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Research Article

An Accelerated Popov's Subgradient Extragradient Method for Strongly Pseudomonotone Equilibrium Problems in a Real Hilbert Space With Applications

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Abstract. In this paper, we introduce a subgradient extragradient method to find the numerical solution of strongly pseudomonotone equilibrium problems with the Lipschitz-type condition on a bifunction in a real Hilbert space. The strong convergence theorem for the proposed method is precisely established on the basis of the standard cost bifunction assumptions. The application of our convergence results is also considered in the context of variational inequalities. For numerical analysis, we consider the well-known Nash-Cournot oligopolistic equilibrium model to support our well-established convergence results.

Keywords. Subgradient extragradient method; Strongly pseudomonotone equilibrium problems; Lipschitz-type condition; Strong convergence theorem

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1. Introduction

Let $\mathbb{C} \subset \mathbb{H}$ be a convex and closed set of a real Hilbert space $\mathbb{H}$. The inner product is denoted by $\langle \cdot, \cdot \rangle$ and the norm is denoted by $\| \cdot \|$. Let f be a bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ with $EP(f, \mathbb{C})$ denotes the solution set of an equilibrium problem over the set $\mathbb{C}$ and p^* is any random element of $EP(f, \mathbb{C})$. Let consider the following definitions of a monotonicity of a bifunction (see [5,6] for details). Let a bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ on $\mathbb{C}$ for $\gamma > 0$ is said to be:

(i) *strongly monotone* if
$$f(\breve{x}, \breve{y}) + f(\breve{y}, \breve{x}) \le -\gamma \|\breve{x} - \breve{y}\|^2, \quad \forall\, \breve{x}, \breve{y} \in \mathbb{C};$$

(ii) *monotone* if
$$f(\breve{x}, \breve{y}) + f(\breve{y}, \breve{x}) \le 0, \quad \forall\, \breve{x}, \breve{y} \in \mathbb{C};$$

(iii) *strongly pseudomonotone* if
$$f(\breve{x}, \breve{y}) \ge 0 \implies f(\breve{y}, \breve{x}) \le -\gamma \|\breve{x} - \breve{y}\|^2, \quad \forall\, \breve{x}, \breve{y} \in \mathbb{C};$$

(iv) *pseudomonotone* if
$$f(\breve{x}, \breve{y}) \ge 0 \implies f(\breve{y}, \breve{x}) \le 0, \quad \forall\, \breve{x}, \breve{y} \in \mathbb{C};$$

(v) satisfying the *Lipschitz-type condition* on $\mathbb{C}$ if two real numbers $c_1, c_2 > 0$, such that
$$f(\breve{x}, \breve{z}) - c_1 \|\breve{x} - \breve{y}\|^2 - c_2 \|\breve{y} - \breve{z}\|^2 \le f(\breve{x}, \breve{y}) + f(\breve{y}, \breve{z}), \quad \forall\, \breve{x}, \breve{y}, \breve{z} \in \mathbb{C}.$$

For given $\mathbb{C}$ to be a nonempty closed and convex subset of a real Hilbert space $\mathbb{H}$ and let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction through $f(\breve{x}, \breve{x}) = 0$, for every $\breve{x} \in \mathbb{C}$. The *equilibrium problem* [6,8] for f over $\mathbb{C}$ defined as follows:

$$\text{Find } p^* \in \mathbb{C} \text{ such that } f(p^*, \breve{y}) \ge 0, \quad \forall\, \breve{y} \in \mathbb{C}. \tag{EP}$$

Equilibrium problem (EP) had various mathematical problems as a particular case especially the *variational inequality problems* (VIP), optimization problems, the fixed point problems, complementarity problems, the Nash equilibrium of non-cooperative games, saddle point and vector minimization problems (for further details see e.g., [6,7,12]). To the best of our knowledge, the term "equilibrium problem" in specific format introduced in 1992 by Muu and Oettli [13] and has been further studied by Blum and Oettli [6]. The problem of equilibrium is also acknowledged as the famous Ky Fan inequality [8]. One of the most interesting and effective research fields in equilibrium problem theory is to construct new iterative schemes and modify the existing methods and also study their convergence analysis. A number of methods have previously developed to approximate the solution of an equilibrium problem in both finite and infinite-dimensional spaces, i.e., the extragradient methods [11,15,16,19,27–31] and others in [1,2,10,18,21–26].

Hieu [9] proposed an extragradient method to solve strongly pseudomonotone equilibrium problems in a real Hilbert space. It is mandatory to solve two minimization problems on a closed convex set for each iteration of the sequence generated by the method in [9], and an appropriate step size sequence is required for each minimization problem. An iterative sequence $\{x_n\}$ generated as follows:

Let $x_n, y_n \in \mathbb{H}$ such that

$$\begin{cases} x_{n+1} = \arg\min_{y \in \mathbb{C}} \left\{ \lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 \right\}, \\ y_{n+1} = \arg\min_{y \in \mathbb{C}} \left\{ \lambda_{n+1} f(y_n, y) + \frac{1}{2}\|x_{n+1} - y\|^2 \right\}, \end{cases} \tag{1}$$

where $\{\lambda_n\} \subset (0, +\infty)$ be a non-increasing sequence having following conditions:

$$(\text{Cd1}): \lim_{n \to +\infty} \lambda_n = 0 \quad \text{and} \quad (\text{Cd2}): \sum_{n=1}^{+\infty} \lambda_n = +\infty. \tag{2}$$

In this work, we study well-established projection methods that are easy to implement due to their easy and smooth numerical calculations. We propose a modified subgradient extragradient method to resolve strongly pseudomonotone equilibrium problems in real Hilbert space in order to improve the convergence speed of the iterative sequence. Our result is based on the two-step inertial subgradient extragradient method for finding a numerical solution to the strongly pseudomonotone equilibrium problems and the strong convergence of the proposed method based on the mild conditions.

