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Representation of Topological Algebras by Projective Limit of Fréchet Algebras

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Abstract. It is shown that every topological Hausdorff algebra (in particular, locally pseudoconvex Hausdorff algebra) A with jointly continuous multiplication is topologically isomorphic to a dense subalgebra of the projective limit of Fréchet (respectively, locally pseudoconvex Fréchet) algebras. In case, when A is complete, A and this projective limit of Fréchet (respectively, locally pseudoconvex Fréchet) algebras are topologically isomorphic. A partly new proof for these results from [11] are given.

1. **Introduction**

It is well-known (published in 1952 in [10, p. 17], and in [5]) that every complete locally m-convex Hausdorff algebra is topologically isomorphic to the projective limit of Banach algebras. This result has been generalized to the case of complete locally m-(k-convex) Hausdorff algebras in [2, Theorem 5]; to the case of complete locally A-convex Hausdorff algebras in [4, Theorem 2.2], and to the case of complete locally m-pseudoconvex Hausdorff algebras in [6, pp. 202–204]. Similar representations of topological algebras (not necessarily with jointly continuous multiplication) by projective limits of topological algebras with more simple structure are considered in [3]. It is known (see [11, Theorem 1]) that every complete topological Hausdorff algebra with jointly continuous multiplication is topologically isomorphic to the projective limit of Fréchet algebras and every complete locally convex Hausdorff algebra with jointly continuous multiplication is topologically isomorphic to the projective limit of locally convex Fréchet algebras. These results in [11] are correct, but the proofs of these are not, because in the proofs there is applied a Lemma which, first of all, is not formulated suitably

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and, secondly, the proof of this Lemma is not correct. Detailed (and partly new) proofs of these results are given in the present paper.

2. **Proof of Müldner's result**

Let M be the dense subset of $\mathbb{R}^+$ which consists of all non-negative rational numbers, having a finite dyadic expansions, i.e, every such number $\rho \in M$ is representable on the form

$$\rho = \sum_{n=0}^{\infty} \delta_n(\rho) \cdot 2^{-n},$$

where[1] $\delta_0(\rho) \in \mathbb{N}_0$, $\delta_n(\rho) \in \{0,1\}$ for each $n \in \mathbb{N}$ and $\delta_n(\rho) = 0$ for n sufficiently large.

Let (A, τ) be a (real or complex) topological algebra with jointly continuous multiplication, $\mathscr{L}_{(A,\tau)}$ a base of neighbourhoods of zero in (A, τ), consisting of closed balanced sets, and $\mathscr{S}_A = \{S_\lambda : \lambda \in \Lambda\}$ the set of all algebraic strings $S_\lambda = (U_n^\lambda)$ in $\mathscr{L}_{(A,\tau)}$, that is, $U_n^\lambda \in \mathscr{L}_{(A,\tau)}$,

$$U_{n+1}^\lambda + U_{n+1}^\lambda \subset U_n^\lambda$$

and

$$U_{n+1}^\lambda U_{n+1}^\lambda \subset U_n^\lambda$$

for each $n \in \mathbb{N}_0$. For each $\lambda \in \Lambda$, $S_\lambda = (U_n^\lambda) \in \mathscr{S}$ and $\rho \in M$ let

$$V_\lambda(\rho) = \underbrace{U_0^\lambda + \cdots + U_0^\lambda}_{\delta_0(\rho)\text{ summands}} + \sum_{n=1}^{\infty} \delta_n(\rho) \cdot U_n^\lambda \tag{2}$$

and

$$q_\lambda(a) = \inf\{\rho \in M : a \in V_\lambda(\rho)\}$$

for each $a \in A$ and $\lambda \in \Lambda$. Then every q_λ is a F-seminorm on A (see [8, pp. 39–40]) and

$$\ker q_\lambda = \bigcap_{n=0}^{\infty} U_n^\lambda$$

(see [3, p. 148]). Let $\mathscr{Q} = \{q_\lambda : \lambda \in \Lambda\}$ and $\tau_{\mathscr{Q}}$ be the initial topology on A, defined by the collection $\mathscr{Q}$. Then (see [3, Theorem 2.2]) $(A, \tau_{\mathscr{Q}})$ is a topological algebra with jointly continuous multiplication and $\tau = \tau_{\mathscr{Q}}$.

To give a new proof for the result of Müldner in [11, Theorem 1], we need (instead of lemma in this) the following result.

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[1] Here and later on $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 2.1. *Let A be a topological T_1-algebra with jointly continuous multiplication, $\mathscr{L}_A$ the base of all closed and balanced neighbourhoods of zero in A and $S_A = (U_n)$ an algebraic string in $\mathscr{L}_A$. Then the kernel*

$$N(S_A) = \bigcap_{n=1}^{\infty} U_n$$

of S_A is a closed two-sided ideal in A.

