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Interrelations between Annihilator, Dual and Pseudo-H-algebras

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Abstract. The annihilator operators play an important rôle in Wedderburn's type decompositions for pseudo-H-algebras. These operators determine the notions of annihilator, resp. dual topological algebras. Thus, it is quite natural to ask for possible relations between the latter topological algebras and those equipped with an *H-structure*. Among other things, we present necessary and sufficient conditions that a modular complemented H-algebra be annihilator. It is known that a dual algebra is annihilator, while the converse is not, in general, true. Our concern here is focused on appropriate conditions on a given H-algebra guaranteeing the coincidence of the notions *dual* and *annihilator*.

1. Introduction and preliminaries

In [12] we developed a Wedderburn's structure theory for certain pseudo-H-algebras where the annihilator operators play an important rôle. The same operators are present in the structure theory of non-normed topological algebras (see, e.g. [3], [6] and [7]). On the other hand, annihilator and dual algebras are defined via these operators; so it is quite natural to seek any possible relations between the classes of modular complemented H-algebras, of pseudo-Hilbert algebras, of annihilator algebras or yet the class of dual topological algebras. In our context, we tried to reach the more general situation; namely, we employ modular complemented H-algebras; here the "H-structure" (see for definition below), stems from Ambrose algebras developed in [5] which constitute a generalization of H^*-(Banach) algebras introduced by W. Ambrose [1]. So, for pseudo-H-algebras, we first give a relation between properly and anti-properly precomplemented H-algebras (Proposition 2.4). Moreover, in certain anti-properly precomplemented H-algebras (resp. precomplemented H-algebras) we characterize annihilator algebras through adjoints of elements (resp. dual

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algebras) (see Theorem 2.5 and Theorem 2.6, respectively). Finally, a relation is given between certain Hilbert-algebras and dual ones (see Theorem 3.7).

All vector spaces and algebras considered here are taken over the field $\mathbb{C}$ of complexes. Let E be an algebra. If $(\emptyset \neq) S \subseteq E$, $\mathscr{A}_l(S)$ (resp. $\mathscr{A}_r(S)$) denotes the left (right) annihilator of S, which is a left (right) ideal of E. In particular, this ideal is 2-sided, if S is a left (right) ideal. In case of a *topological algebra* (separately continuous multiplication) the previous ideals are closed. We denote by $\mathscr{L}_l(E) \equiv \mathscr{L}_l$ ($\mathscr{L}_r(E) \equiv \mathscr{L}_r$, $\mathscr{L}$) the set of all closed left (right, 2-sided) ideals in a topological algebra E, while $\mathfrak{M}_l(E)$ (resp. $\mathfrak{M}_r(E)$) stands for the set of all closed maximal regular left (right) ideals of E. An algebra E is called *left* (resp. *right*) *preannihilator*, if $\mathscr{A}_l(E) = (0)$ (resp. $\mathscr{A}_r(E) = (0)$). If $\mathscr{A}_l(E) = \mathscr{A}_r(E) = (0)$, E is called *preannihilator.* In particular, a topological algebra E is said to be an *annihilator algebra,* if it is preannihilator with $\mathscr{A}_r(I) \neq (0)$ for every $I \in \mathscr{L}_l$, $I \neq E$ and $\mathscr{A}_l(J) \neq (0)$ for every $J \in \mathscr{L}_r$, $J \neq E$ (see [6]). A topological algebra E is called a *right annihilator algebra*, if E is the only closed left ideal having a trivial right annihilator; viz. one has $I = E$ for every $I \in \mathscr{L}_l$ with $\mathscr{A}_r(I) = (0)$ or equivalently, $\mathscr{A}_r(I) \neq 0$ for any proper $I \in \mathscr{L}_l$ (cf. also [17]). A *left annihilator algebra* is defined analogously, by interchanging "left" by "right". Obviously, an annihilator algebra is a left and right annihilator algebra. We note here that Husain and Wong [13] assume moreover, that $\mathscr{A}_r(E)$ (resp. $\mathscr{A}_l(E)$) $= (0)$. We do not put this assumption which is redundant in modular complemented H-algebras, we deal with (see the comments after Definition 2.1). A topological algebra E satisfying $\mathscr{A}_l(\mathscr{A}_r(I)) = I$ for all $I \in \mathscr{L}_l$ and $\mathscr{A}_r(\mathscr{A}_l(J)) = J$ for all $J \in \mathscr{L}_r$ is called a *dual algebra.* The Jacobson radical of E is denoted by $R(E)$; E is *semisimple,* if $R(E) = (0)$. A topological algebra E such that $I \in \mathscr{L}$ and $I^2 = (0)$ implies $I = (0)$ is called *topologically semiprime. Every semisimple topological algebra is topologically semiprime and thus preannihilator* (see Lemma 1.1 in [11]). We denote by $\mathscr{I}d(E)$ the set of all non-zero idempotent elements of an algebra E, namely, the set of all $x \in E$ with $0 \neq x = x^2$. A *minimal* element of an algebra E, is a non-zero idempotent x such that xEx is a division algebra. A non-zero element of E is called *primitive,* if it can not be expressed as the sum of two orthogonal idempotents; viz. of some $y, z \in \mathscr{I}d(E)$ with $yz = zy = 0$. We denote by $\mathfrak{S}_l(E)$ (resp. $\mathfrak{S}_r(E)$) the left (resp. right) socle of an algebra E. If $\mathfrak{S}_l(E) = \mathfrak{S}_r(E) \equiv \mathfrak{S}_E$ the resulted 2-sided ideal $\mathfrak{S}_E$ is called the *socle* of E (see [15]).

