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Research Article

Common Fixed Point Theorems of Quadruple Mappings Satisfying *CLR* Property in G_p Metric Spaces With Applications

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Abstract. The aim of this paper is to establish common fixed point theorems for quadruple of weakly compatible mappings satisfying a new type of common limit range property and involving almost altering distances in G_p metric space. Furthermore, we present an example to validate our main result. Further, we obtain some common fixed point theorems for mappings satisfying contractive conditions of integral type and for φ-contractive mappings.

Keywords. Fixed point; Common fixed point; Weak compatible maps; Almost altering distance; New CLR property

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1. Introduction

Metric fixed point theory plays an important role in mathematics. Banach contraction principle is one of the fundamental results in fixed point theory and is generalized in various direction. Fixed point theorems provide a tool to solve the many problems and have application in nonlinear analysis in various generalized metric spaces. Jungck [7] proved a common fixed point theorem for commuting as a generalization of the Banach's fixed point theorem. Sessa [24] introduced the concept of weakly commuting mappings, Jungck [8] extended this concept to compatible maps. In 1998, Jungck and Rhoades [9] introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but converse not true.

Zand and Nezhad [2] introduced a generalized partial metric space (G_p metric space) by combination of a generalized metric space (G-metric space) due to Mustafa and Sims [13, 14] and a partial metric space introduced by Matthews [12]. Aydi *et al*. [3] established first fixed point result in Gp-metric space. Many authors obtained the results on fixed points in G_p metric space ([4, 5, 15, 21, 22]). Recently, many authors obtained common fixed point theorems under some conditions in G_p metric space ([20, 25]).

2. Preliminaries

We recall some definitions which will be used in the sequel.

Definition 2.1 ([2, 15]). Let X be a nonempty set and function $G : X^3 \to \mathbb{R}_+$ is called a G_p-metric space on X if the following conditions are satisfied

(1) $x = y = z$ if $G_p(x,y,z) = G_p(x,x,x) = G_p(y,y,y) = G_p(z,z,z)$,

(2) $0 \le G_p(x,x,x) \le G_p(x,x,y) \le G_p(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,

(3) $G_p(x,y,z) = G_p(y,z,x) = \dots$ (symmetry in all three variables),

(4) $G_p(x,y,z) \le G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$ for all $x,y,z,a \in X$ (triangle inequality).

The pair (X,G_p) is called a G_p-metric space.

Lemma 2.2 ([3]). *Let (X,G_p) be a G_p metric space. Then*

(1) *if $G_p(x,y,z) = 0$ then $x = y = z$,*

(2) *if $x \neq y$ then $G_p(x,y,y) > 0$.*

Definition 2.3 ([2]). Let (X,G_p) be a G_p metric space and let $\{x_n\}$ be sequence of points in X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m\to\infty} G_p(x,x_n,x_m) = G_p(x,x,x)$. Then the sequence $\{x_n\}$ is called G_p convergent to x.

Lemma 2.4 ([2]). *Let (X,G_p) be a G_p metric space. Then, for any $\{x_n\} \subset X$ and $x \in X$, the following properties are equivalent*

(1) *$\{x_n\}$ is G_p convergent to x,*

(2) *$G_p(x_n,x_n,x) \to G_p(x,x,x)$ as $n \to \infty$,*

(3) *$G_p(x_n,x,x) \to G_p(x,x,x)$ as $n \to \infty$.*

Lemma 2.5 ([15]). *If $x_n \to x$ in a G_p-metric space (X,G_p) and $G_p(x,x,x) = 0$, then for every $y \in Y$*

(1) *$\lim_{n\to\infty} G_p(x_n,y,y) = G_p(x,y,y)$,*

(2) *$\lim_{n\to\infty} G_p(x_n,x_n,y) = G_p(x,x,y)$.*

Definition 2.6 ([16])**.** Let A, S and T be a self mappings of a G_p-metric space (X,G_p). Then the pair (A,S) is said to satisfy the common limit range property with respect to T, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$ with $G_p(z,z,z) = 0$ and $z \in S(X) \cap T(X)$.