Proof. When $N(S_A) = \{\theta_A\}$, then $N(S_A)$ is a closed two-sided ideal in A. Suppose now that $N(S_A) \neq \{\theta_A\}$. Then there are elements $a, b \in N(S_A) \setminus \{\theta_A\}$. Let $n \in \mathbb{N}$ be an arbitrary fixed number. Since $N(S_A) \subset U_{n+1}$ and $U_{n+1} + U_{n+1} \subset U_n$, then $a + b \in U_n$ for each $n \in \mathbb{N}$. Hence, $a + b \in N(S_A)$.

Let next λ be a real or complex number and $a \in N(S_A)$. Then $a \in U_n$ for each $n \in \mathbb{N}$. If $|\lambda| \leqslant 1$, then $\lambda a \in U_n$ for each $n \in \mathbb{N}$, because U_n is balanced. If $|\lambda| > 1$, let $n_0 \in \mathbb{N}$ be a natural number such that[2] $[|\lambda|] + 1 \leqslant 2^{n_0}$ and n an arbitrary fixed natural number. Since $a \in U_{n+n_0}$ and

$$\lambda a = [|\lambda|]\frac{\lambda}{|\lambda|}a + (|\lambda| - [|\lambda|])\frac{\lambda}{|\lambda|}a \in \underbrace{U_{n+n_0} + \cdots + U_{n+n_0}}_{[|\lambda|]+1 \text{ summands}} \subset U_n\,,$$

because $\left|\frac{\lambda}{|\lambda|}\right| = 1$, $\left|(|\lambda| - [|\lambda|])\frac{\lambda}{|\lambda|}\right| < 1$ and every U_n is balanced, then $\lambda a \in U_n$ for each $n \in \mathbb{N}$. Thus, $\lambda a \in N(S_A)$.

Let now $a \in A$, $b \in N(S_A)$, $n \in \mathbb{N}$ and $m = n + 1$. Then there exists a positive number ε_m such that $a \in \varepsilon_m U_m$ (because every neighbourhood of zero absorbs points). If $|\varepsilon_m| \leqslant 1$, then $\varepsilon_m U_m \subset U_m$ because U_m is balanced, and if $|\varepsilon_m| > 1$, then, from $\varepsilon_m b \subset \varepsilon_m N(S_A) \subset N(S_A) \subset U_m$ follows that

$$ab \in (\varepsilon_m U_m)(\varepsilon_m^{-1} U_m) \subset U_m U_m \subset U_n\,.$$

Hence, $ab \in N(S_A)$. Similarly, we can show that $ba \in N(S_A)$. Consequently, $N(S_A)$ is a two-sided ideal in A. □

Theorem 2.2. *For any (real or complex) topological Hausdorff algebra A with jointly continuous multiplication there exists the projective system $\{\tilde{A}_\lambda; \tilde{h}_{\lambda\mu}, \Lambda\}$ of Fréchet algebras and continuous homomorphisms $\tilde{h}_{\lambda\mu}$ from $\tilde{A}_\mu$ to $\tilde{A}_\alpha$ (whenever $\lambda \prec \mu$) such that A is topologicaly isomorphic to a dense subalgebra of the projective limit $\varprojlim \tilde{A}_\lambda$ of this system. In case, when A is complete, then A and $\varprojlim \tilde{A}_\lambda$ are topologically isomorphic.*

Moreover, if A is a locally pseudoconvex[3] *(in particular a locally convex) Hausdorff algebra with jointly continuous multiplication, then A is topologically isomorphic to a dense subalgebra of the projective limit $\varprojlim \tilde{A}_\lambda$ of locally pseudoconvex (respectively,*

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[2]Here $[r]$ denotes the entire part of a real number r.

[3]A topological algebra A is *locally pseudoconvex* if the topology of A is possible to give by a collection of non-homogeneous seminorms (see, for example, [6, pp. 189–198]).

locally convex) Fréchet algebras. In case, when A is complete, then A and $\varprojlim \tilde{A}_\lambda$ are topologically isomorphic.

