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Research Article

Some Results on 2-Vertex Switching in Joints

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Abstract. For a finite undirected graph $G(V,E)$ and a non empty subset $\sigma \subseteq V$, the *switching* of G by σ is defined as the graph $G^{\sigma}(V,E')$ which is obtained from G by removing all edges between σ and its complement V-σ and adding as edges all non-edges between σ and V-σ. For $\sigma = \{v\}$, we write G^{v} instead of $G^{\{v\}}$ and the corresponding switching is called as *vertex switching*. We also call it as $|\sigma|$-vertex switching. When $|\sigma| = 2$, we call it as 2-vertex switching. A subgraph B of G which contains $G[\sigma]$ is called a *joint* at σ in G if B-σ is connected and maximal. If B is connected, then we call B as *c-joint* otherwise *d-joint*. In this paper, we give a necessary and sufficient condition for a c-joint B at $\sigma = \{u,v\}$ in G to be a c-joint and a d-joint at σ in G^{σ} and also a necessary and sufficient condition for a d-joint B at $\sigma = \{u,v\}$ in G to be a c-joint and a d-joint at σ in G^{σ} when $uv \in E(G)$ and when $uv \notin E(G)$.

Keywords. Switching; 2-vertex self switching; $SS_2(G)$; $ss_2(G)$

MSC. 05C60

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1. Introduction

For a finite undirected simple graph $G(V,E)$ with $|V(G)| = p$ and a non-empty set $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^{\sigma}(V,E')$ which is obtained from G by removing all edges between σ and its complement, V-σ and adding as edges all non-edges between σ and V-σ. Switching has been defined by Seidel ([7], [3]) and is also referred to as Seidel switching. We also call it as $|\sigma|$-vertex switching. When $|\sigma| = 1$, we call it as 1-vertex switching [4]. When $|\sigma| = 2$, we call it as 2-vertex switching. A subgraph B of G which contains $G[\sigma]$ is called a *joint* at σ in G if B-σ is connected and maximal. If B is connected, then we call B as *c-joint* otherwise *d-joint*. B is called a *total joint* if B is the join of σ and B-σ, that is $B = \sigma + (B - \sigma)$ [5,6]. When $\sigma = \{v\} \subset V$, the corresponding switching G^v is called as the vertex switching. We also call it $|\sigma|$-vertex switching. A connected graph G is said to be highly irregular, if each of its vertices is adjacent only to vertices with distinct degrees [1]. In [2], it is proved that there is no highly irregular graph with a self vertex switching.

For the graph G given in Figure 1.1, G^σ is given in Figure 1.2, $G[\sigma]$ is given in Figure 1.3 at $\sigma = \{u, v\}$. The c-joint, d-joint and the total joint is given in Figures 1.4, 1.5 and 1.6, respectively.

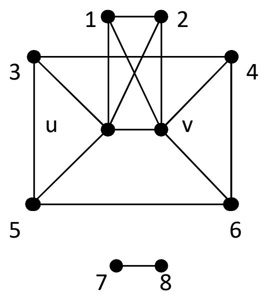

Figure 1.1. G

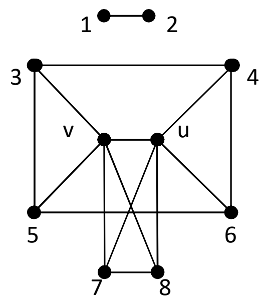

Figure 1.2. G^σ

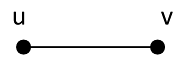

Figure 1.3. $G[\sigma]$

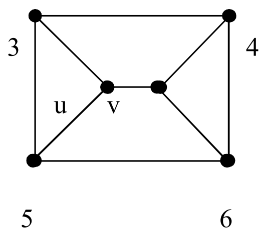

Figure 1.4. c-joint

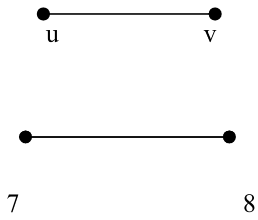

Figure 1.5. d-joint

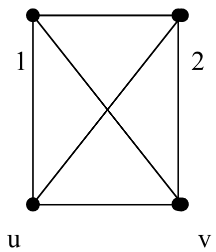

Figure 1.6. total joint

2. 2-Vertex Switching of Connected Joints

In this section,we give necessary and sufficient conditions for a c-joint B at σ in a graph G, B^σ to be a c-joint and a d-joint at σ in G^σ, when uv is an edge and a non-edge. Further the conditions for the graph G itself to be a c-joint are discussed with examples.