Definition 2.7 ([10])**.** An altering distance is a mapping $\psi : [0,\infty) \to [0,\infty)$ such that

(1) ψ is increasing and continuous,

(2) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.8 ([19])**.** A function $\psi : [0,\infty) \to [0,\infty)$ is an almost altering distance if

(1) ψ is continuous,

(2) $\psi(t) = 0$ if and only if $t = 0$.

Lemma 2.9 ([1])**.** *Let f, g be two weakly compatible self mappings of a nonempty set X. If f and g have a unique point of coincidence $w = fx = gx$ for some $x \in X$, then w is the unique common fixed point of f and g.*

3. Implicit Relations

Popa [17, 18] introduced the implicit function to proved the existence of fixed points and found that several fixed point theorems have been considered some general conditions by an implicit function.

We use the following implicit relations in our main result.

Definition 3.1. Let F_{G_p} be the set of all real continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the conditions:

(F_1) $F(t,0,t,0,0,t) > 0$, $\forall\ t > 0$,

(F_2) $F(t,0,0,t,t,0) > 0$, $\forall\ t > 0$.

Now, we present some examples of the function $F : \mathbb{R}_+^6 \to \mathbb{R}$ which satisfy the conditions (F_1), (F_2).

Example 3.2. $F(t_1,t_2,\dots t_6) = t_1 - k\max\left\{\frac{t_2+t_3}{3}, t_4, t_5, t_6\right\}$, where $k \in [0,1)$.

(F_1) $F(t,0,t,0,0,t) = t - k(t) = t(1-k) > 0$, $\forall\ t > 0$,

(F_2) $F(t,0,0,t,t,0) = t - k(t) = t(1-k) > 0$, $\forall\ t > 0$.

Example 3.3. $F(t_1,t_2,\dots t_6) = t_1 - k\max\left\{t_2, \frac{t_3+t_4}{3}, \frac{t_5+t_6}{3}\right\}$, where $k \in [0,1)$.

(F_1) $F(t,0,t,0,0,t) = t - k\left(\frac{t}{3}\right) = t\left(1-\frac{k}{3}\right) > 0$, $\forall\ t > 0$,

(F_2) $F(t,0,0,t,t,0) = t - k\left(\frac{t}{3}\right) = t\left(1-\frac{k}{3}\right) > 0$, $\forall\ t > 0$.

Example 3.4. $F(t_1,t_2,\dots t_6) = t_1^2 - a\{t_2^2 + t_3^2\} - b\max\{t_1.t_3, t_2.t_4, t_5.t_6\}$, where $a,b \in \mathbb{R}$ and $a+b < 1$.

(F_1) $F(t,0,t,0,0,t) = t^2 - at^2 - bt^2 = t^2[1-(a+b)] > 0$, $\forall\ t > 0$,

(F_2) $F(t,0,0,t,t,0) = t^2 - a.0 - b.0 = t^2 > 0$, $\forall\ t > 0$.

Example 3.5. $F(t_1,t_2,\dots t_6) = t_1^2 - at_2.t_4 - b\max\{t_1.t_3,t_5.t_6\}$, where $a,b \in \mathbb{R}$ and $a+b<1$.

(F_1) $F(t,0,t,0,0,t) = t^2 - bt^2 = t^2(1-b) > 0$, $\forall\ t>0$,

(F_2) $F(t,0,0,t,t,0) = t^2 - a.0 - b.0 = t^2 > 0$, $\forall\ t>0$.

Example 3.6. $F(t_1,t_2,\dots t_6) = t_1^2 - at_1.t_3 - bt_2.t_4 - ct_5.t_6$, where $a,b \in \mathbb{R}$ and $a+b+c<1$.

(F_1) $F(t,0,t,0,0,t) = t^2 - at^2 = t^2(1-a) > 0$, $\forall\ t>0$,

(F_2) $F(t,0,0,t,t,0) = t^2 - a.0 - b.0 - c.0 = t^2 > 0$, $\forall\ t>0$.

Example 3.7. $F(t_1,t_2,\dots t_6) = t_1 - a\sqrt{t_1t_3} - b\max\{t_2,t_4,t_5,t_6\}$, where $a,b \in \mathbb{R}$ and $a+b<1$.

