$$+ \varepsilon'\left[e^{-1}\left(1 - e^{-\frac{(1-x)}{\varepsilon'}}\right) + (1-x)\left\{\frac{(e^{-1}+1)}{\varepsilon'}\left(1 + \frac{e^{-\frac{(1-x)}{\varepsilon'}}}{2}\right) - \frac{(1-x)}{2\varepsilon'} + 1\right\}\right], \tag{67}$$

$$y_2 = e^{-x} + \varepsilon' x e^{-x}. \tag{68}$$

The solution which is valid throughout the domain, is given by

$$y = Y_2 + y_2 - (y_2)^i. \tag{69}$$

Substituting eq. (67), eq. (68) and eq. (63) in eq. (69), we get the composite expansion as

$$y = e^{-x} + \varepsilon' x e^{-x} - e^{-\frac{(1-x)}{\varepsilon'}}\left[1 + e^{-1} + \varepsilon' e^{-1} - \frac{(1-x)(1+e^{-1})}{2}\right]. \tag{70}$$

The exact solution of eq. (48) is

$$y = \frac{(1 + e^{m_2})e^{m_1 x} - (e^{m_1} + 1)e^{m_2 x}}{(e^{m_2} - e^{m_1})}, \tag{71}$$

where

$$m_1 = \frac{1 - \sqrt{1 + 4(\varepsilon + \delta)}}{2(\varepsilon + \delta)}, \tag{72}$$

$$m_2 = \frac{1 + \sqrt{1 + 4(\varepsilon + \delta)}}{2(\varepsilon + \delta)}. \tag{73}$$

Results are shown in Tables 3 and 4 and the layer behaviour in Figures 3 and 4.

Table 3. Results for Example 3.2 with $h = 0.01$, $\varepsilon = 0.002$ and $\delta = 0.0003$

x	Present solution	Exact solution
0.0	1.00000000	1.00000000
0.1	0.90499124	0.90504460
0.2	0.81900912	0.81910573
0.3	0.74119603	0.74132722
0.4	0.67077586	0.67093420
0.5	0.60704621	0.60722537
0.6	0.54937142	0.54956605
0.7	0.49717624	0.49738178
0.8	0.44994005	0.45015270
0.9	0.40719171	0.40740827
0.92	0.39914232	0.39935933
0.94	0.39125205	0.39146941
0.96	0.38351776	0.38373533
0.98	0.37592583	0.37592964
1.0	−1.00000000	−1.00000000

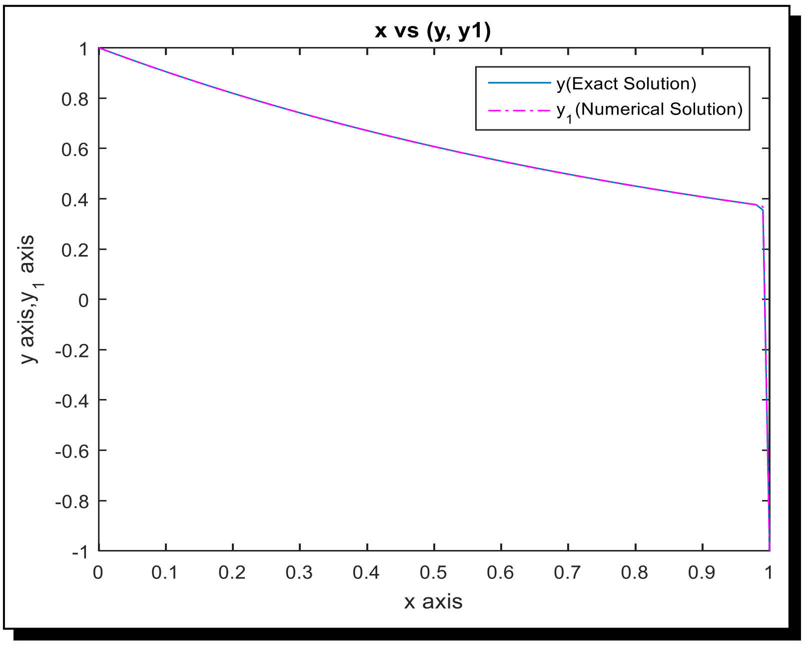

Figure 3. Example 3.2 with $h = 0.01$, $\varepsilon = 0.002$ and $\delta = 0.0003$

