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Research Article

The Arithmetic of Generalization for General Products of Monoids

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Abstract. For A and B arbitrary monoids. In a recent work, Cevik *et al.* (*Hacettepe Journal of Mathematics and Statistics* **50**(1) (2021), 224 – 234) defined new consequence of the general product denoted by $A^{\oplus B}\ {}_{\delta}{\bowtie}_{\psi}\ B^{\oplus A}$ and gave a presentation for this generalization. In this paper, we explore the way in which the structure of the generalization of general product reflects the properties of its associated wreath products.

Keywords. Product; Wreath product; Green's relations; Generalized Green's relations; Congruence; Transitive

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1. Introduction

Wreath products combine any two structures S and T in multiple nonequivalant ways to make a third algebraic structure using action of one algebraic structure T on the T copies on other structure S. The general products have been acknowledged as tools to form algebraic structure from two algebraic structures S and T using actions on each other. These two constructions has been used by group theorists for many years, and in the last years it used widely in the study of semigroups. In [2] the authors bringing together the definitions of general product and (restricted) wreath products of monoids $A^{\oplus B}$ and $B^{\oplus A}$ and then gave a presentation and some other results of the main theories in terms of finite and infinite cases for this generalization "general products of wreath products" which can be thought as a generalization of results in [6] and [9]. We denote this generalization by $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi} B^{\oplus A}$. In this paper, we give some algebraic properties of this generalization. For convenience of the reader we begin by recalling some basic definitions:

Let A and B be monoids the monoid B is said to *act on the left on the monoid A by endomorphisms* if for every $b\in B$, there is a map $(b,a)\mapsto {}^{b}a$ from A to itself satisfying the following two axioms for all $b_1,b_2\in B$ and for all $a_1,a_2\in A$:

(SM1) ${}^{b_1}(a_1a_2)=({}^{b_1}a_1)({}^{b_1}a_2)$;

(SM2) ${}^{(b_1b_2)}a_1={}^{b_1}({}^{b_2}a_1)$;

(SM3) ${}^{1_B}a=a$ for all $a\in A$.

Axioms (SM1) and (SM2) (respectively, (SM1), (SM2) and (SM3) are equivalent to the existence of a homomorphism (respectively, monoid homomorphism) from B to the monoid of endomorphisms of A. Then $A\rtimes B$ is the *semidirect product* of A and B

$$A\rtimes B=\{(a,b):a\in A,b\in B\}$$

with multiplication defined by:

$$(a_1,b_1)(a_2,b_2)=(a_1({}^{b_1}a_2),b_1b_2).$$

Dually, suppose that the monoid A acts on the right on B by endomorphisms; that is for every $a\in A$, there is a map $b\mapsto b^{a}$ from B to itself satisfying axioms

(SM1′) $(b_1b_2)^{a_1}=b_1^{a_1}b_2^{a_1}$;

(SM2′) $(b_1)^{a_1a_2}=(b_1^{a_1})^{a_2}$;

(SM3′) $(b)^{1_A}=b$ for all $b\in B$.

That is there exists a monoid homomorphism from A to the monoid of endomorphisms of B, then we form the *reverse semidirect product*

$$B\ltimes A=\{(a,b):a\in A,b\in B\}$$

with multiplication

$$(a_1,b_1)(a_2,b_2)=(a_1a_2,b_1^{a_2}b_2).$$

For the monoids A and B, the (restricted) wreath product $A\wr B$ is the semidirect product of $A^{\oplus B}\rtimes B$ where $A^{\oplus B}$ is the direct product of B copies of A which can be think of the set of all functions from B to A with finite support, that is, having the property that $(x)f=1_A$ for all but finitely many x in B. The (restricted) wreath product is the set $A^{\oplus B}\times B$ with multiplication defined by

$$(f,b)(g,b')=(f\ {}^{b}g,bb'),$$

where ${}^{b}g:B\to A$ is defined by

$$(y)^{b}g=(yb)g,\quad (y\in B).$$

It is also a well known fact that this wreath product is a monoid with identity $(\overline{1},1_B)$, where $y\overline{1}=1_A$ for all $y\in B$.