A *locally convex algebra* is a topological algebra whose underlying topological vector space is locally convex.

2. Properly precomplemented H-algebras, and annihilator algebras

In this section, we mainly deal with the connection of certain modular complemented H-algebras with annihilator ones. So, we begin by giving some more notation and terminology.

A *pseudo H-space* is a locally convex space E, whose topology is defined by a family $(\langle\, ,\, \rangle_\alpha)_{\alpha\in A}$ of positive semi-definite (: pseudo-)inner products. A *pseudo-H-algebra* is a pseudo H-space and an algebra (which is locally convex) with separately continuous multiplication (or even locally m-convex); see [4, p. 456, Definition 3.1]. The topology of a pseudo-H-algebra E is defined by a family $(p_\alpha)_{\alpha\in A}$ of seminorms so that $p_\alpha(x) = \langle x, x\rangle_\alpha^{1/2}$ for every $x \in E$. Such a topological algebra is denoted by $(E, (p_\alpha)_{\alpha\in A})$ or yet by $(E, (\langle\, ,\, \rangle_\alpha)_{\alpha\in A})$. The "$m$-convex" case will be referred to, accordingly, otherwise, the term *pseudo-H-algebra* will always be employed for the locally convex case. A *locally convex* (resp. *locally m-convex) H^*-algebra* is an algebra E equipped with a family $(p_\alpha)_{\alpha\in A}$ of *Ambrose seminorms* in the sense that $p_\alpha, \alpha \in A$ arises from a positive semi-definite (pseudo-) inner product, denoted by $\langle\, ,\, \rangle_\alpha$, such that the induced topology makes E into a locally convex (resp. locally m-convex) (topological) algebra. Moreover, the following conditions are satisfied: For any $x \in E$, there is an $x^* \in E$, such that

$$\langle xy, z\rangle_\alpha = \langle y, x^*z\rangle_\alpha \quad \text{and} \quad \langle yx, z\rangle_\alpha = \langle y, zx^*\rangle_\alpha$$

for any $y, z \in E$ and $\alpha \in A$. The element x^* (not necessarily unique) is called an *adjoint* of x. If E is proper and Hausdorff, x^* is unique and $* : E \to E : x \mapsto x^*$ is an involution. Thus, in our terminology, *every locally convex H^*-algebra is a pseudo-H-algebra*.

Example. Let I be an arbitrary set of elements. Consider the set $\mathbb{C}^{I\times I}$ of all complex-valued functions a on $I \times I$, such that $\sum_{i,j} |a(i,j)|^2 \in \mathbb{R}_+$. The latter, endowed with "point-wise" defined operations becomes a vector space and an algebra with "*matrix*" multiplication

$$(ab)(i,j) = \sum_k a(i,k)b(k,j),$$

for all $a, b \in \mathbb{C}^{I\times I}$. Take a family of real numbers $(t_\alpha)_{\alpha\in\Lambda}$, such that $t_\alpha \geq 1$. For each $\alpha \in \Lambda$, the mapping $\langle\cdot,\cdot\rangle_\alpha : \mathbb{C}^{I\times I} \times \mathbb{C}^{I\times I} \to \mathbb{C}$ given by

$$\langle a, b\rangle_\alpha = t_\alpha \sum_{i,j} a(i,j)\overline{b}(i,j)$$

defines a pseudo-inner product on $\mathbb{C}^{I\times I}$, where "$-$" denotes complex conjugation. Thus $A \equiv (\mathbb{C}^{I\times I}, (\langle\cdot,\cdot\rangle_\alpha)_{\alpha\in\Lambda})$ *becomes a locally convex pseudo-H-algebra*. Moreover, it is *a locally convex H^*-algebra with an involution* given by $a^*(i,j) = \overline{a}(j,i)$.

The choice of the previous family $(t_\alpha)_{\alpha\in\Lambda}$ is reasonable: indeed, in matrix representations of locally convex H^*-algebras, such a family exists, been a crucial issue in defining a locally convex topology in an algebra of infinite complex matrices that stems from the given algebra; the two algebras actually coincide up to an isomorphism of topological *-algebras (see [9, p. 66, Lemma 1.3, and the comments that follow, and p. 67, relation (2.3) and Theorem 2.1]). This still appears in the classical case of (Banach) H^*-algebras, where in a relevant example all the points of the algebra have "measure" at least one (see [1, p. 367, Example 1, and p. 368, the comments after Example 2]).

Two elements x, y in a pseudo-H-space E are called *orthogonal* if

$$\langle x, y\rangle_\alpha = 0 \quad \text{for all } \alpha \in A. \tag{2.1}$$

Through this, the *orthogonal* $S^\perp$ of a non-empty subset S in E is defined by

$$S^\perp = \{x \in E : \langle x, y\rangle_\alpha = 0 \ \text{ for every } y \in S, \alpha \in A\}, \tag{2.2}$$

being a *closed linear subspace* of E. $S, T \subseteq E$ are *mutually orthogonal* if their elements are pairwise orthogonal. A pseudo H-algebra E which is the *algebraic direct sum* of mutually orthogonal subalgebras E_λ, $\lambda \in \Lambda$, is called the *orthogonal direct sum* of the E_λ's and it is denoted by $E = \oplus^\perp_{\lambda\in\Lambda} E_\lambda$. A (closed) subspace V of a pseudo-H-space E is said to be *orthocomplemented* if

$$E = V \oplus^\perp V^\perp, \tag{2.3}$$

where $V^\perp$ is called the *orthocomplement* of V (with respect to $(\langle\, ,\, \rangle_\alpha)_{\alpha\in A}$); see (2.2). A subspace V of a pseudo-H-space E is closed if $V = (V^\perp)^\perp$. The latter is fulfilled if in particular, E is either *orthocomplemented* in the sense that $E = W \oplus^\perp W^\perp$ for every closed subspace W of E or E is a Hausdorff space and V satisfies the relation $E = V \oplus^\perp V^\perp$ (see Lemma 2.2 below). The respective remarks are valid for ideals too in pseudo-H-algebras, where an orthocomplement of a closed (left, right) ideal is defined.