Theorem 2.1. *Let G be a graph of order p and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. If B and B^σ are c-joints at σ in G and G^σ respectively, then $|V(B)| \geq 4$.*

Proof. Suppose $|V(B)| < 4$. Then $|V(B)| = 3$ and hence $B = P_3$ with $d_B(u) = d_B(v) = 1$. This implies that, $B^\sigma = 3K_1$ which is a d-joint and gives a contradiction to B^σ is a c-joint. Therefore, $|V(B)| \geq 4$. □

Theorem 2.2. *Let G be a graph of order p and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. Let B be a c-joint at σ in G. Then B^σ is a c-joint at σ in G^σ if and only if B-σ is connected, $|V(B)| \geq 4$, $0 < d_B(u) \leq |V(B)| - 3$ and $0 < d_B(v) \leq |V(B)| - 3$.*

Proof. Let B be c-joint at σ in G such that B^σ is a c-joint. By Theorem 2.1, $|V(B)| \geq 4$. Since $uv \notin E(G)$ and B is a c-joint, we have $0 < d_B(u) \leq |V(B)|-2$. Suppose $d_B(u) = |V(B)|-2$. Then all the vertices of $V(B)-\sigma$ are adjacent to u in B, and hence all vertices in $V(B)-\sigma$ are non-adjacent to u in B^σ. Therefore, u is an isolated vertex in B^σ which is a contradiction to B^σ is connected and hence $0 < d_B(u) \leq |V(B)|-3$. Similarly, we can prove that $0 < d_B(v) \leq |V(B)|-3$. By the definition of joints, $B-\sigma$ is connected. Thus, $B-\sigma$ is connected, $|V(B)| \geq 4$, $0 < d_B(u) \leq |V(B)|-3$ and $0 < d_B(u) \leq |V(B)|-3$.

Conversely, let B be a c-joint at σ in G such that $B-\sigma$ is connected, $|V(B)| \geq 4$, $0 < d_B(u) \leq |V(B)|-3$ and $0 < d_B(v) \leq |V(B)|-3$. Now $d_B(v) \leq |V(B)|-3$ implies that there is a vertex, say a, in $V(B)-\sigma$ such that a is non-adjacent to u in B and hence a is adjacent to u in B^σ. Also, $0 < d_B(v) \leq |V(B)|-3$ implies that there is a vertex, say b, in $V(B)-\sigma$ such that b is non-adjacent to v in B and hence adjacent to v in B^σ. Thus ua and vb are edges in B^σ. Now to prove B^σ is connected, we consider the following two cases $a \neq b$ and $a = b$.

Case 1. $a \neq b$

Let x and y be any two vertices in B^σ.

Subcase 1.a. $\{x,y\} \neq \{u,v\}$.

Then $x, y \in V(B)-\sigma$. Since $B-\sigma$ is connected, there exists a x–y path in $B-\sigma$, and hence in B^σ.

Subcase 1.b. $\{x,y\} = \{u,v\}$

Since $uv \notin E(G)$, xy is not an edge in B and B^σ. Since au and bv are edges in B^σ, ax and by are edges in B^σ. Also, $B-\sigma$ is connected and $a, b \in V(B)-\sigma$, implies that there is an a–b path in $B-\sigma$ and hence in B^σ. Now, the edge xa, the path $a-b$ and the edge by form a x–y path in B^σ.

Subcase 1.c. $x = u$ and $y \neq v$

$y \neq v$ implies that $y \in V(B)-\sigma$. Since $B-\sigma$ is connected and $a, y \in V(B)-\sigma$, there exists an a–y path in $B-\sigma$ and hence an a–y path in B^σ. Now the edge xa and the path a–y form a x–y path in B^σ.

Hence there is a x–y path in all the cases. Therefore, B^σ is connected in G^σ and hence B^σ is a c-joint at σ in G^σ.

Case 2. $a = b$

We have au and bv are edges in B^σ. Let x and y be any two vertices in B^σ. We consider the following subcases.

Subcase 2.a. $\{x,y\} \neq \{u,v\}$

By subcase 1.a, there is a x–y path in G^σ.

Subcase 2.b. $\{x,y\} = \{u,v\}$

Since au and av are edges in B^σ, uav is a u–v path in G^σ and hence a x–y path in G^σ.

Subcase 2.c. $x = u$ and $y \neq v$

$y \neq v$ implies that $y \in V(B)-\sigma$. Since $B-\sigma$ is connected and $a, y \in V(B)-\sigma$, there exists an a–y path in $B-\sigma$ and hence an a–y path in B^σ. Now the edge xa and the path a–y form a $x-y$ path in B^σ.