Dually the (restricted) wreath product $B \wr A$ is the semidirect product of $B^{\oplus A} \rtimes A$ where $B^{\oplus A}$ is the direct product of A copies of B which can be think of the set of all functions from A to B with finite support, that is, having the property that $(x)f = 1_B$ for all but finitely many x in A. The (restricted) wreath product is the set $B^{\oplus A} \times A$ with multiplication defined by

$$(a,h)(a',h') = (aa', h^{a'}h')$$

where $h^a : A \to B$ is defined by

$$(x)h^a = (ax)h, \quad (x \in A).$$

It is also a well known fact that this wreath product is a monoid with identity $(\widetilde{1}, 1_A)$, where $x\widetilde{1} = 1_B$ for all $x \in A$.

The general product of the (restricted) monoid $A^{\oplus B}$ by the (restricted) monoid $B^{\oplus A}$ denoted by $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi} B^{\oplus A}$ defined on the set $A^{\oplus B} \times B^{\oplus A}$ with multiplication defined by

$$(f,h)(f',h') = (f\ {}^{h}f', h^{f'}\ h'), \qquad (f,f' \in A^{\oplus B} \text{ and } h,h' \in B^{\oplus A})$$

where $\delta : B^{\oplus A} \to \tau(A^{\oplus B})$, $(f')\delta_h = {}^{h}f'$ and $\psi : A^{\oplus B} \to \tau(B^{\oplus A})$, $(h)\psi_{f'} = h^{f'}$ are defined by, for $a \in A$ and $b \in B$,

$${}^{h}f' = {}^{(h^a)}f' \text{ and } h^{f'} = h^{({}^{b}f')}.$$

Also, for $x \in A$ and $y \in B$, we define

$$(x)h^a = (ax)h \ and\ (y)^{b}f' = (yb)f'$$

such that, for all $c \in A$, $d \in B$,

$$(d)^{(h^a)}f' = (dh^a)f' \text{ and } (c)h^{({}^{b}f')} = ({}^{b}f'c)h.$$

such that the following general axioms are satisfied for all $f,f' \in A^{\oplus B}$ and $h,h' \in B^{\oplus A}$

(GP1) ${}^{(hh')}f = {}^{h}({}^{h'}f)$;

(GP2) ${}^{h}(ff') = ({}^{h}f)({}^{h^f}f')$;

(GP3) $(h^f)^{f'} = h^{(ff')}$;

(GP4) $(hh')^f = h^{({}^{h'}f)}(h')^f$;

(GP5) ${}^{h}\overline{1} = \overline{1}$;

(GP6) $h^{\overline{1}} = h$;

(GP7) ${}^{\widetilde{1}}f = f$;

(GP8) $\widetilde{1}^f = \widetilde{1}$.

It is easy to show that the general products $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi} B^{\oplus A}$ is a monoid with identity $(\overline{1}, \widetilde{1})$, where $\overline{1} : B \to A$, $(b)\overline{1} = 1_A$ and $\widetilde{1} : A \to B$, $(a)\widetilde{1} = 1_B$, for all $a \in A$ and $b \in B$.

Green's relations play crucial role in the theory of semigroups — in particular, for decompositions of such semigroups. As known Green's relations are equivalence relations that characterize the elements of a semigroup in terms of the principle ideals. For more information about them and further references consult [5]. In Section 2, we consider Green's relations on the generalized of general product of wreath products for monoids, we state some results related to $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, and $\mathcal{J}$ relations depending on these relations on the wreath products form our generalized general products. As inspired by Pastijn [14] and El-Qallali-Fountain [4], Ren-Shum

[15] have introduced generalized Green's relations. The relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{H}^*$, and $\mathcal{J}^*$ play an important role in the theory of abundant semigroups which is to some extent analogous to that Green's relations in the theory of semigroups [4]. In Section 3, we characterize generalized Green's relations on the generalized of general product of wreath products for monoids. For more details of the terminologies in this paper the reader is refer to [1], [3], [7], [8] and [10]. We begin by a straightforward lemma.