In the next definition, we gather certain types of pseudo-H-algebras employed in the sequel; see also [12, p. 21, Definition 2.1].

Definition 2.1. Let E be a pseudo-H-algebra. Then

(i) E is called a *left modular complemented H-algebra*, if it satisfies the conditions:

$$\text{Any left or right ideal } I \text{ in } E \text{ with } I^\perp = (0) \text{ is dense in } E. \tag{2.4}$$

(Shortly, *the density property.*)

$$\bigcap_{M\in\mathfrak{M}_l(E)} M = (0), \quad \text{and } M^\perp \text{ is a left ideal for each } M \in \mathfrak{M}_l(E). \tag{2.5}$$

(In short, *the intersection property.*)

(ii) E is a *properly left precomplemented H-algebra*, if

$$E = M \oplus^\perp M^\perp \text{ for every maximal regular left ideal } M \text{ of } E. \tag{2.6}$$

(iii) E is an *anti-properly left precomplemented H-algebra*, if

$$E = I \oplus^\perp I^\perp \text{ for every minimal left ideal } I \text{ of } E. \tag{2.7}$$

(iv) E is a *left precomplemented H-algebra*, if

$$E = I \oplus^\perp I^\perp \text{ for every closed left ideal } I \text{ of } E. \tag{2.8}$$

By (2.5), *a left modular complemented H-algebra is* (*Jacobson*) *semisimple*. We also have the respective notions "on the right".

We state now two easy results useful in the sequel.

Lemma 2.2. *Let E be a Hausdorff pseudo-H-space. Then, each subspace V of E, which is orthocomplemented in E, satisfies the relation $V = V^{\perp\perp}$* (namely, V is ortho-idempotent). *Thus, V is closed.*

Proof. Obviously, $V \subseteq V^{\perp\perp}$. By the Hausdorff property, $V^{\perp} \cap V^{\perp\perp} = \{0\}$, and by $E = V \oplus^{\perp} V^{\perp}$, we finally get the assertion; (see also [11, p. 154, Lemma 3.17]). □

Ortho-idempotentness for closed (left, right) ideals is realized for instance, in pseudo-Hilbert dual algebras (see the comment after Theorem 3.7, below).

We recall some terminology from [8, p. 3723, Definition 2.1] needed in the next proposition. A topological algebra E is called a *left complemented algebra* if there exists a mapping $\perp : \mathscr{L}_l \to \mathscr{L}_l : I \mapsto I^{\perp}$ such that if $I \in \mathscr{L}_l$, then $E = I \oplus I^{\perp}$ ($I^{\perp}$ is called a *complement* of I). Moreover, if $I, J \in \mathscr{L}_l, I \subseteq J$, then $J^{\perp} \subseteq I^{\perp}$ and if $I \in \mathscr{L}_l$, then $(I^{\perp})^{\perp} = I$.

A *right complemented algebra* is defined analogously. A left and right complemented algebra is simply called a *complemented algebra*. $\perp$, as above, is called a *complementor* on E. In the sequel, by the term "complementor" we shall mean a left complementor, unless something else is mentioned.

Proposition 2.3. *Every Hausdorff locally convex H^*-algebra E whose all closed* (*left*) *ideals are orthocomplemented* (*see* (2.3)) *is a left complemented algebra with orthocomplementor the orthogonal map* (see also (2.2))*:*

$$\perp : \mathscr{L}_l \to \mathscr{L}_l : I \mapsto I^{\perp}.$$

Proof. By the very definitions E is a pseudo-H-algebra. Thus, by Lemma 2.2, $I^{\perp\perp} = I$ for every $I \in \mathscr{L}_l$. It is easily seen that for $I \subseteq J$ in $\mathscr{L}_l$, one has $J^{\perp} \subseteq I^{\perp}$. Besides, $I^{\perp} \in \mathscr{L}_l$ (see [4, p. 456, Lemma 3.2]). Thus, the orthomap is well defined, and E is a left complemented algebra. Right complementation is proved similarly. □

The next result gives a relation between properly and anti-properly pre-complemented H-algebras.

Proposition 2.4. *Let E be a pseudo H-algebra, such that every* (*left, right*) *ideal I is ortho-idempotent* (*viz.* $I^{\perp\perp} = I$)*. Consider the assertions:*

(1) *E is a* (*left, right*) *anti-properly precomplemented H-algebra.*
(2) *E is a* (*left, right*) *properly precomplemented H-algebra.*

Then (1)⇒(2). *In particular,* (2)⇒(1) *if $I^{\perp}$ is a maximal regular left* (*right*) *ideal of E, for every minimal left* (*right*) *ideal I of E.*

Proof. (1)⇒(2): Let M be a maximal regular left ideal of E. Claim that $M^{\perp}$ is a minimal left ideal. Suppose there exists some left ideal J of E with $(0) \neq J \subseteq M^{\perp}$. Then $M = M^{\perp\perp} \subseteq J^{\perp}$. If $J^{\perp} = E$, then $J = E^{\perp} = (0)$, a contradiction. The previous argument and the maximality of M imply $M = J^{\perp}$. Thus, $M^{\perp} = J^{\perp\perp}$ and by assumption, $M^{\perp} = J$, namely, $M^{\perp}$ is a minimal left ideal and hence $E = M^{\perp} \oplus M^{\perp\perp}$ with $M^{\perp\perp}$ a maximal left ideal. Since $M = M^{\perp\perp}$ we finally get $E = M \oplus M^{\perp}$. Similarly, on the "right".