(F_1) $F(t,0,t,0,0,t) = t - at - bt = t[1-(a+b)] > 0$, $\forall\ t>0$,

(F_2) $F(t,0,0,t,t,0) = t - bt = t(1-b) > 0$, $\forall\ t>0$.

In this paper we establish a new common fixed point theorem for quadruple of weakly compatible mappings satisfying common limit range property and involving almost altering distances in G_p metric space. Furthermore, we present the example to support our main result. At last part we obtain some common fixed point results for mappings satisfying contractive conditions of integral type and for φ-contractive mappings.

4. Main Results

Now, we present our main result.

Theorem 4.1. *Let A,B,S and T be self mappings of a G_p metric space (X,G_p) satisfying*

$$\begin{aligned}F\Big(&\psi(G_p^2(Ax,By,By)),\psi(G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)),\\&\psi(G_p(Ty,By,By).G_p(Sx,By,By)),\psi(G_p(Ax,By,By).G_p(Ax,Ty,Ty)),\\&\psi(G_p^2(Ax,Ty,Ty)),\psi(G_p^2(Ty,By,By))\Big)\le 0 \qquad (4.1)\end{aligned}$$

for all $x,y \in X$, where $F \in \mathbb{F}_{G_p}$ and ψ is almost altering distance. If (A,S) and T satisfy $CLR_{(A,S)T}$ property, then

(1) $C(A,S) \neq 0$,

(2) $C(B,T) \neq 0$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point z and $G_p(z,z,z)=0$.

Proof. Since (A,S) and T satisfy $CLR_{(A,S)}$-property, then there exists a sequence x_n in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ with $G_p(z,z,z)=0$ and $z \in S(X)\cap T(X)$.

Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$. By (4.1) we obtain

$$\begin{aligned}F\Big(&\psi(G_p^2(Ax_n,Bu,Bu)),\psi(G_p(Sx_n,Tu,Tu).G_p(Sx_n,Sx_n,Ax_n)),\\&\psi(G_p(Tu,Bu,Bu).G_p(Sx_n,Bu,Bu)),\psi(G_p(Ax_n,Bu,Bu).G_p(Ax_n,Tu,Tu)),\\&\psi(G_p^2(Ax_n,Tu,Tu)),\psi(G_p^2(Tu,Bu,Bu))\Big)\le 0. \qquad (4.2)\end{aligned}$$

Since $G_p(Ax_n,Sx_n,Sx_n) \le G_p(Ax_n,z,z)+G_p(z,Sx_n,Sx_n)$, by Lemma 2.5

$$\lim_{n\to\infty} G_p(Ax_n,Sx_n,Sx_n) \le G_p(z,z,z)+G_p(z,z,z)=0.$$

Let n tend to infinity in (4.2), we obtain

$$F\big(\psi(G_p^2(z,Bu,Bu)),0,\psi(G_p(z,Bu,Bu).G_p(z,Bu,Bu)),0,0,\psi(G_p^2(z,Bu,Bu))\big) \le 0,$$

implies

$$F\big(\psi(G_p^2(z,Bu,Bu)),0,\psi(G_p^2(z,Bu,Bu),0,0,\psi(G_p^2(z,Bu,Bu))\big) \le 0,$$

which contradicts (F_1), if $G_p^2(z,Bu,Bu)>0$. Hence, $G_p^2(z,Bu,Bu)=0$ and by Lemma 2.2(1), $z=Bu=Tu$. Therefore $C(B,T)\ne 0$ and $G_p(z,z,z)=0$.