This paper is organized in the following manner: Section 2 includes some definitions and basic results that will be needed in this paper. Section 3 gives an inertial-type algorithm with convergence studies. Section 4 set out some application of our main results. Section 5 sets out experimental investigations to confirm algorithmic behaviour for both standard problems designed based on the Nash-Cournot equilibrium model.

2. Preliminaries

In this section, some basic definitions and important lemmas are provided in order to study the convergence analysis.

A *normal cone* of $\mathbb{C}$ at $\breve{x} \in \mathbb{C}$ is defined by

$$N_{\mathbb{C}}(\breve{x}) = \{w \in \mathbb{H} : \langle w, \breve{y} - \breve{x} \rangle \le 0, \ \forall \ \breve{y} \in \mathbb{C}\}.$$

A projection $P_{\mathbb{C}}(\breve{x})$ of $\breve{x}$ onto a closed, convex subset $\mathbb{C}$ of $\mathbb{H}$ is

$$P_{\mathbb{C}}(\breve{x}) = \arg\min_{\breve{y} \in \mathbb{C}} \{\|\breve{y} - \breve{x}\|\}.$$

Assume that $g : \mathbb{C} \to \mathbb{R}$ is a convex function and *subdifferential* of g at $\breve{x} \in \mathbb{C}$ is defined by

$$\partial g(\breve{x}) = \{w \in \mathbb{C} : g(\breve{y}) - g(\breve{x}) \ge \langle w, \breve{y} - \breve{x} \rangle, \ \forall \ \breve{y} \in \mathbb{C}\}.$$

Lemma 2.1 ([20]). *Let $\mathbb{C}$ be a non-empty, closed and convex subset of a real Hilbert space $\mathbb{H}$ and $g : \mathbb{C} \to \mathbb{R}$ be a convex, subdifferentiable and lower semicontinuous function on $\mathbb{C}$. Then, $\breve{p} \in \mathbb{C}$ is a minimizer of a function g if and only if $0 \in \partial g(\breve{p}) + N_{\mathbb{C}}(\breve{p})$, where $\partial g(\breve{p})$ and $N_{\mathbb{C}}(\breve{p})$ denotes the subdifferential of g at $\breve{p}$ and the normal cone of $\mathbb{C}$ at $\breve{p}$, respectively.*

Lemma 2.2 ([4]). *For $\breve{x}, \breve{y} \in \mathbb{H}$ and $\eth \in \mathbb{R}$, then the following relationship is holds:*

$$\|\eth\breve{x} + (1 - \eth)\breve{y}\|^2 = \eth\|\breve{x}\|^2 + (1 - \eth)\|\breve{y}\|^2 - \eth(1 - \eth)\|\breve{x} - \breve{y}\|^2.$$

Lemma 2.3 ([3]). *Let a_n, b_n and c_n are sequences in $[0, +\infty)$ and*

$$a_{n+1} \le a_n + b_n(a_n - a_{n-1}) + c_n, \ \ \forall \ n \ge 1, \ \textit{with} \ \sum_{n=1}^{+\infty} c_n < +\infty$$

with $b > 0$ *and* $0 \le b_n \le b < 1$, $\forall\ n \in \mathbb{N}$. *Then, the following results are established.*

(i) $\sum_{n=1}^{+\infty} [a_n - a_{n-1}]_+ < \infty$, *with* $[s]_+ := \max\{s, 0\}$*;*

(ii) $\lim_{n\to+\infty} a_n = a^* \in [0, \infty)$.

Lemma 2.4 ([14])**.** *Let* $\{\phi_n\}, \{\psi_n\} \subset [0, +\infty)$ *are sequences and* $\sum_{n=1}^{+\infty} \phi_n = +\infty$ *with* $\sum_{n=1}^{+\infty} \phi_n \psi_n < +\infty$, *then* $\liminf_{n\to+\infty} \psi_n = 0$.

3. Main Results

The proposed algorithm is an inertial algorithm solve strongly pseudomonotone equilibrium problem. However, the advantage of this algorithm is that there is no need to know about the strongly pseudomonotone constant γ and Lipschitz constants c_1, c_2.

Assumption 1. Assume that $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ satisfies the following conditions:

(C1) $f(x,x) = 0$, $\forall\ x \in C$ and f is strongly pseudomontone on $\mathbb{C}$;

(C2) f satisfy the Lipschitz-type condition on $\mathbb{H}$ with constants c_1 and c_2;

(C3) $f(x,\cdot)$ is sub-differentiable and convex on $\mathbb{H}$ for each fixed $x \in \mathbb{H}$.

Algorithm 1 (Two-step algorithm for strongly pseudomonotone equilibrium problem)**.**

Initialization: Choose $x_{-1}, x_0, y_0 \in \mathbb{H}$, $0 \le \vartheta_n \le \vartheta < \sqrt{5} - 2$ and a sequence $\{\lambda_n\}$ satisfying the following conditions:

$$(\text{Cd1}):\ \lim_{n\to+\infty} \lambda_n = 0 \quad \text{and} \quad (\text{Cd2}):\ \sum_{n=1}^{+\infty} \lambda_n = +\infty.$$

Set

$$x_1 = \arg\min_{y\in\mathbb{C}} \left\{ \lambda_0 f(y_0, y) + \frac{1}{2}\|w_0 - y\|^2 \right\}, \quad y_1 = \arg\min_{y\in\mathbb{C}} \left\{ \lambda_1 f(y_0, y) + \frac{1}{2}\|w_1 - y\|^2 \right\},$$

where $w_0 = x_0 + \vartheta_0(x_0 - x_{-1})$ and $w_1 = x_1 + \vartheta_1(x_1 - x_0)$.