Hence in all cases, there exists a x–y path in B^σ. This implies that B^σ is connected and hence B^σ is a c-joint at σ in G^σ. Hence the theorem is proved. □

Corollary 2.3. *Let G be a graph of order p and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. If G itself is a c-joint at σ, then G^σ is a c-joint at σ if and only if $G-\sigma$ is connected, $p \geq 4$, $0 < d_G(u) \leq p-3$ and $0 < d_G(v) \leq p-3$.*

Example 2.4. Consider the graph G of order 9 given in Figure 2.1. Here G is a c-joint at $\sigma = \{u,v\}$ in G and satisfy $0 < d_G(u) = 5 \leq p-3$ and $0 < d_G(v) = 4 \leq p-3$. The graph G^σ is given in Figure 2.2 and is a c-joint at σ.

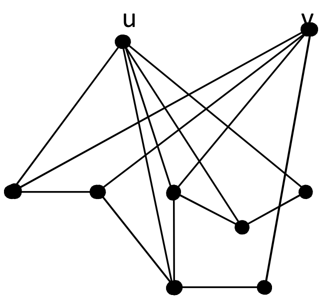

Figure 2.1. G

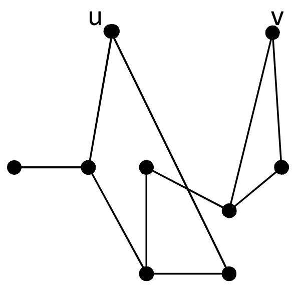

Figure 2.2. G^σ

Theorem 2.5. *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. Let B be a c-joint at σ in G. Then B^σ is a d-joint at σ in G^σ if and only if $B-\sigma$ is connected and either $d_B(u) = |V(B)|-2$ and $0 < d_B(v) \leq |V(B)|-2$ or $d_B(v) = |V(B)|-2$ and $0 < d_B(u) \leq |V(B)|-2$.*

Proof. Let B be a c-joint at σ in G such that B^σ is a d-joint at σ in G^σ. Then $B-\sigma$ is connected. Since $uv \notin E(G)$ and B is a c-joint at σ in G, $d_B(u)$ and $d_B(v)$ cannot be equal to zero and each is at most $|V(B)|-2$ in B. Hence $0 < d_B(u) \leq |V(B)|-2$ and $0 < d_B(v) \leq |V(B)|-2$. If $d_B(u) = |V(B)|-2$, then the proof is over. Otherwise, let $0 < d_B(u) < |V(B)|-2$. Then there exists at least one vertex, say x, in $V(B)-\sigma$ such that x is non-adjacent to u in B. This implies that x is adjacent to u in B^σ and hence xu is an edge in B^σ. We have $0 < d_B(v) \leq |V(B)|-2$. Suppose $d_B(v) < |V(B)|-2$. Then there exists a vertex, say y, in $V(B)-\sigma$ such that y is non-adjacent to v in B and hence adjacent to v in B^σ. This implies that yv is an edge in B^σ. Since $B-\sigma$ is connected, there exists a x–y path in $B-\sigma$ and hence in B^σ. Let a and b be any two vertices in $V(B^\sigma)$. We consider the following three cases.

Case 1. $\{a,b\} \neq \{u,v\}$

Clearly $a,b \in V(B)-\sigma$. Since $B-\sigma$ is connected, there is an a–b path in B^σ.

Case 2. $\{a,b\} = \{u,v\}$

Since ux and vy are edges in B^σ and there is a x–y path in B^σ the edge ux, the path x–y and the edge yv form a u–v path in B^σ and hence an a–b path in B^σ.

Case 3. $a = u$ and $b \neq v$

If $b = x$, then $ux = ab$ is an edge in B^{σ}.

If $b \neq x$, then there exists a x–b path in $B - \sigma$ and hence in B^{σ}. Now the edge ux and the path x–b form a u–b path in B^{σ} and hence an a–b path in B^{σ}.

Thus in all the cases, there is an a–b path in B^{σ} and hence B^{σ} is connected. This is a contradiction to B^{σ} is disconnected. Hence, $d_B(v) = |V(B)| - 2$.