Lemma 1. *Let $G = (A^{\oplus B}, B^{\oplus A})$ be the generalized of general product of a monoid $A^{\oplus B}$ by the a group $B^{\oplus A}$. Then*

(1) $({}^h f)^{-1} = {}^{h^f} f^{-1}$ *for all $f \in A^{\oplus B}$ and $h \in B^{\oplus A}$.*

(2) *The function $\phi_h : (A^{\oplus B})_h \to A^{\oplus B}$ given by $f \mapsto {}^h f$ is a homomorphism.*

(3) *Let $h' = h^f$. Then $(A^{\oplus B})_{h'} = f(A^{\oplus B})_h f^{-1}$ and $\phi_{h'}(f') = ({}^h f)\phi_h(f^{-1}f'f)({}^h f)^{-1}$.*

(4) *If ϕ_h is injective then ϕ_{h_f} is injective.*

(5) *The function from $A^{\oplus B}$ to $A^{\oplus B}$ defined by $f \mapsto {}^h f$ is injective for all $h \in B^{\oplus A}$ iff for all $h \in B^{\oplus A}$, if ${}^h f = \overline{1}$ then $f = \widetilde{1}$.*

Proof. (1) This proved in [17].

(2) Let $f, f' \in (A^{\oplus B})_h$. Then

$$\phi_h(ff') = {}^h ff' = ({}^h f)({}^{h^f} f') = {}^h f \ \ {}^h f' = \phi_h(f)\phi_h(f')$$

using (GP2), which is the required.

(3) We have that $h'^{f'} = h'$ iff $(h^f)^{f'} = h^f$ iff $h^{(f^{-1}f'f)} = h$ iff $f^{-1}f'f \in A^{\oplus B}$. Hence iff $f' \in f(A^{\oplus B})_h f^{-1}$. The proof of the other claim follows by calculating $(A^{\oplus B})_h(f^{-1}f'f)$ using (GP2) and (1) above.

(4) This directly by (3) above.

(5) One direction is obvious. We also prove the other direction. Suppose that for all $h \in B^{\oplus A}$, if ${}^h f = \overline{1}$ then $f = \overline{1}$. We prove that the function from $A^{\oplus B}$ to $A^{\oplus B}$ defined by $f \mapsto {}^h f$ is injective for all $h \in B^{\oplus A}$. Suppose that ${}^h f = {}^h f'$. Then ${}^h f \ ({}^h f')^{-1} = \overline{1}$. By (1) above, $({}^h f')^{-1} = {}^{f'_h} f'^{-1}$. Put $h' = {}^{f'} h$. Then

$$\overline{1} = {}^h f({}^h f')^{-1} = ({}^{f'^{-1}h'} h)({}^{h'} f'^{-1}) = {}^{h'} ff'^{-1}$$

by (GP2). By assumption $ff'^{-1} = \overline{1}$ and so $f = f'$. □

2. Green's Relations on Generalized General Product

We shall rely on basic notations from semigroup theory. Our references for this are [5] and [13]. In particular, we recall the definitions of Green's relations in a monoid S. Define $a\mathcal{L}b$ if and only if $Sa = Sb$, $a\mathcal{R}b$ if and only if $aS = bS$ and $a\mathcal{J}b$ if and only if $SaS = SbS$. The relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. In [16] we characterize Green's relations for generalized general product. The following is general result of [16, Propositions 4.1 and 4.2]. The structure of the principle right, left and two-sided ideals is explained in the following lemma:

Lemma 2. *Let $G=(A^{\oplus B},B^{\oplus A})$ be the generalized of general product of a monoid $A^{\oplus B}$ by a group $B^{\oplus A}$. Then*

(1) $hf\mathcal{R}h'f'$ *iff* $h=h'$.

(2) $hf\mathcal{L}h'f'$ *iff there exists an invertible element g such that* ${}^{g}h'=h$ *and* ${}^{h'}g=ff'^{-1}$.

(3) $hf\mathfrak{J}h'f'$ *iff* $h=h'$ *and there exists* $g\in(B^{\oplus A})_h$ *such that* ${}^{h}g=ff'^{-1}$.

(4) *Let* $h,h'\in B^{\oplus A}$. *Then* $h\mathfrak{J}h'$ *if and only if* $h={}^{f}h'$ *for some* $f\in A^{\oplus B}$.

Proof. (1) If $fh\mathcal{R}f'h'$, then there exist l,l' in $B^{\oplus A}$ such that $(fh)l=(f'h')l,(f'h')l'=(fh)l'$. Hence

$$({}^{f}h)(f^{h})l=(f'h')l \text{ and } ({}^{f'}h')(f'^{h'})l'=(fh)l'.$$

By uniqueness $({}^{f}h)(f^{h})=f'h'$ and $({}^{f'}h')(f'^{h'})=fh$. Hence $h=h'$, the other part is straightforward.