Iterative steps: Given x_{n-1}, y_{n-1}, x_n, y_n for $n \ge 1$. Construct a half space

$$H_n = \{z \in \mathbb{H} : \langle w_n - \lambda_n v_{n-1} - y_n, z - y_n \rangle \le 0\},$$

where $v_{n-1} \in \partial_2 f(y_{n-1}, y_n)$.

Step 1: Compute

$$x_{n+1} = (1 - \beta_n) w_n + \beta_n z_n,$$

where $w_n = x_n + \vartheta_n(x_n - x_{n-1})$ and

$$z_n = \arg\min_{y\in H_n} \left\{ \lambda_n f(y_n, y) + \frac{1}{2}\|w_n - y\|^2 \right\}.$$

Step 2: Compute

$$y_{n+1} = \arg\min_{y\in\mathbb{C}} \left\{ \lambda_{n+1} f(y_n, y) + \frac{1}{2}\|w_{n+1} - y\|^2 \right\},$$

where $w_{n+1} = x_{n+1} + \vartheta_{n+1}(x_{n+1} - x_n)$.

Step 3: If $x_{n+1} = w_n = y_n$, STOP. Otherwise set $n := n + 1$ and go to *Step* 1.

Lemma 3.1. *Let* $f : \mathbb{H}\times\mathbb{H} \to \mathbb{R}$ *be a bifunction satisfying the conditions* (C1)-(C3). *Assume that the EP*$(f,\mathbb{C})$ *is non-empty. Then, for all* $p^* \in EP(f,\mathbb{C})$, *we have*

$$\|z_n - p^*\|^2 \le \|w_n - p^*\|^2 - (1-4c_1\lambda_n)\|w_n - y_n\|^2 - (1-2c_2\lambda_n)\|z_n - y_n\|^2 + 4c_1\lambda_n\|w_n - y_{n-1}\|^2 - 2\gamma\lambda_n\|y_n - p^*\|^2.$$

Proof. By the use of z_n and Lemma 2.1, we have

$$0 \in \partial_2\left\{\lambda_n f(y_n, y) + \frac{1}{2}\|w_n - y\|^2\right\}(z_n) + N_{H_n}(z_n).$$

Thus, for $\omega \in \partial_2 f(y_n, z_n)$ there exists $\overline{\omega} \in N_{H_n}(z_n)$ such that

$$\lambda_n\omega + z_n - w_n + \overline{\omega} = 0.$$

This implies that

$$\langle w_n - z_n, y - z_n\rangle = \lambda_n\langle \omega, y - z_n\rangle + \langle \overline{\omega}, y - z_n\rangle, \quad \forall\ y \in H_n.$$

Since $\overline{\omega} \in N_{H_n}(z_n)$ then $\langle \overline{\omega}, y - z_n\rangle \le 0$ for all $y \in H_n$. It means that

$$\lambda_n\langle \omega, y - z_n\rangle \ge \langle w_n - z_n, y - z_n\rangle, \quad \forall\ y \in H_n. \tag{3}$$

Due to $\omega \in \partial f(y_n, z_n)$, we have

$$f(y_n, y) - f(y_n, z_n) \ge \langle \omega, y - z_n\rangle, \quad \forall\ y \in \mathbb{H}. \tag{4}$$

From (3) and (4) we have

$$\lambda_n f(y_n, y) - \lambda_n f(y_n, z_n) \ge \langle w_n - z_n, y - z_n\rangle, \quad \forall\ y \in H_n. \tag{5}$$

Due to $z_n \in H_n$ implies that $\langle w_n - \lambda_n v_{n-1} - y_n, z_n - y_n\rangle \le 0$. Thus, we get

$$\lambda_n\langle v_{n-1}, z_n - y_n\rangle \ge \langle w_n - y_n, z_n - y_n\rangle. \tag{6}$$

Since $v_{n-1} \in \partial_2 f(y_{n-1}, y_n)$, we have

$$f(y_{n-1}, y) - f(y_{n-1}, y_n) \ge \langle v_{n-1}, y - y_n\rangle, \quad \forall\ y \in \mathbb{H}.$$

By substituting $y = z_n$, we have

$$f(y_{n-1}, z_n) - f(y_{n-1}, y_n) \ge \langle v_{n-1}, z_n - y_n\rangle, \quad \forall\ y \in \mathbb{H}. \tag{7}$$

From (6) and (7) we obtain

$$\lambda_n\{f(y_{n-1}, z_n) - f(y_{n-1}, y_n)\} \ge \langle w_n - y_n, z_n - y_n\rangle. \tag{8}$$

By substituting $y = p^*$ into (5), we obtain

$$\lambda_n f(y_n, p^*) - \lambda_n f(y_n, z_n) \ge \langle w_n - z_n, p^* - z_n\rangle, \quad \forall\ y \in H_n. \tag{9}$$

Since $p^* \in EP(f,\mathbb{C})$ then $f(p^*, y_n) \ge 0$. Thus $f(y_n, p^*) \le -\gamma\|y_n - p^*\|$ due to strong pseudo-monotonicity of a bifunction f. From (8) we get

$$\langle w_n - z_n, z_n - p^*\rangle \ge \lambda_n f(y_n, z_n) + \gamma\lambda_n\|y_n - p^*\|^2. \tag{10}$$