Conversely, assume that B is a c-joint at σ in G such that $B - \sigma$ is connected and either $d_B(u) = |V(B)| - 2$ and $0 < d_B(v) \leq |V(B)| - 2$ or $d_B(v) = |V(B)| - 2$ and $0 < d_B(u) \leq |V(B)| - 2$. If $d_B(u) = |V(B)| - 2$, then u is adjacent to all the vertices of $V(B) - \sigma$ in B. Since $uv \notin E(G)$, u is non-adjacent to all the vertices of $V(B) - \sigma$ in B^{σ}. This implies that u is an isolated vertex in B^{σ} and hence B^{σ} is disconnected in G^{σ}. By a similar argument if $d_B(u) = |V(B)| - 2$, then v is an isolated vertex in B^{σ} and hence B^{σ} is disconnected. Thus, B^{σ} is a d-joint and hence the theorem is proved. □

Corollary 2.6. *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. If G itself is a c-joint at σ, then G^{σ} is a d-joint at σ if and only if $G - \sigma$ is connected and either $d_G(u) = p - 2$ and $0 < d_G(v) \leq p - 2$ or $d_G(v) = p - 2$ and $0 < d_G(u) \leq p - 2$.*

Example 2.7. Consider the graph G of order 8 given in Figure 2.3. Here G is a c-joint at $\sigma = \{u, v\}$ in G and satisfy $d_G(u) = 6 = p - 2$ and $0 < d_G(v) = 3 \leq p - 2$. The graph G^{σ} given in Figure 2.4 is a d-joint.

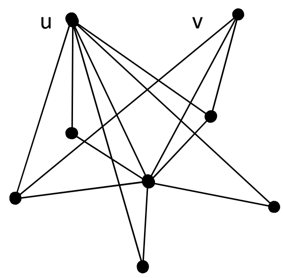

Figure 2.3. G

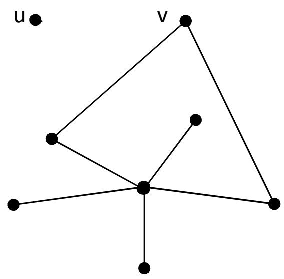

Figure 2.4. G^{σ}

Theorem 2.8. *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. Let B be a c-joint at σ in G. Then B^{σ} is a c-joint if and only $B - \sigma$ is connected and either $0 < d_B(u) \leq |V(B)| - 2$ or $0 < d_B(v) \leq |V(B)| - 2$.*

Proof. Let B be a c-joint such that B^{σ} is a c-joint. By the definition of joints, $B - \sigma$ is connected. Since $uv \in E(G)$ and B is connected, we have $0 < d_B(u) \leq |V(B)| - 1$ and $0 < d_B(v) \leq |V(B)| - 1$. If $d_B(u) \leq |V(B)| - 2$, then the proof is over. So let $d_B(u) = |V(B)| - 1$. This implies that u is adjacent to all the vertices of $V(B) - \sigma$ in B and hence non-adjacent to all the vertices of $V(B) - \sigma$ in B^{σ}. Now, we have $0 < d_B(v) \leq |V(B)| - 1$.

If $d_B(v) = |V(B)| - 1$, then v is adjacent to all the vertices of $V(B) - \sigma$ in B and hence non-adjacent to all the vertices of $V(B) - \sigma$ in B^{σ}. This implies that $B - \sigma$ is a component of B^{σ}. Hence B^{σ} is the union of two components, namely K_2 and $B - \sigma$, where K_2 is the edge uv.

This is a contradiction to B^σ is connected. Hence $0 < d_B(v) \le |V(B)| - 2$. Thus, $B - \sigma$ is connected and either $0 < d_B(u) \le |V(B)| - 2$ or $0 < d_B(v) \le |V(B)| - 2$.

Conversely, assume that B is a c-joint such that $B - \sigma$ is connected and either $0 < d_B(u) \le |V(B)| - 2$ or $0 < d_B(v) \le |V(B)| - 2$. To prove B^σ is a c-joint at $\sigma in G^\sigma$. Without loss of generality, we assume that $0 < d_B(u) \le |V(B)| - 2$. Then there exists at least one vertex, say a, in $V(B) - \sigma$ which is non-adjacent to u in B. Hence u is adjacent to a in B^σ. Let x and y be any two vertices in B^σ. We consider the following three possible cases.

Case 1. $\{x, y\} \ne \{u, v\}$
Then $x, y \in V(B) - \sigma$. Since $B - \sigma$ is a connected, there is a x–y path in $B - \sigma$ and hence in B^σ.

Case 2. $\{x, y\} = \{u, v\}$
Since uv is an edge in B^σ, $uv = xy$ is an edge in B^σ and hence there is a x–y path in B^σ.

Case 3. $x \ne u$ and $y = v$
If $x = a$, then $xu = au$ is an edge in B^σ. Now $xuv = xuy$ is a x–y path in G^σ.

If $x \ne a$, then there exists a x–a path in $B - \sigma$. Now the path x–a, the edges au and $uv = uy$ form a x–y path in B^σ.