(2) Let such g exists. Then

$$g(h'f')=({}^{g}h')({}^{h'}g)f'=hff'^{-1}f'=hf.$$

Similar for the other part. Conversely, suppose that $hf\mathcal{L}h'f'$. Then there is an invertible element g such that $g(h'f')=hf$. The result is follows now.

(3) Follows directly by (1) and (2).

(4) Assume that $h\mathfrak{J}h'$. Then there are group elements $l,l'\in B^{\oplus A}$ such that $h=fh'l'$. But $fh'l'=({}^{f}h')(f^{h'})l'$. By uniqueness, we have that $h=({}^{f}h')$. Conversely, assume that $h=({}^{f}h')$. Notice that $Gfh'G=G(({}^{f}h')(f^{h'}))G=G({}^{f}h')G=GhG$. Hence $GhG=Gh'G$, therefore $h\mathfrak{J}h'$ as required. □

Recall that [12] Green's relation $\mathcal{R}$ is defined by $a\mathcal{R}b$ if and only if $aS=bS$, this is equivalent to the existence of a unit g such that $a=gb$ as known this relation is always left congruence.

Proposition 3. *The action of the group $B^{\oplus A}$ on the monoid $A^{\oplus B}$ is trivial if and only if the relation $\mathcal{R}$ is a right congruence, and thus a congruence.*

Proof. Assume the action is trivial. Let $h\mathcal{R}h'$. We need to show that $hl\mathcal{R}h'l$, where $h=h'g$ and $l=yk$. Then

$$hl=h'gl=h'gyk=h'({}^{g}y)(g^{y})k=(h'yk)^{y}k=h'yk((k^{-1}g)^{y}k)=h'l((k^{-1}g)^{y}k).$$

Hence $hl\mathcal{R}h'l$ which is the result. Conversely, assume that $\mathcal{R}$ is a right congruence. Let $h\in B^{\oplus A}$. Hence $hG=G=\overline{1}G$ and so $h\mathcal{R}\overline{1}$. Let $f\in A^{\oplus B}$ be any function. By assumption, $hf\mathcal{R}f$. Therefore $hfG=fG$. Hence there is a unit l such that $hf=fl$. But $hf=({}^{h}f)(h^{f})$, by uniqueness, $f={}^{h}f$, and so the action is trivial. □

In the next result we reveal the role of the action of the group $B^{\oplus A}$ on the monoid $A^{\oplus B}$ with the features of the principle two-sided ideals. By [17], we know this action is length-preserving. Recall [11] consider when the action of $B^{\oplus A}$ on $A^{\oplus B}$ is transitive. The following result is a generalization of Proposition 3.15 in [13].

Proposition 4. *Let $G=(A^{\oplus B},B^{\oplus A})$ be the generalized of general product of a monoid $A^{\oplus B}$ by a group $B^{\oplus A}$. Then the action of $B^{\oplus A}$ on $A^{\oplus B}$ is transitive if and only if G has a maximal proper principal two-sided ideal.*

Proof. Suppose that for some $f\in A^{\oplus B}$, the two-sided ideal GfG is proper and maximum. Let $f=gh$ where the length of g is one. Thus $GfG\subseteq GgG$. Now GfG is proper and maximal, so either $GgG=G$, which implies that g of length zero, or $GfG=GgG$ which means f of length one. Now, let $g\in A^{\oplus B}$. Then $GgG\subseteq GfG$ and $|g|=|f|$, hence $g={}^{l}f$ for some l. Since the choice of g was arbitrary, we can conclude that the action of $B^{\oplus A}$ on $A^{\oplus B}$ is transitive. Conversely, suppose that the action of $B^{\oplus A}$ on $A^{\oplus B}$ is transitive. Then for any two $f,g\in A^{\oplus B}$ of length one, we have $f={}^{l}g$ and so by Lemma 4.3 in [11] $GfG=GgG$. Now suppose GkG be any two-sided ideal of such that the length of k is greater than or equal two. Since $k=fg$ where the length of f is one. Then $GkG\subseteq GfG$. Therefore GfG is a proper maximal two sided ideal as required. □