Table 4. Results for Example 3.2 with $h = 0.01$, $\varepsilon = 0.002$ and $\delta = 0.00001$

x	Present solution	Exact solution
0.0	1.00000000	1.00000000
0.1	0.90501748	0.90501858
0.2	0.81905660	0.81905863
0.3	0.74126048	0.74126328
0.4	0.67085362	0.67085704
0.5	0.60713415	0.60713808
0.6	0.54946691	0.54947125
0.7	0.49727704	0.49728169
0.8	0.45004429	0.45004917
0.9	0.40729782	0.40730286
0.92	0.39924864	0.39925371
0.94	0.39135854	0.39136362
0.96	0.38362436	0.38362946
0.98	0.37598453	0.37598413
1.0	−1.00000000	−1.00000000

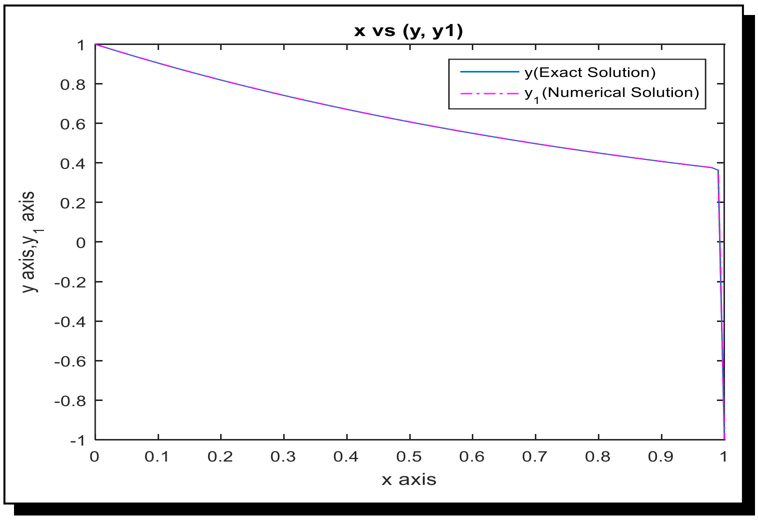

Figure 4. Example 3.2 with $h = 0.01$, $\varepsilon = 0.002$ and $\delta = 0.00001$

Example 3.3. Consider the differential-differential equation having left boundary layer

$$\varepsilon y''(x) + y'(x) - 3y(x) + 2y(x+\eta) = 0, \quad 0 \le x \le 1, \tag{74}$$

with the boundary conditions

$$y(0) = 1, \quad -\delta \le x \le 0, \tag{75}$$

and

$$y(1) = 1, \quad 1 \le x \le 1+\eta, \tag{76}$$

Using Taylor's series expansion, we have:

$$y(x+\eta) = y(x) + \eta y'(x) + \frac{\eta^2}{2} y''(x). \tag{77}$$

Substitute eq. (77) in eq. (74), we get singularly perturbed ordinary differential equation as:

$$\varepsilon' y''(x) + (1+2\eta) y'(x) - y(x) = 0, \tag{78}$$

where $\varepsilon' = \varepsilon + \eta^2$.

The outer solution: To obtain the outer solution, we write eq. (78) as:

$$F(y'', y', y, \varepsilon') = \varepsilon' y''(x) + (1+2\eta) y'(x) - y(x) = 0. \tag{79}$$

In this problem, boundary layer is located at $x = 0$. Therefore, the outer solution will not satisfy the boundary condition at $x = 0$.