Thus in all the cases, there is a x–y path in B^σ. This implies that B^σ is connected and therefore, a c-joint. Hence the theorem is proved. □

Corollary 2.9. *Let G be a graph of order $p \ge 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. If G itself is a c-joint at σ, then G^σ is a c-joint at σ if and only $G - \sigma$ is connected and either $0 < d_G(u) \le p - 2$ or $0 < d_G(v) \le p - 2$.*

Example 2.10. Consider the graph G of order 9 given in Figure 2.5. Here G is a c-joint at $\sigma = \{u, v\}$ in G and satisfy $0 < d_G(u) = 6 \le p - 2$ and $0 < d_G(v) = 5 \le p - 2$. The graph G^σ given in Figure 2.6 is a c-joint.

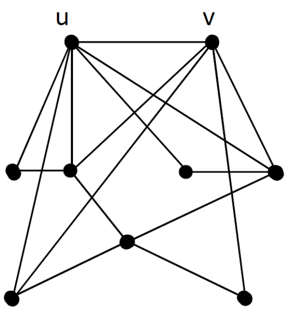

Figure 2.5. G

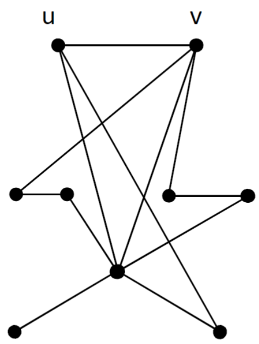

Figure 2.6. G^σ

Theorem 2.11. *Let G be a graph of order $p \ge 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. Let B be a c-joint at σ in G. Then B^σ is a d-joint at σ in G^σ if and only if $B - \sigma$ is connected and $d_B(u) = d_B(v) = |V(B)| - 1$.*

Proof. Let B be a c-joint at σ in G such that B^σ is d-joint at σ in G^σ. Clearly, $B - \sigma$ is connected. Since $uv \in E(G)$, $d_B(u)$ and $d_B(v)$ cannot be equal to zero and each is at most $|V(B)| - 1$ in G. Hence $0 < d_B(u) \le |V(B)| - 1$ and $0 < d_B(v) \le |V(B)| - 1$.

Case 1. $d_B(u) = |V(B)| - 1$ and $d_B(v) < |V(B)| - 1$.
$d_B(u) = |V(B)| - 1$ and $uv \in E(G)$ implies that u is adjacent to all the vertices of $V(B) - \sigma$ in B. Hence u is non-adjacent to all the vertices of $V(B) - \sigma$ in B^σ. $uv \in E(G)$ and $d_B(v) < |V(B)| - 1$ implies that there exists at least one vertex in $V(B) - \sigma$ which is non-adjacent to v in B. This implies that there exists at least one vertex adjacent to v in B^σ, say a. Hence av is an edge in B^σ. Let x and y be any two vertices in B^σ.

Subcase 1.a. $\{x, y\} \neq \{u, v\}$
Then $x, y \in V(B) - \sigma$. Since $B - \sigma$ is connected, there exists a x–y path in $B - \sigma$ and hence in B^σ.

Subcase 1.b. $\{x, y\} = \{u, v\}$
Since $uv \in E(G)$, $uv = xy$ is an edge in B^σ.

Subcase 1.c. $x = u$ and $y \neq v$
Then $y \in V(B) - \sigma$. If $a = y$, then the edges $uv = xv$ and $av = yv$ in B^σ form a x–y path in B^σ. If $a \neq y$, then there exists an a–y path in $B - \sigma$ and hence in B^σ. Now, the edges $uv = xv$, va and the path a–y form a x–y path in B^σ.

Thus in all cases, there is a x–y path in B^σ and hence B^σ is connected which is a contradiction to B^σ is disconnected.

Case 2. $d_B(u) < |V(B)| - 1$ and $d_B(v) < |V(B)| - 1$
Since $uv \in E(G)$ and $d_B(u) < |V(B)| - 1$, there exists at least one vertex, say a, in $V(B) - \sigma$ such that a is non-adjacent to u in B and hence adjacent to u in B^σ. This implies that au is an edge in B^σ. Also $d_B(v) < |V(B)| - 1$ implies that there exists at least one vertex, say b, in $V(B) - \sigma$ such that b is non-adjacent to v in B^σ and hence adjacent to v in B^σ. This implies that bv is an edge in B^σ. Since $B - \sigma$ is connected, there exists an a–b path in $B - \sigma$ and hence in B^σ. Let x and y be any two vertices in B^σ.

Subcase 2.a. $\{x, y\} = \{u, v\}$
Clearly, $uv = xy$ is an edge in B^σ.