(2)⇒(1): Let I be a minimal left ideal of E. By assumption, $I^{\perp\perp} = I$, and $I^{\perp}$ is a maximal regular left ideal. Thus, $E = I^{\perp} \oplus^{\perp} I^{\perp\perp}$ and $E = I^{\perp} \oplus^{\perp} I$ which completes the proof on the left. Similarly, on the "right". □

Concerning the previous result, we note that in [12, Theorem 2.3, Lemma 2.6] we give pseudo-H-algebras for which minimal (left, right) ideals do exist. Moreover, the proof of (1)⇒(2) is still valid when $I^{\perp\perp} = I$ is not assumed for the maximal regular (left, right) ideals. Indeed, by modifying slightly the proof, one can take the idempotentness for M.

Let $(E, \langle\, , \rangle_{\alpha})_{\alpha\in A}$ be a pseudo-H-algebra. An element x^l is a *left adjoint* of $x \in E$ if $\langle xy, z\rangle_{\alpha} = \langle y, x^l z\rangle_{\alpha}$ for all $y, z \in E$, $\alpha \in A$. It is easily seen that x^l is unique, if there exists. A *right adjoint* x^r is defined analogously; see also [16]. Notice that if the algebra E is preannihilator and $x \neq 0$, then x^l (resp. x^r)$\neq 0$.

All the results in the rest of the paper are also valid, by interchanging "left properties" by "right ones".

The next result concerns characterizations of annihilator algebras over certain anti-properly precomplemented H-algebras. The same result implies Theorem 3.1 in [13]. In our proof we apply arguments analogous to those given in [ibid., p. 458]. For convenience, we present the details, adapted to our case.

Theorem 2.5. *Let E be a semisimple Hausdorff, anti-properly and left properly precomplemented H-algebra with continuous quasi-inversion, satisfying the density property. Then the following are equivalent:*

(1) *E is an annihilator algebra.*
(2) *E is a left annihilator algebra.*
(3) *Every element in the socle $\mathfrak{S}_E$ of E has a left adjoint.*
(4) *Every nonzero right ideal of E contains an element with a left adjoint.*

Proof. We first note that E is actually a properly precomplemented H-algebra (see Corollary 2.7 in [12]).

(1)⇒(2): It is obvious by the very definitions.

(2)⇒(3): If I is a minimal right ideal of E, we show that $I^{\perp}$ is a (closed) maximal right ideal. Take $M \neq E$ a closed right ideal with $I^{\perp} \subseteq M$. If $M \cap I \neq (0)$, then by the minimality of I, $I = M \cap I$, but then, $I \subseteq M$ and since $E = I \oplus^{\perp} I^{\perp}$ we get $E \subseteq M + M \subseteq M$ and $E = M$, a contradiction. The above argument shows that $M \cap I = (0)$. Thus, by Lemma 3.17 in [11, p. 154], $M = I^{\perp}$. Namely, $I^{\perp}$ is a closed maximal right ideal. Thus, by the assumption for E, $\mathscr{A}_l(I^{\perp}) \neq (0)$, but then $I^{\perp} = (1-x)E \equiv \{y - xy : y \in E\}$ with x a minimal idempotent element in E (see [19, p. 38, Lemma 3.3]). Moreover, $I^{\perp}$ is a maximal regular right ideal. Now, by [8, p. 3729, Theorem 3.9], $I = xE$. Claim that each element in xE has a left adjoint and *a fortiori* this will be true for the elements in $\mathfrak{S}_E$. Since E is topologically semiprime, and $I = xE$, we get that x is a minimal primitive idempotent (and I is a minimal right ideal of E) (see [6, p. 154, Corollary 3.7]). Thus, by Lemma 2.6 in [12] (see also its proof) any element of I has a left adjoint, as asserted.

(3)⇒(4): Let J be a nonzero right ideal of E. By Theorem 2.3 in [12], $E = \overline{\mathfrak{S}_E}$ and hence $\mathfrak{S}_E \cap J \neq (0)$. From here we get that J contains an element with a left adjoint.

(4)⇒(2): Let I be a proper closed right ideal of E. If $I^{\perp} = (0)$, then the density property yields $E = I$, which is a contradiction. Thus, $I^{\perp} \neq (0)$ and by assumption, $I^{\perp}$ has a nonzero element, say x, with a left adjoint x^l. Here, we also note that $I^{\perp}$ is a right ideal (see Lemma 2.6 in [12]). Claim that $x^l I = (0)$, and thus $\mathscr{A}_l(I) \neq (0)$. Indeed, $\langle E, x^l I\rangle_\alpha = \langle xE, I\rangle_\alpha$. Since $xE \subseteq I^{\perp}$, the above argument shows that $\langle E, x^l I\rangle_\alpha = 0$ for all α's. In particular, $x^l I = (0)$, as asserted.