3. Generalized Green's Relations on Generalized General Products

In [18] Zenab describe generalized Green's relations for the general product of semigroups and monoids. In this section we aim to characterize generalized Green's relations on generalized general product. We consider generalized general products of semigroups and monoids and study the roles of the relations $\mathcal{R}^*$, $\mathcal{L}^*$, $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$. Recall the relations $\mathcal{R}^*$, $(\mathcal{L}^*)$ on semigroups S is defined by the rule that for any $s,t\in S$, $s\mathcal{R}^*t$ $(s\mathcal{L}^*t)$ if and only if for all $u,v\in S^1$ (the extension of S by adding the identity element)

$$us=vs \text{ iff } ut=vt \quad (su=sv \text{ iff } tu=tv).$$

We observe that $\mathcal{L}^*$ and $\mathcal{R}^*$ are equivalence relations. It is obvious that $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence. Dissimilar Green's relations, the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ need not commute. We note that an idempotent e of S acts as a right identity for its $\mathcal{L}^*$-class and left identity for its $\mathcal{R}^*$-class. Also, note that $\mathcal{L}\subseteq\mathcal{L}^*$ and $\mathcal{R}\subseteq\mathcal{R}^*$, and if S is regular then $\mathcal{L}=\mathcal{L}^*$ and $\mathcal{R}=\mathcal{R}^*$. Now, we move to the relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$. Let S be a semigroup and let E be a subset of $E(S)$, where $E(S)$ denotes the set of idempotents of S. The relation $s\widetilde{\mathcal{L}}_Et$ if and only s and t have same set of right identities in E. Dually, $\widetilde{\mathcal{R}}_E$ on S is defined as for any $s,t\in S$, $s\widetilde{\mathcal{R}}_Et$ if and only if for all $e\in E$, $es=s$ if and only if $et=t$, that is, s and t have same set of left identities in E. Certainly $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are equivalence relations.

The result can be obtained from the definition:

Lemma 5. *Let $G=\left(A^{\oplus B},B^{\oplus A}\right)$ be the generalized of general product of monoids $A^{\oplus B}$ and $B^{\oplus A}$, then $(f,h)\mathcal{R}^*(f',h')$ if and only if for all $g,g'\in A^{\oplus B}$ and for all $l,l'\in B^{\oplus A}$*

$$g({}^{l}f)=g'({}^{l'}f) \quad \text{and} \quad lfh=l'fh$$
$$\Leftrightarrow \quad g({}^{l}f')=g'({}^{l'}f') \quad \text{and} \quad lf'h'=l'f'h'.$$

Similarly for the $\mathcal{L}^*$ relation.

Proposition 6. *Let $G=(A^{\oplus B},B^{\oplus A})$ be the generalized of general product of monoids $A^{\oplus B}$ and $B^{\oplus A}$. Then the following hold:*

(1) *if* $h, h' \in B^{\oplus A}$ *and* $h\mathcal{R}^* h'$ *then* $(f,h)\mathcal{R}^*(f,h')$ *in* G *for* $f \in A^{\oplus B}$*;*
(2) *if* $f, f' \in A^{\oplus B}$ *and* $f\mathcal{L}^* f'$*, then* $(f,h)\mathcal{L}^*(f',h)$ *in* G *for* $h \in B^{\oplus A}$.

Proof. (1) Suppose that $h\mathcal{R}^* h'$ for some $h, h' \in B^{\oplus A}$. Let $g, g' \in A^{\oplus B}$ and $l, l' \in B^{\oplus A}$, then for $f \in A^{\oplus B}$

$$\begin{aligned} & g({}^l f) = g'({}^{l'} f) \quad \text{and} \quad l^f\ h = l'^f\ h \\ \Leftrightarrow\quad & g({}^l f) = g'({}^{l'} f) \quad \text{and} \quad l^f\ h' = l'^f\ h' \text{ because } h\mathcal{R}^* h'. \end{aligned}$$

By previous Lemma $(f,h)\mathcal{R}^*(f,h')$ in G. Similar proof for part (2). □

Proposition 7. *Let* $G = (A^{\oplus B}, B^{\oplus A})$ *be the generalized of general product of monoids* $A^{\oplus B}$ *and* $B^{\oplus A}$*. Then*
(1) $(f,h)\mathcal{R}^*(f',h')$ *in* G *implies* $f\mathcal{R}^* f'$ *in* $A^{\oplus B}$*;*
(2) $(f,h)\mathcal{L}^*(f',h')$ *in* G *implies* $h\ \mathcal{L}^*\ h'$ *in* $B^{\oplus A}$.