Subcase 2.b. $\{x, y\} \neq \{u, v\}$
Then $x, y \in V(B) - \sigma$. Since $B - \sigma$ is connected, there exists a x–y path in $B - \sigma$ and hence in B^σ.

Subcase 2.c. $x = u$ and $y \neq v$
Then $y \notin V(B) - \sigma$. If $y = a$, then $ua = xy$ is an edge in B^σ.
If $y = b$, then $uv = xv$ and $vb = vy$ are edges in B^σ, and hence xvy is a x–y path in B^σ.
If $y \neq \{a, b\}$, then $uv = xv$ and vb are edges in B^σ and b, $y \in V(B) - \sigma$ implies that there is a b–y path in $B - \sigma$ and hence in B^σ. Now, the edges xv, vb and the b–y path in B^σ form a x––y path in B^σ.

Thus in all the above subcases, we get a x–y path in B^σ and hence B^σ is connected, which is a contradiction to B^σ is disconnected.

From *Case* 1 and *Case* 2, we conclude that $d_B(u) = d_B(v) = |V(B)| - 1$.

Conversely, let B be a c-joint at σ in G such that $B - \sigma$ is connected and $d_B(u) = d_B(v) = |V(B)| - 1$. Since B is a c-joint at σ in G, any two vertices in $V(B)$ are connected by a path in B

and hence in G. Now, $uv \in E(G)$ and $d_B(u) = d_B(v) = |V(B)| - 1$ implies that u and v are adjacent to all the vertices of $V(B) - \sigma$ in B. This implies that u and v are non-adjacent to all the vertices of $V(B) - \sigma$ in B^σ and hence B^σ is the union of two components namely, $B - \sigma$ and K_2. Therefore, B^σ is disconnected and hence a d-joint. □

Corollary 2.12. *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. If G itself is a c-joint at σ, then G^σ is a d-joint at σ if and only if $G - \sigma$ is connected and $d_G(u) = d_G(v) = p - 1$.*

Example 2.13. Consider the graph G of order 6 given in Figure 2.7. Here G is a c-joint at $\sigma = \{u, v\}$ in G and satisfy $d_G(u) = d_G(v) = 5 = p - 1$. The graph G^σ given in Figure 2.8 is a d-joint.

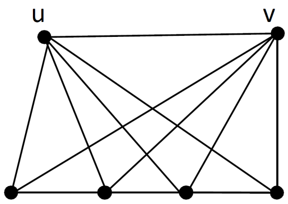

Figure 2.7. G

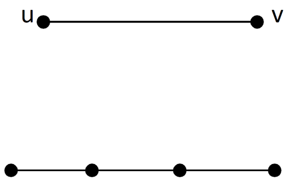

Figure 2.8. G^σ

3. 2-Vertex Switching of Disconnected Joints

In this section, we give necessary and sufficient conditions for a d-joint B at $\sigma = \{u, v\}$ in a graph G, B^σ to be a c-joint or a d-joint or a total joint at σ in G^σ, when uv is either an edge or a non-edge. Further, the conditions when the graph G itself is a d-joint are also discussed with suitable examples.

Theorem 3.1. *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. Let B be a d-joint at σ in G. Then B^σ is a c-joint at σ in G^σ if and only if $B - \sigma$ is connected and either $d_B(u) = 0$ and $0 \leq d_B(v) \leq |V(B)| - 3$ or $d_B(v) = 0$ and $0 \leq d_B(u) \leq |V(B)| - 3$.*

Proof. Let B be a d-joint at σ in G such that B^σ is a c-joint at σ in G^σ. By the definition of joints at σ in G, $B - \sigma$ is connected. Since $uv \notin E(G)$ and $B - \sigma$ is connected, we have either u or v or both is/are an isolated vertex in B. Without loss of generality, let us assume that u is an isolated vertex in B. Hence $d_B(u) = 0$. Now $0 \leq d_B(v) \leq |V(B)| - 2$. If $d_B(v) = |V(B)| - 2$, then v is adjacent to all the vertices of $V(B) - \sigma$ in B and hence v is non-adjacent to all the vertices of $V(B) - \sigma$ in B^σ. This implies that v is an isolated vertex in B^σ and hence B^σ is disconnected which is a contradiction to B^σ is connected. Hence $0 \leq d_B(v) < |V(B)| - 2$. If v is an isolated vertex, then by a similar argument, we can prove that $d_B(v) = 0$ and $0 \leq d_B(u) < |V(B)| - 2$. Thus $B - \sigma$ is connected and either $d_B(u) = 0$ and $0 \leq d_B(v) \leq |V(B)| - 3$ or $d_B(v) = 0$ and $0 \leq d_B(u) \leq |V(B)| - 3$.