Since $z\in S(X)$, there exists $v\in X$ such that $z=Sv$. By (4.1) we get

$$\begin{aligned}F\Big(&\psi(G_p^2(Av,Bu,Bu)),\psi(G_p(Sv,Tu,Tu).G_p(Sv,Sv,Av)),\\ &\psi(G_p(Tu,Bu,Bu).G_p(Sv,Bu,Bu)),\psi(G_p(Av,Bu,Bu).G_p(Av,Tu,Tu)),\\ &\psi(G_p^2(Av,Tu,Tu)),\psi(G_p^2(Tu,Bu,Bu))\Big) \le 0,\end{aligned}$$

or

$$F\big(\psi(G_p^2(Av,z,z)),0,0,\psi(G_p(Av,Bu,Bu).G_p(Av,Tu,Tu)),\psi(G_p^2(Av,Tu,Tu)),0\big) \le 0,$$

implies

$$F\big(\psi(G_p^2(Av,z,z)),0,0,\psi(G_p^2(Av,z,z)),\psi(G_p^2(Av,z,z)),0\big) \le 0,$$

which contradicts (F_2), if $G_p^2(Av,z,z)>0$. Hence, $G_p^2(Av,z,z)=0$ and by Lemma 2.2(1), $z=Av=Sv$. Therefore $C(A,S)\ne 0$ and $G_p(z,z,z)=0$.

So, $z=Av=Sv=Bu=Tu$ and z is a coincidence point of A, S and B, T.

Now, we prove that z is the unique point of coincidence of A and S and of B and T. Suppose that there exists another point of coincidence of A and S, $t=Aw=Sw$. Then by (4.1) we get

$$\begin{aligned}F\Big(&\psi(G_p^2(Aw,Bu,Bu)),\psi(G_p(Sw,Tu,Tu).G_p(Sw,Sw,Aw)),\psi(G_p(Tu,Bu,Bu).G_p(Sw,Bu,Bu)),\\ &\psi(G_p(Aw,Bu,Bu).G_p(Aw,Tu,Tu)),\psi(G_p^2(Aw,Tu,Tu)),\psi(G_p^2(Tu,Bu,Bu))\Big) \le 0,\end{aligned}$$

or

$$F\big(\psi(G_p^2(Sw,Bu,Bu)),0,0,\psi(G_p(Aw,Bu,Bu)(G_p(Aw,Tu,Tu)),\psi(G_p^2(Aw,Tu,Tu)),0)\big) \le 0,$$

implies

$$\big(\psi(G_p^2(Sw,Tu,Tu)),0,0,\psi(G_p^2(Sw,Tu,Tu)),\psi(G_p^2(Sw,Tu,Tu)),0)\big) \le 0,$$

which contradicts (F_2), if $G_p^2(Sw,Tu,Tu)>0$. Hence, $G_p^2(Sw,Tu,Tu)=0$ and by Lemma 2.2(1), $Sw=Tu=z$. Hence $t=z$ and z is the unique point of coincidence of A and S. Similarly, by (4.1), (F_1) and (F_2), we get that z is the unique point of coincidence of B and T. Hence, z is the unique point of coincidence of (A,S) and (B,T). Moreover, if (A,S) and (B,T) are weakly compatible, by Lemma 2.9, z is the unique common fixed point of A, B, S and T and $G_p(z,z,z)=0$. This completes the proof. □

If $\psi(t)=t$, we get

Theorem 4.2. *Let A,B,S and T be self mappings of a G_p metric space (X,G_p) satisfying*

$$F\Big(G_p^2(Ax,By,By),G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax),G_p(Ty,By,By).G_p(Sx,By,By),$$
$$G_p(Ax,By,By).G_p(Ax,Ty,Ty),G_p^2(Ax,Ty,Ty),G_p^2(Ty,By,By)\Big)\le 0,$$

for all $x,y\in X$, where $F\in F_{G_p}$ and ψ is almost altering distance. If (A,S) and T satisfy $CLR_{(A,S)T}$ property, then

(1) $C(A,S)\neq 0$,

(2) $C(B,T)\neq 0$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point z and $G_p(z,z,z)=0$.

Example 4.3. Let $X=[0,1]$ and $G_p(x,y,z)=\max\{x,y,z\}$. Then (X,G_p) is a G_p metric space. Define the following mappings:

$$Ax=\frac{\sqrt{x}}{2},\quad Sx=\sqrt{x},\quad Bx=0,\quad Tx=\frac{\sqrt{x}}{4}.$$

Then $S(x)=[0,1]$, $T(X)=\left[0,\frac{1}{4}\right]$ and $S(X)\cap T(X)=\left[0,\frac{1}{4}\right]$.