(2)⇒(1): We show that E is a right annihilator algebra. By Theorem 2.3 in [12], E has a dense socle. Consider a minimal right ideal, say J. By the proof of (2)⇒(3), $J^{\perp}$ is a maximal regular right ideal. Put $M = J^{\perp}$. By the proof of Lemma 2.6 in [12] (see also the comments preceding it), $M^{\perp} = xE$, $M = (1-x)E$ with x a left self-adjoint element of E. Moreover, any element in the minimal right ideal xE, which obviously, coincides with J, has a left adjoint. Thus, any element in the right socle has a left adjoint. Starting now, with a minimal left ideal, and arguing in an analogous way as before, we take that the left socle has a right adjoint. Namely, any element in the socle $\mathfrak{S}_E$ has a left and a right adjoint. Indeed, let I be a minimal left ideal. Then $I^{\perp}$ is a (closed) maximal left ideal. For this, take a left ideal $M \neq E$ with $I^{\perp} \subseteq M$. If $M \cap I \neq (0)$, the minimality of I implies $I = M \cap I$. Thus $I \subseteq M$. By hypothesis, $E = I \oplus^{\perp} I^{\perp}$ and hence $E \subseteq M + M$ and $M = E$, a contradiction. Thus, $M \cap I = (0)$ and by [11, p. 154, Lemma 3.17] $M = I^{\perp}$. Namely, $I^{\perp}$ is a maximal right ideal. Put $I^{\perp} = N$. Since N is maximal, $N^{\perp}$ is a minimal left ideal (see [4, p. 965, the proof of Theorem 2.3]). Moreover, by [12, Lemma 2.5], $N^{\perp} = Ex$ with x an idempotent, right self-adjoint element in E. By Peirce decomposition, $E = Ex \oplus E(1-x)$ and by hypothesis, $E = Ex \oplus (Ex)^{\perp}$. Obviously, $E(1-x) \subseteq (Ex)^{\perp}$, so since $Ex \cap (Ex)^{\perp} = (0)$, we get $(Ex)^{\perp} = E(1-x)$ (see [11, p. 154, Lemma 3.17]). By the previous argument, $N \subseteq N^{\perp\perp} = E(1-x)$, so the maximality of N implies $N = E(1-x)$. So, N is a maximal regular left ideal and by [12, Lemma 2.6; see also its proof], $N^{\perp} = Ex_0$ with x_0 an idempotent, right self-adjoint element of E, and every element of $N^{\perp}$ has a right adjoint. By the relations $I \subseteq I^{\perp\perp} = N^{\perp} = Ex_0$, we finally take $I = Ex_0$. Thus, every element in the left socle of E has a right adjoint.

Now, let I be a proper closed left ideal of E. By the density property and (ibid.), $I^{\perp}$ is a nonzero left ideal. Obviously, $\mathfrak{S}_E I^{\perp} \subseteqq \mathfrak{S}_E \cap I^{\perp}$ and $\mathfrak{S}_E I^{\perp} \neq (0)$, otherwise, $EI^{\perp} = (0)$ and $I = E$, a contradiction. Thus $\mathfrak{S}_E \cap I^{\perp} \neq (0)$, and hence $I^{\perp}$ contains a nonzero element, say w with a (nonzero) right adjoint w^r. For this w and for any $z \in I, y \in \mathfrak{S}_E$, we get

$$\langle yw, z\rangle_\alpha = \langle w, y^l z\rangle_\alpha = 0.$$

The continuity of the pseudo-inner product in both variables, the separate continuity of multiplication in E, and the density of $\mathfrak{S}_E$ in E, imply $\langle Ew, z\rangle_\alpha = \{0\}$. In particular, $p_\alpha(zw^r) = 0$ for all α's, and hence $zw^r = 0$ for all $z \in I$. Thus, $\mathscr{A}_r(I) \neq (0)$. $\square$

By the very definitions, *every dual algebra is an annihilator one*. The converse is not in general true even in the normed case; for references see [6, p. 151]. However, we do know special cases for which the converse is also true (ibid.). For non-normed contexts, see [10, p. 226, Theorem 3.1]. Here, we give one more non-normed case for which the two classes of "annihilator" and "dual" algebras coincide. First, we recall from [12] that a pseudo-H-algebra E has the *Peirce property* if it satisfies the condition:

$$\left.\begin{array}{l}\text{If } x_0 \text{ is a right (left) unit for } E \text{ modulo}\\ \text{a maximal regular left (right) ideal } M \text{ of } E,\\ \text{then } x_0 \in M^{\perp}, \text{ and } M^{\perp} \text{ is a left (right) ideal.}\end{array}\right\} \tag{2.9}$$

In the sequel, by a *deep algebra* we mean an algebra in which every non-zero left (right) ideal contains a minimal left (right) ideal (see [6, p. 151, Definition 3.1]). Moreover, a *left* (resp. *right*) *regular annihilator algebra* is an algebra in which $\mathscr{A}_l(M) \neq (0)$ (resp. $\mathscr{A}_r(M) \neq (0)$) for every maximal regular right (left) ideal M; if both conditions hold, we speak about a *regular annihilator algebra* [8, 18]. A topological algebra E is called a *Q'-algebra*, if every maximal regular left or right ideal in E is closed (see [6, p. 148, Definition 1.1]).

Theorem 2.6. *Let E be a semisimple Hausdorff precomplemented, and anti-properly precomplemented H-algebra with continuous quasi-inversion. Suppose E has the Peirce and the density properties. Then the following are equivalent:*