Due to the Lipschitz-type continuity of bifunction f we have

$$f(y_{n-1}, z_n) \le f(y_{n-1}, y_n) + f(y_n, z_n) + c_1\|y_{n-1} - y_n\|^2 + c_2\|y_n - z_n\|^2. \tag{11}$$

From (10) and (11) we get

$$\langle w_n - z_n, z_n - p^*\rangle \ge \lambda_n\{f(y_{n-1}, z_n) - f(y_{n-1}, y_n)\} - c_1\lambda_n\|y_{n-1} - y_n\|^2 - c_2\lambda_n\|y_n - z_n\|^2 + \gamma\lambda_n\|y_n - p^*\|^2. \tag{12}$$

Combining expressions (8) and (12), we obtain

$$\langle w_n - z_n, z_n - p^* \rangle \geq \langle w_n - y_n, z_n - y_n \rangle - c_1\lambda_n \|y_{n-1} - y_n\|^2 - c_2\lambda_n \|y_n - z_n\|^2 + \gamma\lambda_n \|y_n - p^*\|^2. \tag{13}$$

We have the following facts:

$$-2\langle w_n - z_n, z_n - p^* \rangle = -\|w_n - p^*\|^2 + \|z_n - w_n\|^2 + \|z_n - p^*\|^2,$$
$$2\langle w_n - y_n, z_n - y_n \rangle = \|w_n - y_n\|^2 + \|z_n - y_n\|^2 - \|w_n - z_n\|^2.$$

We also have the following inequality

$$\|y_{n-1} - y_n\|^2 \leq \big(\|y_{n-1} - w_n\| + \|w_n - y_n\|\big)^2 \leq 2\|y_{n-1} - w_n\|^2 + 2\|w_n - y_n\|^2.$$

The above two facts and last inequality, completes the proof. □

Next, we can prove the strong convergence of Algorithm 1.

Theorem 3.2. *Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction satisfying the conditions* (C1)-(C3)*. Assume that $\{x_n\}$ is a sequences in $\mathbb{H}$ generated by Algorithm* 1*. Moreover, the sequence ϑ_n is non-decreasing with $0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2$ and β_n is non-increasing with $0 < \beta \leq \beta_n \leq 1$. Then, $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ strongly converge to an element p^* in $EP(f, \mathbb{C})$.*

Proof. Since $\lambda_n \to 0$, there is an $n_0 \in \mathbb{N}$ such that

$$0 < \lambda_n < \frac{\frac{1}{2} - 2\vartheta - \frac{1}{2}\vartheta^2 - \delta}{\frac{b}{2}(1-\vartheta)^2 + 2c_1(1 + \vartheta + \vartheta^2 + \vartheta^3)} \quad \text{and} \quad 0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2, \tag{14}$$

where $0 < \delta < \frac{1}{2} - 2\vartheta - \frac{1}{2}\vartheta^2$ and $b = \max\{4c_1, 2c_2\}$. By value of x_{n+1} gives that

$$\begin{aligned}\|x_{n+1} - p^*\|^2 &= \|(1-\beta_n)(w_n - p^*) + \beta_n(z_n - p^*)\|^2 \\ &\leq (1-\beta_n)\|w_n - p^*\|^2 + \beta_n\|z_n - p^*\|^2 \\ &= \|w_n - p^*\|^2 - \beta_n(1 - 4c_1\lambda_n)\|w_n - y_n\|^2 - \beta_n(1 - 2c_2\lambda_n)\|z_n - y_n\|^2 \\ &\quad + 4c_1\lambda_n\beta_n\|w_n - y_{n-1}\|^2 - 2\gamma\lambda_n\beta_n\|y_n - p^*\|^2.\end{aligned} \tag{15}$$

By the use of w_n and Lemma 2.2, we obtain

$$\begin{aligned}\|w_n - p^*\|^2 &= \|(1+\vartheta_n)(x_n - p^*) - \vartheta_n(x_{n-1} - p^*)\|^2 \\ &= (1+\vartheta_{n+1})\|x_n - p^*\|^2 - \vartheta_n\|x_{n-1} - p^*\|^2 + \vartheta_n(1+\vartheta_n)\|x_n - x_{n-1}\|^2.\end{aligned} \tag{16}$$

By the use of w_{n+1} and Lemma 2.2, we obtain

$$\begin{aligned}\|w_{n+1} - y_n\|^2 &= \|x_{n+1} + \vartheta_{n+1}(x_{n+1} - x_n) - y_n\|^2 \\ &= (1+\vartheta_{n+1})\|x_{n+1} - y_n\|^2 - \vartheta_{n+1}\|x_n - y_n\|^2 + \vartheta_{n+1}(1+\vartheta_{n+1})\|x_{n+1} - x_n\|^2 \\ &\leq (1+\vartheta_{n+1})\|x_{n+1} - y_n\|^2 + \vartheta_{n+1}(1+\vartheta_{n+1})\|x_{n+1} - x_n\|^2 \\ &\leq (1+\vartheta)\big[\|w_n - y_n\|^2 + \|z_n - y_n\|^2\big] + \vartheta(1+\vartheta)\|x_{n+1} - x_n\|^2.\end{aligned} \tag{17}$$