Substituting eq. (79) in eq. (8) and get:

$$(1+2\eta) y_n' - y_n - \varepsilon'(y_c)_n + (1+2\eta)\varepsilon'(y_c')_n + \varepsilon' y_n'' = 0, \quad n = 0, 1, 2, \dots. \tag{80}$$

Initial guess is taken as:

$$y_0 = 1. \tag{81}$$

Substituting eq. (81) in eq. (80) for $n = 0$ and solving we get:

The first iteration solution is

$$y_1 = e^{\frac{(x-1)}{(1+2\eta)}}. \tag{82}$$

Similarly, the second iteration result for $n = 1$, we get y_2 is

$$y_2 = e^{\frac{(x-1)}{(1+2\eta)}} + \varepsilon' \frac{(1-x)}{(1+2\eta)^3} e^{\frac{(x-1)}{(1+2\eta)}}. \tag{83}$$

This solution does not satisfy the boundary condition at left side i.e. $y_2(0) \neq 1$.

The inner solution: To obtain the inner solution, the stretching variable is defined as

$$\xi = \frac{x}{\varepsilon'}. \tag{84}$$

Substitute eq. (84) into eq. (78), the transformed equation is considered as

$$F(Y'', Y', Y, \varepsilon') = Y'' + (1+2\eta) Y' - \varepsilon' Y = 0, \tag{85}$$

where $Y = Y(\xi)$. Substitute eq. (85) into eq. (8) and re-arrange the terms, then:

$$Y_n'' + (1+2\eta) Y_n' + (1+2\eta)\varepsilon'(Y_C')_n + \varepsilon'(Y_C'')_n - \varepsilon' Y_n = 0, \quad n = 0, 1, 2, \dots. \tag{86}$$

Initial guess is taken as:

$$Y_0 = 1 \quad (\text{for } = 0). \tag{87}$$

Following the same procedure, the successive two iteration are

$$Y_1 = 1 + \varepsilon' c_3 (1 - e^{-(1+2\eta)\xi}) + \varepsilon' \frac{\xi}{(1+2\eta)}, \tag{88}$$

$$Y_2 = 1 + \varepsilon' c_3 (1 - e^{-(1+2\eta)\xi}) + \varepsilon' \frac{\xi}{(1+2\eta)}$$
$$+ \varepsilon' \left[c_6 (1 - e^{-(1+2\eta)\xi}) + \varepsilon' \frac{\xi}{(1+2\eta)} \left\{ c_3 \left(1 + \frac{e^{-(1+2\eta)\xi}}{2} \right) + \frac{\xi}{2(1+2\eta)} - \frac{1}{(1+2\eta)^2} \right\} \right]. \tag{89}$$

The arbitrary constants involved in eq. (88) and eq. (89) are determined by the Matching Principle.

Matching Principle. Taking approximation up to two terms only for fixed value of ξ.

$$(y_2)^i \cong e^{-\frac{1}{(1+2\eta)}} \left(1 + \frac{x}{(1+2\eta)} \right) + \varepsilon' \frac{e^{-\frac{1}{(1+2\eta)}}}{(1+2\eta)^3} + \dots. \tag{90}$$

From eq. (89), inner solution in terms of the outer variable x is

$$(Y_2)^o = 1 + \varepsilon' c_3 \left(1 - e^{-(1+2\eta)x/\varepsilon'} \right) + \frac{x}{(1+2\eta)}$$
$$+ \varepsilon' \left[c_6 \left(1 - e^{-(1+2\eta)x/\varepsilon'} \right) + \frac{x}{(1+2\eta)} \left\{ c_3 \left(1 + \frac{e^{-(1+2\eta)x/\varepsilon'}}{2} \right) + \frac{x/\varepsilon'}{2(1+2\eta)} - \frac{1}{(1+2\eta)^2} \right\} \right]. \tag{91}$$

Taking approximation upto two terms only, as the terms $e^{-\frac{(1+2\eta)(x)}{\varepsilon'}}$ and x^2 can be neglected, we get:

$$(Y_2)^o = \left(1 + \frac{x}{(1+2\eta)} \right) (1 + \varepsilon' c_3) + \varepsilon' c_6. \tag{92}$$