(1) *E is a dual algebra.*
(2) *E is a (left) annihilator algebra.*

Proof. We only have to prove that (2)⇒(1): By [12, Theorem 2.4; see also the comments before Theorem 2.3], E is a regular annihilator, properly precomplemented H-algebra. By Theorem 2.5, *E is actually an annihilator algebra*. We show that E is a dual algebra on the left. Namely, if I is a proper closed left ideal in E, then $\mathscr{A}_l(\mathscr{A}_r(I)) = I$. Since I is proper, $I^\perp \neq (0)$. By [12, Theorem 2.3] E is a Q' modular complemented H-algebra with dense socle. On the other hand, E, as a semisimple annihilator Q'-algebra, is a deep one (and a regular annihilator algebra); see [6, p. 153, Theorem 3.6]. Therefore, $I^\perp$ contains a minimal left ideal, say L. But then $I \subseteq L^\perp$; applying now a proof analogous to that of (2)⇒(3) of Theorem 2.5, we get that $L^\perp$ is a closed maximal regular left ideal. Put $S = \cap M$, the intersection is taken over all closed maximal regular left ideals of E containing I, which by the previous argument, is non-empty. Then $I \subseteq S$. Suppose there exists some $x \in S$ with $x \notin I$. By assumption, $x = y + z$ with $y \in I$ and $z \in I^\perp$. Therefore, $0 \neq z = x - y \in I^\perp \cap S$. Thus, the nonzero left ideal $I^\perp \cap S$ again as above, contains a minimal left ideal, say K, such that $K^\perp$ is a closed maximal regular left ideal. Thus, K has the form $K = Ex_0$ with x_0 a minimal (idempotent) element of E (see [6, p. 156, Section 4]). Therefore, $I^\perp \cap S$ contains x_0, which also is right selfadjoint (see [12, Lemma 2.5 and its proof]). Now, $Ex_0 \subseteq I^\perp$ and $I \subseteq I^{\perp\perp} \subseteq (Ex_0)^\perp = E(1 - x_0)$ (by the Peirce property). Thus, $E(1 - x_0)$ is one of the M's above. This means that $S \subseteq E(1 - x_0)$. Therefore, $x_0 \in E(1 - x_0)$, a contradiction. Thus, finally, $I = S = \cap M \subseteq M$. Each M has the form $M = E(1 - x), x \in \mathscr{I}d(E)$, from which $M = \mathscr{A}_l(\mathscr{A}_r(M))$. Finally, $I \subseteq \mathscr{A}_l(\mathscr{A}_r(I)) \subseteq \mathscr{A}_l(\mathscr{A}_r(M)) = M$ for all M's. Thus, $I \subseteq \mathscr{A}_l(\mathscr{A}_r(I)) \subseteq \cap M = S = I$, and hence $I = \mathscr{A}_l(\mathscr{A}_r(I))$. Similarly, E is a right dual algebra. □

Lemma 2.2, Proposition 2.4, Theorems 2.5, 2.6, and [12, Theorem 2.4] yield the next.

Corollary 2.7. *Let E be a Hausdorff modular complemented, anti-properly precomplemented, and precomplemented H-algebra with continuous quasi-inversion. Then the assertions* (1)-(4) *of Theorem* 2.5 *are equivalent to the following*
(5) *E is a dual algebra.*

Theorem 2.9 below, concerns equivalent conditions, under which a certain modular complemented H-algebra be an annihilator one (cf. also [20, p. 264, Theorem 2.4]). For its proof we use among others, the next result which is a direct consequence of [12, Theorems 2.3 and 2.11].

Corollary 2.8. *Let E be a Hausdorff pseudo-H-algebra. The following are equivalent.*

(1) *E is a semisimple properly precomplemented H-algebra, having the density property.*
(2) *E is a Q' modular complemented H-algebra with continuous quasi-inversion.*

In that case, E has a dense socle.

Theorem 2.9. *Let E be a Hausdorff Q' modular complemented H-algebra with continuous quasi-inversion. Then the following assertions are equivalent:*

(1) *$L^\perp$ is finite-dimensional for every maximal regular left ideal L of E.*
(2) *$R^\perp$ is finite-dimensional for every maximal regular right ideal R of E.*

If anyone of the above equivalent conditions holds, then E is an annihilator algebra.

Proof. We only prove that (1)⇒(2). The inverse implication is proved similarly. By Corollary 2.8, E is properly precomplemented with dense socle. Let R be a maximal regular right ideal of E. By ([12, Lemma 2.6; see also its proof]) $R^{\perp}$ is a right ideal of the form $R^{\perp} = x_0 E$ for some left self-adjoint idempotent element x_0 in E, and every element in it has a left adjoint. Notice that $R^{\perp} \neq (0)$, otherwise $E = \overline{R}(= R)$, a contradiction (see [6, p. 149, Proposition 1.3]). Since $E(1 - x_0)$ is a maximal regular left ideal, $E(1 - x_0) = E(1 - x)$ for some right self-adjoint idempotent $x \in E$ (see [12, Lemma 2.5]). Moreover, any element in Ex has a right adjoint (ibid, Lemma 2.6). Obviously, $Ex \subseteq E(1 - x)^{\perp}$. By the Peirce decomposition and [11, p. 154, Lemma 3.17], we get $Ex = E(1 - x)^{\perp}$. Thus, by hypothesis, Ex is finite-dimensional. Now, we show that $xE = R^{\perp}(= x_0 E)$. Indeed, by $E(1 - x_0) = E(1 - x)$, we get $x = x x_0 x$ and $x_0 = x_0 x x_0$. The latter yield $(x - x_0 x)^2 = 0$, and by semisimplicity of E, $x = x_0 x$. Thus $xE = x_0 x E \subseteq x_0 E$. The minimality of $x_0 E$ implies $xE = x_0 E$. As noticed above, the elements in Ex have right adjoints. Thus, for any $z \in Ex$, $z^r = (zx)^r = xz^r \in R^{\perp}$, so that the "right-adjoint" map identifies (conjugate-linearly) the finite-dimensional space Ex with a finite-dimensional subspace of $R^{\perp}$. We prove that, $(Ex)^r = R^{\perp}$ which yields the finite-dimensionality of $R^{\perp}$. Indeed, by [12, Lemma 2.6] the algebraic sum $\mathscr{S} = \sum M^{\perp}$, where M runs over all maximal regular left ideals is dense in E. Using this and the fact that any element in $\mathscr{S}$ has a right adjoint (ibid.), we get the assertion. So, if $w \in \mathscr{S}$, w^r exists and thus, $xw = (w^r x)^r$ and $x\mathscr{S} \subseteq (Ex)^r$. From here, $\overline{x\mathscr{S}} \subseteq \overline{(Ex)^r}$ or $x\overline{\mathscr{S}} \subseteq \overline{(Ex)^r}$. The previous implies $xE \subseteq (Ex)^r$, and finally, $(Ex)^r = xE = R^{\perp}$. The preceding argument also shows that *any element in $R^{\perp}$ has a right adjoint as well*. Thus, *if any one of* (1) *or* (2) *holds, the elements in $\mathscr{S}$ (resp. $\mathscr{S}_r$) where M runs over all maximal regular left* (*resp. right*) *ideals have left and right adjoints*.