Combining (15), (16) and (17), we obtain

$$\begin{aligned}&\|x_{n+1} - p^*\|^2 + 4c_1\lambda_n\beta_{n+1}\|w_{n+1} - y_n\|^2 \\ &\leq (1+\vartheta_{n+1})\|x_n - p^*\|^2 - \vartheta_n\|x_{n-1} - p^*\|^2 + \vartheta(1+\vartheta)\|x_n - x_{n-1}\|^2 \\ &\quad + 4c_1\lambda_n\beta_n\|w_n - y_{n-1}\|^2 - \beta_n(1 - 4c_1\lambda_n)\|w_n - y_n\|^2 - \beta_n(1 - 2c_2\lambda_n)\|z_n - y_n\|^2\end{aligned}$$

$$+4c_1\lambda_n\beta_n(1+\vartheta)\big[\|w_n-y_n\|^2+\|z_n-y_n\|^2\big]+4c_1\lambda_n\beta_n\vartheta(1+\vartheta)\|x_{n+1}-x_n\|^2 \tag{18}$$

$$\begin{aligned}&\le(1+\vartheta_{n+1})\|x_n-p^*\|^2-\vartheta_n\|x_{n-1}-p^*\|^2+\vartheta(1+\vartheta)\|x_n-x_{n-1}\|^2\\&\quad+4c_1\lambda_n\beta_n\|w_n-y_{n-1}\|^2+4c_1\lambda_n\vartheta(1+\vartheta)\|x_{n+1}-x_n\|^2\\&\quad-\frac{1}{2}(1-b\lambda_n-4c_1\lambda_n(1+\vartheta))\|x_{n+1}-w_n\|^2,\end{aligned} \tag{19}$$

where $b=\max\{4c_1,2c_2\}$ and

$$\|x_{n+1}-w_n\|^2=\gamma_n^2\|z_n-w_n\|^2.$$

By using Cauchy inequality, we have

$$\begin{aligned}\|x_{n+1}-w_n\|^2&=\|x_{n+1}-x_n-\vartheta_n(x_n-x_{n-1})\|^2\\&=\|x_{n+1}-x_n\|^2+\vartheta_n^2\|x_n-x_{n-1}\|^2-2\vartheta_n\langle x_{n+1}-x_n,x_n-x_{n-1}\rangle \qquad (20)\\&\ge\|x_{n+1}-x_n\|^2+\vartheta_n^2\|x_n-x_{n-1}\|^2-2\vartheta_n\|x_{n+1}-x_n\|\|x_n-x_{n-1}\|\\&\ge(1-\vartheta_n)\|x_{n+1}-x_n\|^2+(\vartheta_n^2-\vartheta_n)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{21}$$

From (19) and (21), we have

$$\begin{aligned}&\|x_{n+1}-p^*\|^2-\vartheta_{n+1}\|x_n-p^*\|^2+4c_1\lambda_n\beta_{n+1}\|w_{n+1}-y_n\|^2\\&\le\|x_n-p^*\|^2-\vartheta_n\|x_{n-1}-p^*\|^2+\vartheta(1+\vartheta)\|x_n-x_{n-1}\|^2+4c_1\lambda_n\beta_n\|w_n-y_{n-1}\|^2\\&\quad+4c_1\lambda_n\vartheta(1+\vartheta)\|x_{n+1}-x_n\|^2-\sigma_n\big[(1-\vartheta)\|x_{n+1}-x_n\|^2+(\vartheta^2-\vartheta)\|x_n-x_{n-1}\|^2\big],\end{aligned} \tag{22}$$

where $\sigma_n:=\frac{1}{2}(1-b\lambda_n-4c_1\lambda_n(1+\vartheta))\ge0$, for all $n\ge n_0$. Let consider that

$$\Phi_n=\|x_n-p^*\|^2-\vartheta_n\|x_{n-1}-p^*\|^2+4c_1\lambda_n\beta_n\|w_n-y_{n-1}\|^2. \tag{23}$$

The expression (22) implies that

$$\Phi_{n+1}\le\Phi_n+R_n\|x_n-x_{n-1}\|^2-Q_n\|x_{n+1}-x_n\|^2, \tag{24}$$

where $R_n:=\vartheta(1+\vartheta)+\sigma_n\vartheta(1-\vartheta)\ge0$ for all $n\ge n_0$, and

$$Q_n:=\sigma_n(1-\vartheta)-4c_1\lambda_n\vartheta(1+\vartheta).$$

Furthermore, we also take

$$\Psi_n=\|x_n-p^*\|^2-\vartheta_n\|x_{n-1}-p^*\|^2+4c_1\lambda_n\beta_n\|w_n-y_{n-1}\|^2+R_n\|x_n-x_{n-1}\|^2.$$

It follows from (14) and (24) such that

$$\Psi_{n+1}-\Psi_n\le-\delta\|x_{n+1}-x_n\|^2\le0,\quad n\ge n_0. \tag{25}$$

The above means that $\{\Psi_n\}$ is nonincreasing for $n\ge n_0$. By Ψ_n we have

$$\begin{aligned}\|x_n-p^*\|^2&\le\Psi_n+\alpha_n\|x_{n-1}-p^*\|^2\\&\le\Psi_{n_0}+\alpha\|x_{n-1}-p^*\|^2\\&\le\cdots\le\Psi_{n_0}(\alpha^{n-n_0}+\cdots+1)+\alpha^{n-n_0}\|x_{n_0}-p^*\|^2\\&\le\frac{\Psi_{n_0}}{1-\alpha}+\alpha^{n-n_0}\|x_{n_0}-p^*\|^2.\end{aligned} \tag{26}$$