Let x_n be a sequence in X such that $\lim_{n\to\infty}x_n=0$. Then,

$$\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=0=z\in S(X)\cap T(X).$$

Hence, (A,S) and T satisfy $CLR_{(A,S)T}$-property with $G_p(0,0,0)=0$.

$Ax=Sx$ implies $C(A,S)=0$ and $Bx=Tx$ implies $C(B,T)=0$.

Moreover, $AS0=SA0=0$ and $BT0=TB0=0$. Hence $(A,S)$and (B,T) are weakly compatible.

On the other hand,

$$G_p^2(Ax,By,By)=\frac{x}{4}\quad\text{and}\quad G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)=x.$$

Hence,

$$G_p^2(Ax,By,By)\le kx,\quad\text{where }k\in\left[\frac{1}{4},1\right),$$

which implies

$$\begin{aligned}G_p^2(Ax,By,By)\le k\max\Big\{&G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax),\\ &\frac{1}{3}\big(G_p(Ty,By,By).G_p(Sx,By,By)+G_p(Ax,By,By).G_p(Ax,Ty,Ty)\big),\\ &\frac{1}{3}\big(G_p^2(Ax,Ty,Ty)+G_p^2(Ty,By,By)\big)\Big\},\end{aligned}$$

where $k\in\left[\frac{1}{4},1\right)$. By Theorem 4.1 and Example 3.3, A, B, S and T have a unique common fixed point $z=0$ and $G_p(z,z,z)=0$.

5. Applications

In this section we obtain some common fixed point results for mappings satisfying contractive conditions of integral type and for φ-contractive mappings.

5.1 Fixed points for mappings satisfying contractive conditions of integral type in G_p metric spaces

Branciari [6], established the following theorem which opened the way to the study of fixed points for mappings satisfying a contractive condition of integral type.

Theorem 5.1 ([6])**.** *Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ be a mapping such that for all $x, y, \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \le c\int_0^{d(x,y)} h(t)dt,$$

where $h : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping, integrable on each compact subset of $[0,\infty)$, such that $\int_0^{\varepsilon} h(t)dt > 0$ for $\varepsilon > 0$. Then f has a unique fixed point $z \in X$ and $z = \lim_{n\to\infty} f^n x$ for all $x \in X$.

Lemma 5.2 ([23])**.** *Let $h:[0,\infty)\to[0,\infty)$ be as in Theorem 5.1. Then $\psi(x) = \int_0^x h(t)dt$ is an almost altering distance.*

Now, we apply our main result to obtain the following theorem.

Theorem 5.3. *Let A,B,S and T be self mappings of a G_p metric space (X,G_p) such that*

$$F\Big(\int_0^{G_p^2(Ax,By,By)} h(t)dt, \int_0^{G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)} h(t)dt, \int_0^{G_p(Ty,By,By).G_p(Sx,By,By)} h(t)dt,$$
$$\int_0^{G_p(Ax,By,By).G_p(Ax,Ty,Ty)} h(t)dt, \int_0^{G_p^2(Ax,Ty,Ty)} h(t)dt, \int_0^{G_p^2(Ty,By,By)} h(t)dt\Big) \le 0 \tag{5.1}$$

for all $x, y \in X$, where $F \in \mathscr{F}_{G_p}$ and $h(t)$ is as in Theorem 5.1. If (A,S) and T satisfy $CLR_{(A,S)T}$ property, then

(1) *$C(A,S) \neq 0$,*

(2) *$C(B,T) \neq 0$.*

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point z and $G_p(z,z,z) = 0$.

Proof. Taking $\psi(x) = \int_0^x h(t)dt$, we get

$$\psi(G_p^2(Ax,By,By)) = \int_0^{G_p^2(Ax,By,By)} h(t)dt,$$
$$\psi(G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)) = \int_0^{G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)} h(t)dt,$$
$$\psi(G_p(Ty,By,By).G_p(Sx,By,By)) = \int_0^{G_p(Ty,By,By).G_p(Sx,By,By)} h(t)dt,$$
$$\psi(G_p(Ax,By,By).(G_p(Ax,Ty,Ty)) = \int_0^{G_p(Ax,By,By).G_p(Ax,Ty,Ty)} h(t)dt,$$
$$\psi(G_p^2(Ax,Ty,Ty)) = \int_0^{G_p^2(Ax,Ty,Ty)} h(t)dt,$$