Equating eq. (90) and eq. (92)

$$\varepsilon' c_3 + 1 = e^{-\frac{1}{(1+2\eta)}} \quad \text{and} \quad c_6 = \frac{e^{-\frac{1}{(1+2\eta)}}}{(1+2\eta)^3}. \tag{93}$$

Therefore, inner and outer solutions, in terms of original variable x can be written as

$$Y_2 = 1 + (e^{-\frac{1}{(1+2\eta)}} - 1)(1 - e^{-(1+2\eta)x/\varepsilon'}) + \frac{x}{(1+2\eta)} + \varepsilon' \left[\frac{e^{-\frac{1}{(1+2\eta)}}}{(1+2\eta)^3} (1 - e^{-(1+2\eta)x/\varepsilon'}) \right.$$
$$\left. + \frac{x}{(1+2\eta)} \left\{ \frac{(e^{-\frac{1}{(1+2\eta)}} - 1)}{\varepsilon'} \left(1 + \frac{e^{-(1+2\eta)x/\varepsilon'}}{2} \right) + \frac{x/\varepsilon'}{2(1+2\eta)} - \frac{1}{(1+2\eta)^2} \right\} \right]. \tag{94}$$

$$y_2 = e^{\frac{(x-1)}{(1+2\eta)}} + \varepsilon' \frac{(1-x)}{(1+2\eta)^3} e^{\frac{(x-1)}{(1+2\eta)}}. \tag{95}$$

The solution which is valid throughout the domain is given by

$$y = Y_2 + y_2 - (y_2)^i. \tag{96}$$

Substituting eq. (94), eq. (95) and eq. (90) in eq. (96), we get the composite expansion is

$$y = e^{\frac{(x-1)}{(1+2\eta)}} - e^{(-\frac{(1+2\eta)x}{\varepsilon'} - \frac{1}{(1+2\eta)})} \left(1 - \frac{x}{2(1+2\eta)} \right) + e^{-\frac{(1+2\eta)x}{\varepsilon'}} + \varepsilon' \frac{e^{-\frac{1}{(1+2\eta)}}}{(1+2\eta)^3} (1 - e^{-\frac{(1+2\eta)x}{\varepsilon'}}) - \frac{x e^{-\frac{(1+2\eta)x}{\varepsilon'}}}{2(1+2\eta)}. \tag{97}$$

The exact solution of eq. (74) is

$$y = \frac{(-1 + e^{m_2}) e^{m_1 x} + (1 - e^{m_1}) e^{m_2 x}}{(e^{m_2} - e^{m_1})}, \tag{98}$$

where

$$m_1 = \frac{-(1+2\eta)+\sqrt{(1+2\eta)^2+4\varepsilon}}{2\varepsilon}, \tag{99}$$

$$m_2 = \frac{-(1+2\eta)-\sqrt{(1+2\eta)^2+4\varepsilon}}{2\varepsilon}. \tag{100}$$

Results are shown in Tables 5 and 6 and the layer behaviour in Figures 5 and 6.

Table 5. Results for Example 3.3 with $h = 0.01$, $\varepsilon = 0.001$ and $\eta = 0.0001$

x	Present solution	Exact solution
0.0	1.00000000	1.00000000
0.02	0.37575238	0.37575175
0.04	0.38333412	0.38333326
0.06	0.39106900	0.39106774
0.08	0.39896009	0.39895828
0.1	0.40701057	0.40700802
0.2	0.44976858	0.44975957
0.3	0.49702255	0.49700169
0.4	0.54924522	0.54920605
0.5	0.60695903	0.60689388
0.6	0.67074139	0.67064117
0.7	0.74123039	0.74108437
0.8	0.81913123	0.81892684
0.9	0.90522324	0.90494576
1.0	1.00036773	1.00000000