By the use of Ψ_{n+1} and (25) we obtain

$$\begin{aligned}-\Psi_{n+1}&\le\alpha_{n+1}\|x_n-p^*\|^2\\&\le\alpha\|x_n-p^*\|^2\end{aligned}$$

$$
\begin{aligned}
&\le \alpha \frac{\Psi_{n_0}}{1-\alpha} + \alpha^{n-n_0+1}\|x_{n_0} - p^*\|^2 \\
&\le \alpha \frac{\Psi_{n_0}}{1-\alpha} + \|x_{n_0} - p^*\|^2.
\end{aligned} \tag{27}
$$

It is the result from (25) and (27) that

$$
\begin{aligned}
\delta \sum_{n=n_0}^{k} \|x_{n+1} - x_n\|^2 &\le \Psi_{n_0} - \Psi_{k+1} \\
&\le \Psi_{n_0} + \alpha \frac{\Psi_{n_0}}{1-\alpha} + \|x_{n_0} - p^*\|^2 \\
&\le \frac{\Psi_{n_0}}{1-\alpha} + \|x_{n_0} - p^*\|^2.
\end{aligned} \tag{28}
$$

By letting $k \to \infty$ implies that

$$
\sum_n \|x_{n+1} - x_n\|^2 < +\infty \quad \text{implies that} \quad \|x_{n+1} - x_n\| \to 0, \text{ as } n \to \infty. \tag{29}
$$

From the expressions (20) and (29) we obtain

$$
\|x_{n+1} - w_n\| \to 0 \text{ as } n \to \infty. \tag{30}
$$

By (27) implies that

$$
-\Phi_{n+1} \le \alpha \frac{\Psi_{n_0}}{1-\alpha} + \|x_{n_0} - p^*\|^2 + R_{n+1}\|x_{n+1} - x_n\|^2. \tag{31}
$$

Since $0 < \beta \le \beta_n \le 1$ with $0 \le \alpha_n \le \alpha < \sqrt{5} - 2$, we can re-write (19) for $n \ge n_0$, such that

$$
\begin{aligned}
&\beta(1 - b\lambda_{n_0} - 4c_1\lambda_{n_0}(1+\alpha))\left[\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right] \\
&\quad \le \Phi_n - \Phi_{n+1} + \alpha(1+\alpha)\|x_n - x_{n-1}\|^2 + 4c_1\lambda_n\alpha(1+\alpha)\|x_{n+1} - x_n\|^2.
\end{aligned} \tag{32}
$$

Fix $k > n_0$ and (32) for $n = n_0, n_0 + 1, \cdots, k$. Summing up them, we obtain

$$
\begin{aligned}
&\beta(1 - b\lambda_{n_0} - 4c_1\lambda_{n_0}(1+\alpha)) \sum_{n=n_0}^{k} \left[\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right] \\
&\quad \le \Phi_1 - \Phi_{k+1} + \alpha(1+\alpha) \sum_{n=n_0}^{k} \|x_n - x_{n-1}\|^2 + 4c_1\lambda_{n_0}\alpha(1+\alpha) \sum_{n=n_0}^{k} \|x_{n+1} - x_n\|^2 \\
&\quad \le \Phi_1 + \alpha \frac{\Psi_{n_0}}{1-\alpha} + \|x_{n_0} - p^*\|^2 + (2\alpha + \alpha^2)\|x_{k+1} - x_k\|^2 \\
&\qquad + \alpha(1+\alpha) \sum_{n=n_0}^{k} \|x_n - x_{n-1}\|^2 + 4c_1\lambda_{n_0}\alpha(1+\alpha) \sum_{n=n_0}^{k} \|x_{n+1} - x_n\|^2 \\
&\quad = M_3.
\end{aligned} \tag{33}
$$

By letting $k \to +\infty$ implies that

$$
\sum_n \|z_n - y_n\|^2 < +\infty \quad \text{and} \quad \sum_n \|w_n - y_n\|^2 < +\infty, \tag{34}
$$

and

$$
\lim_{n\to\infty} \|z_n - y_n\| = \lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{35}
$$

By using (17), (29) and (34) gives that

$$
\sum_n \|w_{n+1} - y_n\|^2 < +\infty. \tag{36}
$$

By using expressions (15), (16), (29), (36) and Lemma 2.3 implies that

$$\lim_{n\to\infty} \|x_n - p^*\| = l. \tag{37}$$

Next, we show that the sequence $\{x_n\}$ strongly converges to p^*. For all $n \geq n_0$, the expression (15) gives that

$$\begin{aligned}
2\gamma\lambda_n\|y_n - p^*\|^2 &\leq -\|x_{n+1} - p^*\|^2 + (1+\alpha_n)\|x_n - p^*\|^2 - \alpha_n\|x_{n-1} - p^*\|^2 \\
&\quad + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 + 4c_1\lambda_n\|w_n - y_{n-1}\|^2 \\
&\leq (\|x_n - p^*\|^2 - \|x_{n+1} - p^*\|^2) + 2\alpha\|x_n - x_{n-1}\|^2 \\
&\quad + (\alpha_n\|x_n - p^*\|^2 - \alpha_{n-1}\|x_{n-1} - p^*\|^2) + 4c_1\lambda_n\|w_n - y_{n-1}\|^2.
\end{aligned} \tag{38}$$