Proof. Suppose $(f,h)\mathcal{R}^*(f',h')$ in G. To show that $f\mathcal{R}^* f'$ in $A^{\oplus B}$, let $g, g' \in A^{\oplus B}$ be such that $gf = g'f$. Then

$$g({}^{\overline{1}} f) = g'({}^{\overline{1}} f) \quad \text{and} \quad \widetilde{1}^f\ h = \widetilde{1}^f\ h,$$

so by previous lemma, $g({}^{\overline{1}} f') = g'({}^{\overline{1}} f')$, that is, $gf' = g'f'$. Thus together with the converse direction, we obtain $f\mathcal{R}^* f'$ in $A^{\oplus B}$. Similar proof for $\mathcal{L}^*$. □

Recall [5] $\ker\alpha = \{(a,b) \in A \times B : a\alpha = b\alpha\}$ where $\alpha : A \to B$ is a function from A to B. We will record now the behavior of the generalized Green's relations when we have one of the actions is trivial action (action by identity map) in this case we have the semidirect product $A^{\oplus B}\ {}_{\delta}\!\rtimes B^{\oplus A}$ (if ψ acts trivially) and $A^{\oplus B} \ltimes_{\psi} B^{\oplus A}$ (if δ acts trivially).

Proposition 8. *Suppose* $T = A^{\oplus B} \ltimes_{\psi} B^{\oplus A}$ *is a semidirect product of monoids* $A^{\oplus B}$ *and* $B^{\oplus A}$*, where* $B^{\oplus A}$ *is right cancellative. Also, suppose that for any* $f, f' \in A^{\oplus B}$*,* $f\mathcal{R}^* f'$ *implies* $\operatorname{Ker} f = \operatorname{Ker} f'$ *(where* $\operatorname{Ker} f$ *is the kernel of the map induced by the right action of* f*). Then* $f\mathcal{R}^* f'$ *in* $A^{\oplus B}$ *implies that* $(f,h)\mathcal{R}^*(f',h')$ *in* T *for all* $h, h' \in T$.

Proof. Suppose $f\mathcal{R}^* f'$ in $A^{\oplus B}$ and let $g, g' \in A^{\oplus B}$ and $l, l' \in B^{\oplus A}$. Then $gf = g'f$ and $l^f\ h = l'^f\ h$ if and only if $gf = g'f$ and $l^f = l'^f$ because $B^{\oplus A}$ is cancellative if and only if $gf' = g'f'$ and $l^{f'} = l'^{f'}$ because $f\mathcal{R}^* f'$ and $\operatorname{Ker} f = \operatorname{Ker} f'$ if and only if $gf' = g'f'$ and $l^{f'}h' = l'^{f'}h'$ because $B^{\oplus A}$ is cancellative. Thus by previous lemma yields $(f,h)\mathcal{R}^*(f',h')$ in T. □

Corollary 9. *Suppose* $T = A^{\oplus B} \ltimes_{\psi} B^{\oplus A}$ *is a semidirect product of monoids* $A^{\oplus B}$ *and* $B^{\oplus A}$*, where* $B^{\oplus A}$ *is right cancellative. Also, suppose that* $A^{\oplus B}$ *acts on* $B^{\oplus A}$ *injectively. Then* $f\mathcal{R}^* f'$ *in* $A^{\oplus B}$ *implies that* $(f,h)\mathcal{R}^*(f',h')$ *in* T *for all* $h, h' \in T$.

Theorem 10. *Let* $G = (A^{\oplus B}, B^{\oplus A})$ *be the generalized of general product of monoids* $A^{\oplus B}$ *and* $B^{\oplus A}$*, where* $B^{\oplus A}$ *is right cancellative. Suppose* $A^{\oplus B}$ *acts faithfully on the right of* $B^{\oplus A}$ *and for any* $f, f' \in A^{\oplus B}$*,* $f\mathcal{R}^* f'$ *implies* $\operatorname{Ker} f = \operatorname{Ker} f'$*. Then* $f\mathcal{R}^* f'$ *in* $A^{\oplus B}$ *implies that* $(f,h)\mathcal{R}^*(f',h')$ *in* G *for all* $h, h' \in B^{\oplus A}$.