$$\psi(G_p^2(Ty,By,By)) = \int_0^{G_p^2(Ty,By,By)} h(t)dt.$$

Then, by (5.1) we get

$$\begin{aligned}
F\Big(&\psi(G_p^2(Ax,By,By)),\psi(G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)),\\
&\psi(G_p(Ty,By,By).G_p(Sx,By,By)),\psi(G_p(Ax,By,By).G_p(Ax,Ty,Ty)),\\
&\psi(G_p^2(Ax,Ty,Ty)),\psi(G_p^2(Ty,By,By))\Big) \le 0,
\end{aligned}$$

which is inequality (4.1) of Theorem 4.1. Also, by Lemma 5.2, ψ is an almost altering distance and the conditions of Theorem 5.3 are same as Theorem 4.1. Hence, the proof is similar. □

By Theorem 5.3 and Example 3.3 we get the following:

Theorem 5.4. *Let A,B,S and T be self mappings of a G_p metric space (X,G_p) such that*

$$\begin{aligned}
\int_0^{G_p^2(Ax,By,By)} h(t)dt \le k\max\Big\{&\int_0^{G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)} h(t)dt,\\
&\frac{1}{3}\Big(\int_0^{G_p(Ty,By,By).G_p(Sx,By,By)} h(t)dt + \int_0^{G_p(Ax,By,By).G_p(Ax,Ty,Ty)} h(t)dt\Big),\\
&\frac{1}{3}\Big(\int_0^{G_p^2(Ax,Ty,Ty)} h(t)dt + \int_0^{G_p^2(Ty,By,By)} h(t)dt\Big)\Big\}
\end{aligned}$$

for all $x,y \in X$, where $k \in [0,1)$ and $h(t)$ is as in Theorem 5.1. If (A,S) and T satisfy $CLR_{(A,S)T}$ property,then

(1) $C(A,S) \neq 0$,

(2) $C(B,T) \neq 0$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point z and $G_p(z,z,z) = 0$.

Example 5.5. Let $X = [0,\infty)$ and $G_p(x,y,z) = \max\{x,y,z\}$. Then (X,G_p) is a G_p metric space. Define the following mappings:

$$Ax = \frac{\sqrt{x}}{3},\quad Sx = \frac{\sqrt{x}}{2},\quad Bx = 0,\quad Tx = \sqrt{x}.$$

Then $S(x) = [0,\infty)$, $T(X) = [0,\infty)$ and $S(X) \cap T(X) = [0,\infty)$.

Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = 0$. Then,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0 = z \in S(X) \cap T(X).$$

Hence, (A,S) and T satisfy $CLR_{(A,S)T}$-property with $G_p(0,0,0) = 0$. $Ax = Sx$ implies $C(A,S) = 0$ and $Bx = Tx$ implies $C(B,T) = 0$.

Moreover, $AS0 = SA0 = 0$ and $BT0 = TB0 = 0$. Hence $(A,S)$and (B,T) are weakly compatible.

On other hand,

$$G_p^2(Ax,By,By) = \frac{x}{9} \quad\text{and}\quad G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax) = \frac{x}{2}.$$

Moreover,

$$\int_0^{\frac{x}{9}} t\,dt \le k \int_0^{\frac{x}{2}} t\,dt$$

for $k \ge \frac{4}{81}$. Thus for $h(t) = t$, we get

$$\int_0^{G_p^2(Ax,By,By)} h(t)dt \le k \int_0^{G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)} h(t)dt,$$

where $0 < \frac{4}{81} \le k < 1$. Hence,

$$\begin{aligned}\int_0^{G_p^2(Ax,By,By)} h(t)dt \le k \max\Big\{ &\int_0^{G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)} h(t)dt,\\ &\frac{1}{3}\Big(\int_0^{G_p(Ty,By,By).G_p(Sx,By,By)} h(t)dt + \int_0^{G_p(Ax,By,By).G_p(Ax,Ty,Ty)} h(t)dt\Big),\\ &\frac{1}{3}\Big(\int_0^{G_p^2(Ax,Ty,Ty)} h(t)dt + \int_0^{G_p^2(Ty,By,By)} h(t)dt\Big)\Big\},\end{aligned}$$

where $k \in \left[\frac{4}{81}, 1\right)$. By Theorem 5.4, A, B, S and T have a unique common fixed point $z = 0$.