From expression (38) implies that

$$\begin{aligned}
&\sum_{n=n_0}^{k} 2\gamma\lambda_n\|y_n - p^*\|^2 \\
&\leq (\|x_{n_0} - p^*\|^2 - \|x_{k+1} - p^*\|^2) + 2\alpha\sum_{n=n_0}^{k}\|x_n - x_{n-1}\|^2 \\
&\quad + (\alpha_k\|x_k - p^*\|^2 - \alpha_{n_0-1}\|x_{n_0-1} - p^*\|^2) + \frac{4c_1}{2c_2+4c_1}\sum_{n=n_0}^{k}\|w_n - y_{n-1}\|^2 \\
&\leq \|x_{n_0} - p^*\|^2 + \alpha\|x_k - p^*\|^2 + 2\alpha\sum_{n=n_0}^{k}\|x_n - x_{n-1}\|^2 + \frac{4c_1}{2c_2+4c_1}\sum_{n=n_0}^{k}\|w_n - y_{n-1}\|^2 \\
&\leq M_4,
\end{aligned} \tag{39}$$

for some $M_4 \geq 0$. It gives that

$$\sum_{n=1}^{+\infty} 2\gamma\lambda_n\|y_n - p^*\|^2 < +\infty. \tag{40}$$

By Lemma 2.4 and (40) such that

$$\liminf \|y_n - p^*\| = 0. \tag{41}$$

Finally, (37) and (41) gives $\lim_{n\to\infty}\|x_n - p^*\| = 0$. This complete the proof. □

4. Application to Variational Inequality Problem

An operator $F : \mathbb{C} \to \mathbb{H}$ is define by

(1) *strongly pseudomonotone* on $\mathbb{C}$ if for $\eta > 0$ such that

$$\langle F(x_1), x_2 - x_1\rangle \geq 0 \Longrightarrow \langle F(x_2), x_1 - x_2\rangle \leq -\eta\|x_1 - x_2\|^2, \quad \forall\, x_1, x_2 \in \mathbb{C};$$

(2) *L-Lipschitz continuous* on $\mathbb{C}$ if

$$\|F(x_1) - F(x_2)\| \leq L\|x_1 - x_2\|, \quad \forall\, x_1, x_2 \in \mathbb{C}.$$

A variational inequality problem is defined as follows:

$$p^* \in \mathbb{C} \text{ such that } \langle F(p^*), y - p^*\rangle \geq 0, \quad \forall\, y \in \mathbb{C}.$$

Note. If $f(x,y) := \langle F(x), y - x\rangle$ for all $x, y \in \mathbb{C}$, then equilibrium problem turn to variational inequality problem with $\frac{L}{2} = c_1 = c_2$. The value of z_n rewritten as

$$
\begin{aligned}
z_n &= \operatorname*{argmin}_{y \in H_n} \left\{ \lambda_n f(y_n, y) + \frac{1}{2}\|w_n - y\|^2 \right\} \\
&= \operatorname*{argmin}_{y \in H_n} \left\{ \lambda_n \langle F(y_n), y - y_n \rangle + \frac{1}{2}\|w_n - y\|^2 \right\} \\
&= \operatorname*{argmin}_{y \in H_n} \left\{ \lambda_n \langle F(y_n), y - w_n \rangle + \frac{1}{2}\|w_n - y\|^2 + \lambda_n \langle F(y_n), w_n - y_n \rangle \right\} \\
&= \operatorname*{argmin}_{y \in H_n} \left\{ \frac{1}{2}\|y - (w_n - \lambda_n F(y_n)\|^2 \right\} - \frac{\lambda_n^2}{2}\|F(y_n)\|^2 \\
&= P_{H_n}(w_n - \lambda_n F(y_n)). \qquad (42)
\end{aligned}
$$

Corollary 4.1. *Assume that $F : \mathbb{C} \to \mathbb{H}$ is strongly pseudomonotone and L-Lipschitz continuous on $\mathbb{C}$ with solution set VI$(F,\mathbb{C})$ is non-empty. Choose x_{-1}, x_0, y_0 and compute*

$$x_1 = P_{\mathbb{C}}(w_0 - \lambda_0 F(y_0)), \quad y_1 = P_{\mathbb{C}}(w_1 - \lambda_1 F(y_0)),$$

where $w_0 = x_0 + \vartheta_0(x_0 - x_{-1})$ and $w_1 = x_1 + \vartheta_1(x_1 - x_0)$.

(i) *Given x_{n-1}, x_n, y_{n-1}, y_n for $n \geq 1$. Set $w_n = x_n + \vartheta_n(x_n - x_{n-1})$ and compute*

$$x_{n+1} = (1-\beta_n)w_n + \beta_n z_n,$$

where $z_n = P_{H_n}(w_n - \lambda_n F(y_n))$ and

$$H_n = \{z \in \mathbb{H} : \langle w_n - \lambda_n F w_n - y_n, z - y_n \rangle \leq 0\}.$$

(ii) *Compute*

$$y_{n+1} = P_{\mathbb{C}}(w_{n+1} - \lambda_{n+1} F(y_n)),$$

where $w_{n+1} = x_{n+1} + \vartheta_{n+1}(x_{n+1} - x_n)$ and $\beta_n \in (0,1]$ with λ_n satisfy the condition (2). Moreover, $c_1 = c_2 = \frac{L}{2}$ and $0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2$. Then the sequence $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ strongly converge to the solution p^* of VI$(F,\mathbb{C})$.

5. Numerical Illustration

Numerical findings are presented in this segment to demonstrate the performance of our proposed methodology. The MATLAB code have been operating in MATLAB edition 9.5 (R2018b) on the Intel(R) Core(TM)i5-6200 Processor PC @ 2.30GHz 2.40GHz, RAM 8.00 GB.