Now, based on the latter information, we prove that E is an annihilator algebra, if and only if, (1) or (2) holds. Since E is semisimple, we only have to prove that $\mathscr{A}_r(I) \neq (0)$ (resp. $\mathscr{A}_l(I) \neq (0)$) for every closed left (right) ideal of E (see also the arguments in the proof of Theorem 2.5). So, suppose that I is a proper closed left ideal of E. By the density property, $I^{\perp} \neq (0)$. Now, $\mathscr{S}_r I^{\perp} \subseteq I^{\perp} \cap \mathscr{S}_r$. By [12, Lemma 2.6], $\mathscr{S}_r$ is dense in E. If $\mathscr{S}_r I^{\perp} = (0)$, then $EI^{\perp} = (0)$ that yields $I^{\perp} = (0)$, a contradiction. Thus, there exists a nonzero element, say x, in $I^{\perp} \cap \mathscr{S}_r$, so that for any $y \in I$, $z \in \mathscr{S} = \sum M^{\perp}$ (see above), we get $\langle zx, y\rangle_\alpha = \langle x, z^l y\rangle_\alpha = 0$ and by continuity, $\langle Ex, y\rangle_\alpha = \{0\}$ for all α's. Thus, $\langle E, yx^r\rangle_\alpha = 0$. In particular, $yx^r = 0$ for all y's in I. Thus, $0 \neq x^r \in \mathscr{A}_r(I)$. Namely, $\mathscr{A}_r(I) \neq (0)$ (see also the comments after Proposition 2.4). Likewise, E is a left annihilator algebra. □

Note. Concerning the previous proof, we note that *the finite-dimensionality of $R^{\perp}$ is implied also without using the "right-adjoint" map*. Indeed, $Ex_0 \cong x_0 E(= R^{\perp})$ within an isomorphism φ of linear spaces. φ is given by $\varphi(x)=\varphi(xx_0)=x_0x$. Hence $R^{\perp}$ is finite-dimensional.

3. **Pseudo-Hilbert algebras**

The following notion generalizes Hilbert algebras (called also unitary algebras) (see [2, p. 51] and [14]).

Definition 3.1. An algebra E is called a *pseudo-Hilbert algebra* if it is a pseudo H-space equipped with an involution $x\to x^*$ having the properties:

$$\langle xy,z\rangle_\alpha=\langle y,x^*z\rangle_\alpha \quad \text{for all } x,y,z\in E, \tag{3.1}$$

$$\langle x,y\rangle_\alpha=\langle y^*,x^*\rangle_\alpha \quad \text{for all } x,y\in E, \tag{3.2}$$

$$\text{the left multiplication } y\to xy \text{ is continuous on } E \tag{3.3}$$

for every $x\in E$.

$$\text{The set } \{xy : x,y\in E\} \text{ is dense in } E. \tag{3.4}$$

By (3.1) and (3.2), we easily take

$$\langle yx,z\rangle_\alpha=\langle y,zx^*\rangle_\alpha \quad \text{for all } x,y,z\in E. \tag{3.5}$$

By (3.2), *the involution of a pseudo-Hilbert algebra is continuous*.

If I is an orthocomplemented left (right) ideal in a pseudo-Hilbert algebra, then $I^{\perp}$ is a left (right) ideal of E. Indeed, for $z\in EI^{\perp}$, $z=xy$ with $x\in E$, $y\in I^{\perp}$ and for all $i\in I$, $\langle z,i\rangle_\alpha=\langle xy,i\rangle_\alpha=\langle y,x^*i\rangle_\alpha=0$. Thus, $z\in I^{\perp}$.

Following [4, p. 459], a pseudo H-algebra (a f o r t i o r i a locally convex H^*-algebra) E is called *square-complemented* (in E) if $\overline{[E^2]}$ is orthocomplemented (in E); namely $E=\overline{[E^2]}\oplus^{\perp}\overline{[E^2]}^{\perp}$. Here $\overline{[E^2]}$ denotes the closed 2-sided ideal of E, generated by E^2.

Lemma 3.2. *Any pseudo-Hilbert algebra has the properties:*

(i) *E is proper* (viz. $\mathscr{A}_r(E)=(0)$).

(ii) *E is preannihilator.*

(iii) *E is square-complemented.*

Proof. (i) Using (3.1), (3.2) and (3.5) we get $\mathscr{A}_l(E)=\mathscr{A}_r(E)$ (see also the proof of Theorem 1.2 in [4, p. 452]). Next, we apply arguments as in the proof of Theorem 1.3 in [4, p. 452]. So, if $Ey=(0)$ for some $0\neq y\in E$ and $x\in E$, then (see also (3.5)) $\langle w,z(x^*+y)\rangle_\alpha=\langle w,zx^*\rangle_\alpha=\langle wx,z\rangle_\alpha$ for all $w,z\in E$, $\alpha\in A$. Similarly, x^*+y satisfies (3.1), which means that x has two adjoints, a contradiction.