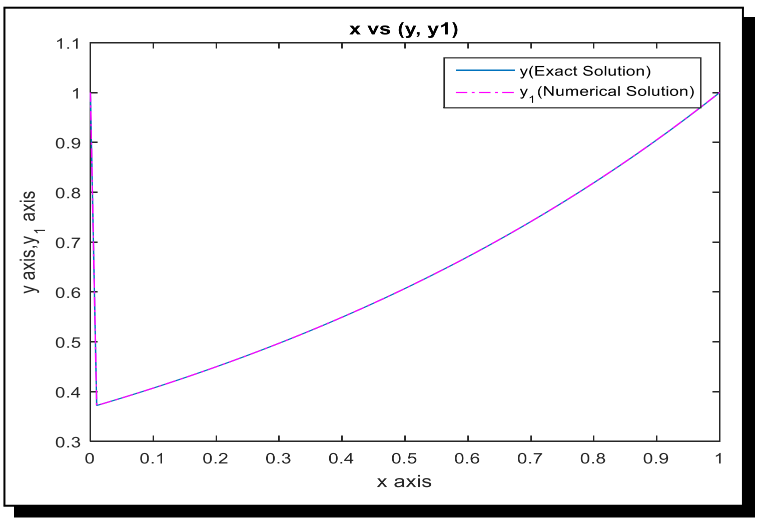

Figure 5. Example 3.3 with $h = 0.01$, $\varepsilon = 0.001$ and $\eta = 0.0001$

Table 6. Results for Example 3.3 with $h = 0.01$, $\varepsilon = 0.002$ and $\eta = 0.0003$

x	Present solution	Exact solution
0.0	1.00000000	1.00000000
0.02	0.37629483	0.37629214
0.04	0.38384827	0.38384539
0.06	0.39158298	0.39157932
0.08	0.39947386	0.39946906
0.1	0.40752404	0.40751778
0.2	0.45027947	0.45026025
0.3	0.49752869	0.49748577
0.4	0.54974403	0.54966452
0.5	0.60744744	0.60731605
0.6	0.67121575	0.67101434
0.7	0.74168640	0.74139363
0.8	0.81956385	0.81915465
0.9	0.90562658	0.90507163
1.0	1.00073490	1.00000000

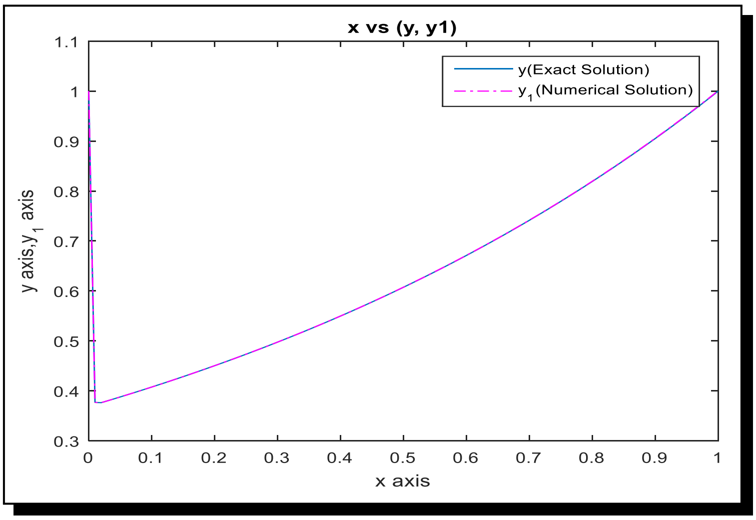

Figure 6. Example 3.3 with $h = 0.01$, $\varepsilon = 0.002$ and $\eta = 0.0003$

4. Discussion and Conclusions

We have described the Perturbation iteration method for solving differential-difference equations having boundary layer. Firstly, the given differential-difference equation having boundary layer is converted into a singularly perturbed differential equation using Taylor's transformations. Then perturbation iteration method applied to solve the resulting singularly perturbed differential equation.We have implemented this on three model examples, (i) a delay differential equation having left boundary layer, (ii) a delay differential equation having right boundary layer and (iii) a differential-difference equation having left boundary layer. Computational results and layer behaviour are presented in tables and figures and for different values of the parameters. It is observed from tables that our solutions approximate the exact solutions very well.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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