5.1 Nash-Cournot oligopolistic equilibrium model

We take into account the enhanced version of the Nash-Cournot oligopolistic equilibrium model [17]. Assume there are n companies that manufacture the same commodity. Let x represent a vector where each element x_i specifies the quantity of the commodity generated by the company i. The price function P for each individual company is define as $P_i(S) = \phi_i - \psi_i S$, where $\phi_i > 0$, $\psi_i > 0$ and $S = \sum_{i=1}^{m} x_i$. The function of income $F_i(x) = P_i(S)x_i - t_i(x_i)$, while $t_i(x_i)$ is the value tax and fee for producing x_i. The strategy framework for the entire concept is taking the form of $\mathbb{C} := \mathbb{C}_1 \times \mathbb{C}_2 \times \cdots \times \mathbb{C}_n$, where $\mathbb{C}_i = [x_i^{\min}, x_i^{\max}]$. In addition, each firm strives to achieve its optimum profit by taking into account the subsequent amount of demand on the basis that the output of all the other companies would be an input parameter. A point $p^* \in \mathbb{C} = \mathbb{C}_1 \times \mathbb{C}_2 \times \cdots \times \mathbb{C}_n$ is an equilibrium point of the model if

$$F_i(p^*) \geq F_i(p^*[x_i]), \ \forall \ x_i \in \mathbb{C}_i, \ \forall \ i = 1, 2, \cdots, n,$$

where $p^*[x_i]$ represent the vector get from p^* by taking x_i^* with x_i. Let $f(x,y) := \varphi(x,y) - \varphi(x,x)$ with $\varphi(x,y) := -\sum_{i=1}^{n} F_i(x[y_i])$, and the problem of finding the Nash equilibrium point is

$$\text{Find } p^* \in \mathbb{C} : f(p^*, y) \geq 0, \quad \forall \ y \in \mathbb{C}.$$

The bifunction f could be taken in the following form

$$f(x,y) = \langle Px + Qy + q, y - x \rangle,$$

while $q \in \mathbb{R}^n$ and P, Q are matrices of order n and Q is symmetric positive semi-definite and $Q - P$ is symmetric negative definite through Lipschitz constants $c_1 = c_2 = \frac{1}{2}\|P - Q\|$ (see [16]). Two matrices P,Q are randomly generated[1] and vector q randomly generated $[-n,n]$. The feasible set $\mathbb{C} \subset \mathbb{R}^n$ is

$$\mathbb{C} := \{x \in \mathbb{R}^n : -2 \leq x_i \leq 5\}.$$

We use $x_{-1} = x_0 = y_0 = (1,1,\cdots,1,1)^T$. The findings are seen in the Table 1–2 with with Algorithm 1(Algo1) and Algorithm 3.1 (Algo3.1) in [9].

Table 1. Experiment 5.1: Comparison of Algorithm 1 and Algorithm 3.1 in [9]

					Algo3.1		Algo1	
n	λ_n	ϑ_n	β_n	TOL	Iter.	CPU(s)	Iter.	CPU(s)
5	$(n+1)^{-1}$	0.12	0.80	10^{-6}	270	3.6617	190	2.4359
10	$(n+1)^{-1}$	0.12	0.80	10^{-6}	365	5.2656	240	4.1639
20	$(n+1)^{-1}$	0.12	0.80	10^{-6}	441	6.9567	342	5.7361
50	$(n+1)^{-1}$	0.12	0.80	10^{-6}	586	7.5834	416	6.1619

Table 2. Experiment 5.1: Comparison of Algorithm 1 and Algorithm 3.1 in [9]

					Algo3.1		Algo1	
n	λ_n	ϑ_n	β_n	TOL	Iter.	CPU(s)	Iter.	CPU(s)
5	$(n+1)^{-1}$	0.12	0.85	10^{-9}	1192	25.3000	788	20.5985
5	$(n+1)^{-0.5}$	0.12	0.85	10^{-9}	2631	53.2534	1630	41.4862
5	$(n+1)^{-1}\log^{1}(n+3)$	0.12	0.85	10^{-9}	1305	27.5130	1010	25.1062
5	$(n+1)^{-1}\log^{0.5}(n+3)$	0.12	0.85	10^{-9}	1935	42.7673	1554	36.4381
5	$\log^{-2}(n+3)$	0.12	0.80	10^{-9}	2596	58.4369	2133	50.6567
5	$\log^{-1}(n+3)$	0.12	0.80	10^{-9}	3186	72.7256	2411	54.3991

Discussion About Numerical Experiments: The following observation was obtained from Table 1–2.

(i) No previous knowledge for Lipschitz-constant c_1, c_2 is needed for Matlab running equations.

[1]Choosing two diagonal matrices randomly A_1 and A_2 with entries from $[1,n]$ and $[-n,0]$, respectively. Two random orthogonal matrices B_1 and B_2 are able to generate a positive semi definite matrix $M_1 = B_1 A_1 B_1^T$ and negative semi definite matrix $M_2 = B_2 A_2 B_2^T$. Finally, set $Q = M_1 + M_1^T$, $S = M_2 + M_2^T$ and $P = Q - S$.

(ii) Indeed, the convergence rate of algorithms probably depends on the convergence rate of the step-size sequences λ_n. For certain instances, the step-size sequence converges quickly to zero which would be more efficient for our situation.

(iii) The convergence rate of the iterative sequence often depends on the nature of the problem and the scale of the problem.

(iv) Due to the variable step-size sequence, a different step-size value that is not suitable for the current iteration of the process also creates ambiguity and hump in the actions of the iterative sequence.

6. Conclusion

This paper proposes a new algorithms to solve problems of strong pseudomonotone equilibrium. The primary advantage of this algorithm is that the stepsize, in this case, is independent of the constants of the Lipschitz type and strongly pseudomonotone. The reasonable explanation is that we use a stepsize sequence that is non-summable and non-increasing. Numerical experiments have also been considered for looking at the overall impact of the stepsize sequence on the convergence of an iterative sequence.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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