(ii) The assertion follows from (i) (see also its proof).

(iii) By hypothesis, $\overline{[E^2]}=E$. Moreover, $\overline{[E^2]}^{\perp}=(0)$. Indeed, since $\overline{[E^2]}^{\perp}\subseteq[E^2]^{\perp}\subseteq\{E^2\}^{\perp}$, it is enough to show that $\{E^2\}^{\perp}=\{0\}$. So, let z be an element in $\{E^2\}^{\perp}$. Then $\langle xy,z\rangle_\alpha=0$ for any $xy\in E^2$ and every α, and from (3.1),

$\langle y, x^*z\rangle_\alpha = 0$ for any $x, y \in E$ and $\alpha \in A$. In particular, $p_\alpha(x^*z) = 0$ for any $x \in E$, and by the Hausdorff property, $x^*z = 0$ or $z^*x = 0$ for any $x \in E$. By Lemma 3.2, E is preannihilator, hence $z^* = 0$ and $z = 0$. Namely, E is trivially square complemented and this completes the proof. □

Now, we get the following characterization of pseudo-Hilbert algebras.

Proposition 3.3. *For any Hausdorff pseudo-H-algebra E the following assertions are equivalent:*

(1) *E is a proper, square-complemented locally convex H^*-algebra.*
(2) *E is a pseudo-Hilbert algebra.*

Thus, the involution is continuous.

Proof. (1)⇒(2): Immediate by the very definitions and [4, p. 459, Corollary 3.12, Lemma 3.14 and Theorem 3.15]. Notice that "properness" is redundant here (see Lemma 3.2).
(2)⇒(1): Apply Lemma 3.2 (see also Definition 3.1). □

In what follows, we employ the term *orthocomplemented pseudo-Hilbert algebra* for a pseudo-Hilbert algebra in which every closed left (right) ideal is orthocomplemented.

Any proper Hausdorff orthocomplemented locally convex H^*-algebra is dual (see [4, p. 457, Theorem 3.9]; for the classical case of (Banach) H^*-algebras see [15, p. 273, Theorem 4.10.30]). Proposition 3.3 yields "duality" for orthocomplemented pseudo-Hilbert algebras. Here, for the sake of completeness, we give a proof and for this, we present first some easy results. Analogous statements hold for right ideals.

Lemma 3.4. *Let E be a proper algebra and I a left ideal of E such that there exists a left ideal I' of E with $E = I \oplus I'$. If $Ex \subseteq I$ (resp. $Ex \subseteq I'$) for some $x \in E$, then $x \in I$ (resp. $x \in I'$).*

Proof. Suppose that $Ex \subseteq I$. Since $E = I \oplus I'$, $x = y + z$ with $y \in I$, $z \in I'$. Thus, $wz = wx - wy \in I \cap I'$ for every $w \in E$ and hence $z = 0$. Therefore, $x = y \in I$. □

Lemmas 3.2 and 3.4 lead to the next.

Corollary 3.5. *Let E be a pseudo-Hilbert algebra and I a (closed) left ideal, which is orthocomplemented. Suppose that $Ex \subseteq I$ (resp. $Ex \subseteq I^\perp$) for some $x \in E$. Then $x \in I$ (resp. $x \in I^\perp$).*

Lemma 3.6. *Let E be a Hausdorff pseudo-Hilbert algebra and I (resp. J) a (closed) left (resp. right) orthocomplemented ideal of E. Then $\mathscr{A}_r(I) = (I^\perp)^*$ (resp. $\mathscr{A}_l(J) = (J^\perp)^*$).*

Proof. We prove the assertion only for left ideals. Take $x \in E$ with $Ix = (0)$. We have $\langle I, Ex^* \rangle_\alpha = \langle Ix, E \rangle_\alpha = \{0\}$ for all α's (see also (3.5)). Then $Ex^* \subseteq I^\perp$ and by Corollary 3.5, $x^* \in I^\perp$. Thus, $\mathscr{A}_r(I) \subseteq (I^\perp)^*$. On the other hand, for $y \in (I^\perp)^*$, $y^* \in I^\perp$ and, since $I^\perp$ is a left ideal (see the comments after Definition 3.1), we get $Ey^* \subseteq I^\perp$ and thus $\langle Iy, E \rangle_\alpha = \langle I, Ey^* \rangle_\alpha = \{0\}$ for all α's. Thus, if $z \in Iy \cap E$, then $\langle z, z \rangle_\alpha = 0$ for all α's and $z = 0$. Namely, $Iy = Iy \cap E = (0)$ and thus $(I^\perp)^* \subseteq \mathscr{A}_r(I)$. □

Theorem 3.7. *Every Hausdorff orthocomplemented pseudo-Hilbert algebra E is a dual algebra.*

Proof. If I is a closed left ideal then, by Lemmas 2.2 and 3.6, we get in turn, $\mathscr{A}_l(\mathscr{A}_r(I)) = \mathscr{A}_l((I^\perp)^*) = (\mathscr{A}_r(I^\perp))^* = (I^{\perp\perp})^{**} = I$. Arguing similarly, we get that E is a dual algebra with respect to closed right ideals. □

Relative to the previous result, and based on Lemma 3.6, we get that *the closed left (right) ideals of a dual pseudo-Hilbert algebra, are ortho-idempotent* (see also Lemma 2.2).

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