Proof. Suppose $(f,h),(f',h') \in G$ and $f\mathcal{R}^*f'$ in $A^{\oplus B}$. To show that $(f,h)\mathcal{R}^*(f',h')$ in G, let $g,g' \in A^{\oplus B}$ and $l,l' \in B^{\oplus A}$ be such that $g({}^l f) = g'({}^{l'} f)$ and $l^f h = l'^f h$. Then $l^f = l'^f$, as $B^{\oplus A}$ is right cancellative. Also as $\operatorname{Ker} f = \operatorname{Ker} f'$, we have $l^{f'} = l'^{f'}$ and thus $l^{f'}\ h' = l'^{f'}\ h'$. Now for any $k \in B^{\oplus A}$,

$$(k^g l)^f = k^{g({}^l f)}\ l^f = k^{g'({}^{l'} f)}\ l'^f = (k^{g'} l')^f$$

and so as $\operatorname{Ker} f = \operatorname{Ker} f'$, $(k^g l)^{f'} = (k^{g'} l')^{f'}$, gives $k^{g({}^{l'} f')} l^{f'} = k^{g'({}^{l'} f')} l'^{f'}$. But $l^{f'} = l'^{f'}$ and $B^{\oplus A}$ is right cancellative, therefore $k^{g({}^{l'} f')} = k^{g'({}^{l'} f')}$. As this is true for any $k \in B^{\oplus A}$, and $A^{\oplus B}$ acts faithfully, we have $g({}^l f') = g'({}^{l'} f')$. Together with the opposite result of previous Lemma gives $(f,h)\mathcal{R}^*(f',h')$ in G. □

Recall $E(S) = \{s \in S : s^2 = s\}$ is the set of idempotent elements of any semigroup S. In the next result we record the behavior of $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ depending on the set of idempotents in $G = (A^{\oplus B}, B^{\oplus A})$ the generalized of general product of monoids $A^{\oplus B}$ and $B^{\oplus A}$.

Theorem 11. *Let $G = (A^{\oplus B}, B^{\oplus A})$ the generalized of general product of monoids $A^{\oplus B}$ and $B^{\oplus A}$. Let $A \subseteq E(A^{\oplus B})$ and $B \subseteq E(B^{\oplus A})$. Put*

$$\overline{A} = \{(a,\overline{1}) : a \in A\} \ \ and \ \ \overline{B} = \{(\widetilde{1}, b) : b \in B\}.$$

Then $\overline{A}$ and $\overline{B}$ are sets of idempotents in G and

(1) *$(f,h)\widetilde{\mathcal{R}}_{\overline{A}}(f',h')$ in G if and only if $f\widetilde{\mathcal{R}}_A f'$ in $A^{\oplus B}$;*

(2) *$(f,h)\widetilde{\mathcal{L}}_{\overline{B}}(f',h')$ in G if and only if $h\ \widetilde{\mathcal{L}}_B\ h'$ in $B^{\oplus A}$.*

Proof. (1) Suppose $(f,h)\widetilde{\mathcal{R}}_{\overline{A}}(f',h')$ in G. Let $a \in A$. Then $af = f$ if and only if $a({}^{\overline{1}} f) = f$ by action of $B^{\oplus A}$ on $A^{\oplus B}$ if and only if $(a,\overline{1})(f,h) = (f,h)$ if and only if $(a,\overline{1})(f',h') = (f',h')$ because $(f,h)\widetilde{\mathcal{R}}_{\overline{A}}(f',h')$ if and only if $a({}^{\overline{1}} f') = f'$ if and only if $af' = f'$. Hence $f\widetilde{\mathcal{R}}_A f'$ in $A^{\oplus B}$. Conversely, suppose that $f\widetilde{\mathcal{R}}_A f'$ in $A^{\oplus B}$ and let $(f,h) \in G$. Then for $(f,\overline{1}) \in A$ we have $(f,\overline{1})(f,h) = (f,h)$ if and only if $(a({}^{\overline{1}} f), \overline{1}^f h) = (f,h)$ if and only if $(af,h) = (f,h)$ if and only if $af = f$ if and only if $af' = f'$ because $f\widetilde{\mathcal{R}}_A f\,'$ if and only if $(af',h') = (f',h')$ if and only if $(a({}^{\overline{1}} f'), \overline{1}^{f'}\ h') = (f',h')$ if and only if $(a,\overline{1})(f',h') = (f',h')$. Hence $(f,h)\widetilde{\mathcal{R}}_{\overline{A}}(f',h')$. Similar proof for $\widetilde{\mathcal{L}}_B$. □