Conversely, let B be a d-joint at σ in G such that $B - \sigma$ is connected and either $d_B(u) = 0$ and $0 \leq d_B(v) \leq |V(B)| - 3$ or $d_B(v) = 0$ and $0 \leq d_B(u) \leq |V(B)| - 3$. Without loss of generality, let us assume that $d_B(u) = 0$ and $0 \leq d_B(v) \leq |V(B)| - 3$. Since $d_B(u) = 0$, u is non-adjacent to all the vertices of $V(B)-\sigma$ in B. This implies that u is adjacent to all the vertices of $V(B)-\sigma$ in B^{σ}. Now $0 \le d_B(v) \le |V(B)|-3$ implies that v is non-adjacent to at least one of the vertices of $V(B)-\sigma$ in B and hence adjacent to at least one vertex, say b, of $V(B)-\sigma$ in B^{σ}. Therefore, bv is an edge in B^{σ}. Since u is adjacent to all the vertices of $V(B)-\sigma$ in B^{σ}, $B-\sigma$ is connected and bv is an edge in B^{σ}, we have B^{σ} is connected and hence a c-joint. Hence the theorem is proved. □

Corollary 3.2. *Let G be a graph of order $p \ge 3$ and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. If G itself is a d-joint at σ, then G^{σ} is a c-joint at σ if and only if $G-\sigma$ is connected and either $d_G(u) = 0$ and $0 \le d_G(v) \le p-3$ or $d_G(v) = 0$ and $0 \le d_G(u) \le p-3$.*

Example 3.3. Consider the graph G of order 6 given in Figure 3.1. Here G is a d-joint at $\sigma = \{u,v\}$ in G satisfying $d_G(u) = 0$ and $0 < d_G(v) = 2 \le p-3$. The graph G^{σ} given in Figure 3.2 is a c-joint.

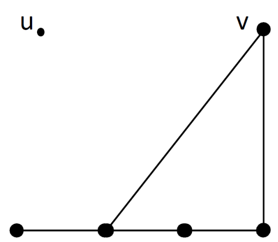

Figure 3.1. G

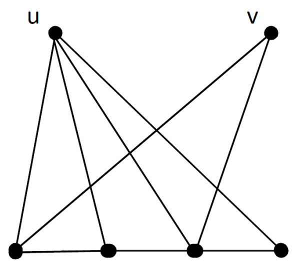

Figure 3.2. G^{σ}

Theorem 3.4. *Let G be a graph of order $p \ge 3$ and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. Let B be a d-joint at σ in G. Then B^{σ} is a d-joint at σ in G^{σ} if and only if $B-\sigma$ is connected and $\{d_B(u), d_B(v)\} = \{0, |V(B)|-2\}$.*

Proof. Let B be a d-joint at σ in G such that B^{σ} is a d-joint at σ in G^{σ}. Since $uv \notin E(G)$, $d_B(u) \ge 0$ and $d_B(v) \ge 0$. If $d_B(u) = d_B(v) = 0$, then $B = 2K_1 \cup (B-\sigma)$ and hence $B^{\sigma} = (B-\sigma)+2K_1$ which is connected. This is a contradiction to B^{σ} is disconnected and hence $d_B(u)$ and $d_B(v)$ cannot be zero simultaneously. Also, if both $d_B(u) > 0$ and $d_B(v) > 0$, then there exist vertices, say a and b, in $V(B)-\sigma$ such that a is adjacent to u and b is adjacent to v in B. This implies that B is connected which is a contradiction to B is disconnected. Therefore, either $d_B(u) = 0$ and $d_B(v) > 0$ or $d_B(v) = 0$ and $d_B(u) > 0$. Without loss of generality, assume that $d_B(u) = 0$ and $d_B(v) > 0$. Since $uv \notin E(G)$, $0 < d_B(v) \le |V(B)|-2$. Suppose $d_B(v) < |V(B)|-2$. Then v is non-adjacent to at least one vertex of $V(B)-\sigma$ in B and hence adjacent to at least one vertex in B^{σ}. Let it be a. Hence av is an edge in B^{σ}. Let x and y be any two vertices in B^{σ}.

Case 1. $\{x,y\} \ne \{u,v\}$
Then $x,y \in V(B)-\sigma$. Since $B-\sigma$ is connected, there exists a x–y path in $B-\sigma$ and hence in B^{σ}.