Remark 5.6. By Theorem 5.3 and Example 3.2, 3.4-3.7 we obtain new particular results.

5.2 Fixed points for mappings satisfying contractive conditions in G_p metric spaces

Definition 5.7 ([11])**.** Let Φ be the set of real continuous nondecreasing functions $\varphi : [0,\infty) \to [0,\infty)$ with $\lim_{n\to\infty} \varphi^n(t) = 0$. If $\varphi \in \Phi$, then

(1) $\varphi(t) < t$ for all $t \in (0,\infty)$,

(2) $\varphi(0) = 0$.

The following functions $F : \mathbb{R}_+^6 \to \mathbb{R}_+$ satisfy conditions (F_1), (F_2).

Example 5.8. $F(t_1, t_2, \dots t_6) = t_1 - \varphi\left(\max\left\{\frac{t_2+t_3}{3}, t_4, t_5, t_6\right\}\right)$.

(F_1) $F(t,0,t,0,0,t) = t - \varphi(t) > 0$, $\forall\ t > 0$,

(F_2) $F(t,0,0,t,t,0) = t - \varphi(t) > 0$, $\forall\ t > 0$.

Example 5.9. $F(t_1, t_2, \dots t_6) = t_1 - \varphi\left(\max\{t_2, \frac{t_3+t_4}{3}, \frac{t_5+t_6}{3}\}\right)$.

(F_1) $F(t,0,t,0,0,t) = t - \varphi\left(\frac{t}{3}\right) > 0$, $\forall\ t > 0$,

(F_2) $F(t,0,0,t,t,0) = t - \varphi\left(\frac{t}{3}\right) > 0$, $\forall\ t > 0$.

Example 5.10. $F(t_1, t_2, \dots t_6) = t_1^2 - \varphi\left(t_2^2 + b\max\{t_3^2, t_4^2, t_5^2, t_6^2\}\right)$, where $b \in \mathbb{R}$ and $b < 1$.

(F_1) $F(t,0,t,0,0,t) = t^2 - \varphi(bt^2) = t^2 - bt^2 = t^2(1-b) > 0$, $\forall\ t > 0$,

(F_2) $F(t,0,0,t,t,0) = t^2 - \varphi(bt^2) = t^2 - bt^2 = t^2(1-b) > 0$, $\forall\ t > 0$.

Example 5.11. $F(t_1, t_2, \dots t_6) = t_1 - \varphi\left(a\sqrt{t_1 t_3} + b\max\{t_2, t_4, t_5, t_6\}\right)$, where $a, b \in \mathbb{R}$ and $a + b < 1$.

(F_1) $F(t,0,t,0,0,t) = t - \varphi[t(a+b)] = t[1-(a+b)] > 0$, $\forall\ t > 0$,

(F_2) $F(t,0,0,t,t,0) = t - \varphi(bt) = t(1-b) > 0$, $\forall t > 0$.

By Theorem 4.1 and Example 5.9 we get the following:

Theorem 5.12. *Let A, B, S and T be self mappings of a G_p metric space (X,G_p) such that*

$$G_p^2(Ax,By,By) \leq \varphi\Big(\max\{\psi\big(G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)\big),$$
$$\frac{1}{3}\big(\psi\big(G_p(Ty,By,By).G_p(Sx,By,By)\big)+\psi\big(G_p(Ax,By,By).G_p(Ax,Ty,Ty)\big)\big),$$
$$\frac{1}{3}\big(\psi\big(G_p^2(Ax,Ty,Ty)\big)+\psi\big(G_p^2(Ty,By,By)\big)\big)\}\Big)$$

for all $x,y \in X$, where $\varphi \in \Phi$ and ψ is an almost altering distance. If (A,S) and T satisfy $CLR_{(A,S)T}$ property, then

(1) $C(A,S) \neq 0$,

(2) $C(B,T) \neq 0$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point z and $G_p(z,z,z)=0$.