In the following result let there exists a right identity a for $A^{\oplus B}$ and a left identity b for $B^{\oplus A}$ such that

$${}^l a = a, l^a = l \quad \text{for all } \ l \in B^{\oplus A}, \tag{LI}$$

$${}^b g = g, b^g = b \quad \text{for all } \ g \in A^{\oplus B} \tag{RI}$$

and then we put $\overline{A} = \{(f,b) : f \in A\}$ and $\overline{B} = \{(a,h) : h \in B\}$. We have the following result:

Theorem 12. *Let $G = (A^{\oplus B}, B^{\oplus A})$ be the generalized of general product of monoids $A^{\oplus B}$ and $B^{\oplus A}$. Let $A \subseteq E(A^{\oplus B})$ and $B \subseteq E(B^{\oplus A})$. Then*

(1) *$\overline{A}$ and $\overline{B}$ are sets of idempotents in G;*

(2) *$(f,h)\widetilde{\mathcal{L}}_{\overline{B}}(f',h')$ implies $h\widetilde{\mathcal{L}}_B h'$ and $(f,h)\widetilde{\mathcal{R}}_{\overline{A}}(f',h')$ implies $f\widetilde{\mathcal{R}}_A f'$;*

(3) *$(f,h)\mathcal{R}^*(f',h')$ implies $f\mathcal{R}^* f'$ in $A^{\oplus B}$ and $(f,h)\mathcal{L}^*(f',h')$ implies $h\ \mathcal{L}^*\ h'$ in $B^{\oplus A}$.*

Proof. (1) To prove that $\overline{A}$ and $\overline{B}$ are sets of idempotents in G. Let $(f,b)\in\overline{A}$. Then $(f,b)(f,b)=(f\,{}^{b}f,b^{f}b)=(ff,bb)$ (by RI). Hence $(f,b)(f,b)=(f,b)$. Therefore, $\overline{A}\subseteq E(G)$ similarly for $\overline{B}$.

(2) Let $(f,h),(f',h')\in G$ such that $(f,h)\widetilde{\mathcal{L}}_{\overline{B}}(f',h')$. Let $k\in B$ such that $hk=h$. Then $h^{a}k=h$ (by LI). Now as ${}^{h}a=a$ and $fa=f$, we have $(f,h)(a,k)=(f,h)$ which implies $(f',h')(a,k)=(f',h')$ because $(f,h)\widetilde{\mathcal{L}}_{\overline{B}}(f',h')$ which implies that $f'({}^{h'}a)=f'$ and $h'^{a}k=h'$ which would give $h'k=h'$. Together with the other direction, we have $h\widetilde{\mathcal{L}}_{B}h'$. Similarly, $(f,h)\widetilde{\mathcal{R}}_{\overline{A}}(f',h')$ implies $f\widetilde{\mathcal{R}}_{A}f'$.

(3) Suppose that $(f,h)\mathcal{R}^{*}(f',h')$ and let $g,g'\in A^{\oplus B}$ be such that $gf=g'f$. Then $g({}^{b}f)=g'({}^{b}f)$ (by RI) which implies that $(g,b)(f,h)=(g',b)(f,h)$ which implies that $(g,b)(f',h')=(g',b)(f',h')$ because $(f,h)\mathcal{R}^{*}(f',h')$ which implies that $g({}^{b}f')=g'({}^{b}f')$ which implies that $gf'=g'f'$. Moreover, $f=g'f=g'({}^{b}f)$ which implies that $(f,h)=(g',b)(f,h)$ which implies $(f',h')=(g',b)(f',h')$ because $(f,h)\mathcal{R}^{*}(f',h')$ hence $f'=g'({}^{b}f')$ that is $f'=g'f'$. Together with the dual, we get $f\mathcal{R}^{*}f'$ in $A^{\oplus B}$. Similar proof for $(f,h)\mathcal{L}^{*}(f',h')$ implies $h\ \mathcal{L}^{*}\ h'$ in $B^{\oplus A}$. □

4. Conclusions

In this paper, we investigated some arithmetic properties of Generalization for General Products $A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ of wreath product of monoids such as Green's relations and generalized Green's relations which play an important role in the theory of abundant semigroups which is to some extent analogous to that Green's relations in the theory of semigroups. We characterize generalized Green's relations on the generalized of general product of wreath products for monoids. Of course, there are still so many different structure properties that can be checked on this important product.

For future work the author will investigate special kind of semigroups such as inverse semigroups and study how this would interrupt with this new definition.

Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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