Case 2. $\{x,y\} = \{u,v\}$
$d_B(u) = 0$ implies that u is adjacent to all the vertices of $V(B)-\sigma$ in B^{σ} and hence au is an edge in B^{σ}. Now, the edges au and av form a u–v path in B^{σ} and hence a x–y path in B^{σ}.

Case 3. $x \neq u$ and $y = v$

If $x = a$, then $av = xy$ is an edge in B^{σ}.

If $x \neq a$, then there exists an a–x path in $B - \sigma$ and hence in B^{σ}. Now, the edge ya and the path a–x form a x–y path in B^{σ}.

Hence in all cases, there is a x–y path in B^{σ}, and hence B^{σ} is connected, which is a contradiction to B^{σ} is disconnected. Therefore, $d_B(v) < |V(B)| - 2$ is not possible and hence $d_B(v) = |V(B)| - 2$. Thus, we have $d_B(u) = 0$ and $d_B(v) = |V(B)| - 2$. Similarly, we can prove that $d_B(v) = 0$ and $d_B(u) = |V(B)| - 2$ if we take $d_B(v) = 0$ and $d_B(u) > 0$. Thus, $B - \sigma$ is connected and $\{d_B(u), d_B(v)\} = \{0, |V(B)| - 2\}$.

Conversely, let B be a d-joint at σ in G such that $B - \sigma$ is connected and $\{d_B(u), d_B(v)\} = \{0, |V(B)| - 2\}$. Let $d_B(u) = 0$ and $d_B(v) = |V(B)| - 2$. $uv \notin E(G)$ and $d_B(v) = |V(B)| - 2$ implies that v is adjacent to all the vertices of $V(B) - \sigma$ in B and hence non-adjacent to all the vertices of $V(B) - \sigma$ in B^{σ}. This implies that v is an isolated vertex in B^{σ} and hence B^{σ} is disconnected. Therefore, B^{σ} is a d-joint. Hence the theorem is proved. □

Corollary 3.5. *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. If G itself is a d-joint at σ, then G^{σ} is a d-joint at σ if and only if $G - \sigma$ is connected and $\{d_G(u), d_G(v)\} = \{0, p-2\}$.*

Example 3.6. Consider the graph G of order 8 given in Figure 3.3. Here G is a d-joint at $\sigma = \{u, v\}$ in G and satisfy $d_G(u) = 0$ and $0 < d_G(v) = p - 2 = 6$. The graph G^{σ} is given in Figure 3.4 which is a also d-joint.

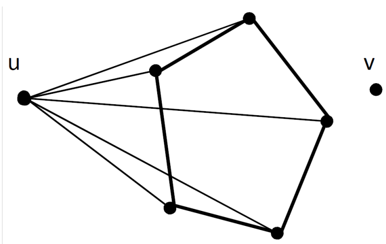

Figure 3.3. G

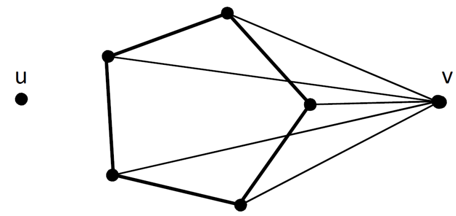

Figure 3.4. G^{σ}

Theorem 3.7. *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. Then B is a d-joint at σ in G if and only if B^{σ} is a total joint at σ in G^{σ}.*

Proof. Let B be a d-joint at σ in G. Since $B - \sigma$ is connected and $uv \in E(G)$ we have $B = (B - \sigma) \cup K_2$. By definition $B^{\sigma} = (B - \sigma) + K_2$ which is a total joint. Thus, B is a d-joint at σ in G if and only if B^{σ} is a total joint at σ in G^{σ}. □

Corollary 3.8. *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$, then G is a d-joint at σ if and only if G^{σ} is a total joint at σ.*

Note 3.9. Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. Let B be a d-joint at σ in G. Then B^{σ} is a total joint which implies that G^{σ} is always a c-joint.

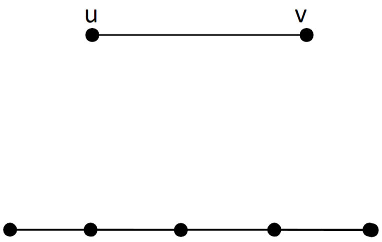

Figure 3.5. G

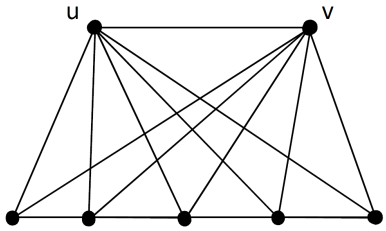

Figure 3.6. G^σ

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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