Example 5.13. Let $X=[0,\infty)$ and $G_p(x,y,z)=\max\{x,y,z\}$. Then (X,G_p) is a G_p metric space. Put $\varphi(t)=\frac{t}{16}$ and $\varphi \in \Phi$.

Define the following mappings:

$$Ax=\frac{\sqrt{x}}{4}, \quad Sx=4\sqrt{x}, \quad Bx=0, \quad Tx=\frac{\sqrt{x}}{6}.$$

Then $S(x)=[0,\infty)$, $T(X)=[0,\infty)$ and $S(X)\cap T(X)=[0,\infty)$.

Let x_n be a sequence in X such that $\lim_{n\to\infty} x_n=0$. Then,

$$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=0=z\in S(X)\cap T(X).$$

Hence, (A,S) and T satisfy $CLR_{(A,S)T}$-property with $G_p(0,0,0)=0$. $Ax=Sx$ implies $C(A,S)=0$ and $Bx=Tx$ implies $C(B,T)=0$. Moreover, $AS0=SA0=0$ and $BT0=TB0=0$. Hence (A,S) and (B,T) are weakly compatible. On other hand,

$$G_p^2(Ax,By,By)=\frac{x}{16} \quad \text{and} \quad G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)=16x,$$

which implies

$$\begin{aligned}
G_p^2(Ax,By,By) &\leq \frac{1}{16}G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)\\
&\leq \frac{1}{16}\max\Big\{G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax),\\
&\qquad \frac{1}{3}\big(G_p(Ty,By,By).G_p(Sx,By,By)+G_p(Ax,By,By).G_p(Ax,Ty,Ty)\big),\\
&\qquad \frac{1}{3}\big(G_p^2(Ax,Ty,Ty)+G_p^2(Ty,By,By)\big)\Big\}\\
&=\varphi\Big(\max\{G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax),\\
&\qquad \frac{1}{3}\big(G_p(Ty,By,By).G_p(Sx,By,By)+G_p(Ax,By,By).G_p(Ax,Ty,Ty)\big),\\
&\qquad \frac{1}{3}\big(G_p^2(Ax,Ty,Ty)+G_p^2(Ty,By,By)\big)\}\Big).
\end{aligned}$$

By Theorem 5.12, A, B, S and T have a unique common fixed point $z=0$.

By Theorem 5.3 and Example 5.9, we get the following:

Theorem 5.14. *Let A,B,S and T be self mappings of a G_p metric space (X,G_p) such that*

$$\int_0^{G_p^2(Ax,By,By)} h(t)dt \le \varphi\Big(\max\Big\{\int_0^{G_p(Sx,Ty,Ty).G_p(Sx,Sx,Ax)} h(t)dt,$$
$$\frac{1}{3}\Big(\int_0^{G_p(Ty,By,By).G_p(Sx,By,By)} h(t)d + \int_0^{G_p(Ax,By,By).G_p(Ax,Ty,Ty)} h(t)dt\Big),$$
$$\frac{1}{3}\Big(\int_0^{G_p^2(Ax,Ty,Ty)} h(t)dt + \int_0^{G_p^2(Ty,By,By)} h(t)dt\Big)\Big\}\Big)$$

for all $x,y \in X$, where $h(t)$ is as in Theorem 5.1. If (A,S) and T satisfy $CLR_{(A,S)T}$ property, then

(1) $C(A,S) \neq 0$,

(2) $C(B,T) \neq 0$.

Moreover, if (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point z and $G_p(z,z,z) = 0$.

Remark 5.15. By Theorem 5.3 and Examples 5.8, 5.10 and 5.11, we get new results.

6. Conclusion

From our investigations, we conclude that the self mappings on a G_p metric space with CLR property and weak compatibility have a unique common fixed point with certain conditions. Fixed points also occurs for mappings satisfying a contractive condition of integral type in G_p metric spaces. Our investigations and results obtained were supported by the suitable examples which provides new path for the researchers in the concerned field.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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