Proof. Let A be a topological Hausdorff algebra with jointly continuous multiplication, $\mathscr{L}_A$ the base of all closed and balanced neighbourhoods of zero in A and $\mathscr{S}_A = \{S_\lambda : \lambda \in \Lambda\}$ the collection of all algebraic strings in $\mathscr{L}_A$. That is, every $S_\lambda \in \mathscr{S}_A$ is a sequence (O_n^λ) in $\mathscr{L}_A$, members O_n^λ of which satisfy the conditions

$$O_{n+1}^\lambda + O_{n+1}^\lambda \subset O_n^\lambda$$

and

$$O_{n+1}^\lambda O_{n+1}^\lambda \subset O_n^\lambda$$

for each $n \in \mathbb{N}_0$. We define the ordering $\prec$ in Λ in the following way: we say that $\lambda \prec \mu$ in Λ if and only if $S_\mu \subset S_\lambda$, that is, if $S_\lambda = (O_n^\lambda)$ and $S_\mu = (O_n^\mu)$, then $O_n^\mu \subset O_n^\lambda$ for each $n \in \mathbb{N}_0$. It is easy to see that $(\Lambda, \prec)$ is a partially ordered set. To show that $(\Lambda, \prec)$ is a directed set let $S_{\lambda_1} = (O_n^{\lambda_1})$ and $S_{\lambda_2} = (O_n^{\lambda_2})$ be arbitrary fixed algebraic strings in $\mathscr{S}_A$ and let $S_\mu = (O_n^\mu)$ be the algebraic string in $\mathscr{L}_A$, which we define in the following way: let $O_0^\mu \in \mathscr{L}_A$ be such that $O_0^\mu \subset O_0^{\lambda_1} \cap O_0^{\lambda_2}$. Further, for each $n \geqslant 0$, let U_{n+1} be a neighbourhood of zero in $\mathscr{L}_A$ such that $U_{n+1} + U_{n+1} \subset O_n^\mu$ and $U_{n+1}U_{n+1} \subset O_n^\mu$. Let now O_{n+1}^μ be a neighbourhood in $\mathscr{L}_A$ such that

$$O_{n+1}^\mu \subset U_{n+1} \cap O_{n+1}^{\lambda_1} \cap O_{n+1}^{\lambda_2}.$$

Then

$$O_{n+1}^\mu + O_{n+1}^\mu \subset U_{n+1} \cap O_{n+1}^{\lambda_1} \cap O_{n+1}^{\lambda_2} + U_{n+1} \cap O_{n+1}^{\lambda_1} \cap O_{n+1}^{\lambda_2} \subset U_{n+1} + U_{n+1} \subset O_n^\mu$$

and

$$O_{n+1}^\mu O_{n+1}^\mu \subset (U_{n+1} \cap O_{n+1}^{\lambda_1} \cap O_{n+1}^{\lambda_2})^2 \subset U_{n+1}U_{n+1} \subset O_n^\mu$$

for each $n \in \mathbb{N}$. Since $S_\mu \in \mathscr{S}_A$ and $O_n^\mu \subset O_n^{\lambda_1} \cap O_n^{\lambda_2}$ for each $n \in \mathbb{N}_0$, then $\lambda_1 \prec \mu$ and $\lambda_2 \prec \mu$. It means that $(\Lambda, \prec)$ is a directed set.

Let q_λ be the F-seminorm on A, which is defined by the string $S_\lambda = (O_n^\lambda)$ for each $\lambda \in \Lambda$ and let $\mathscr{Q}_A = \{q_\lambda : \lambda \in \Lambda\}$. As it was mentioned above, we can consider on A the topology, which has been defined by $\mathscr{Q}_A$. Then $\ker q_\lambda$ is a closed two-sided ideal in A by Lemma 2.1. For each $\lambda \in \Lambda$ let $A_\lambda = A/\ker q_\lambda$ and let π_λ be the canonical homomorphism of A onto A_λ. Moreover, let $\overline{q}_\lambda(\pi_\lambda(a)) = q_\lambda(a)$ for each $a \in A$. Then the multiplication in A_λ is jointly continuous. Indeed, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $2^{-n_\varepsilon} < \varepsilon$. Since

$$U = \left\{ x \in A_\lambda : \overline{q}_\lambda(x) < \frac{1}{2^{n_\varepsilon+1}} \right\}$$

is a neighbourhood of zero in A_λ and from $x, y \in U$ follows that

$$xy \in V_\lambda\left(\frac{1}{2^{n_\varepsilon+1}}\right)^2 = U_{n_\varepsilon+1}(\lambda)^2 \subset U_{n_\varepsilon}(\lambda) = V_\lambda\left(\frac{1}{2^{n_\varepsilon}}\right),$$

then $\overline{q}_\lambda(xy) \leqslant 2^{-n_\varepsilon} < \varepsilon$ for each $x, y \in U$. It means that the multiplication in A_λ is continuous at $(\theta_{A_\lambda}, \theta_{A_\lambda})$. Consequently, the multiplication in A_λ is jointly continuous.

Let $\tilde{A}_\lambda$ be the completion[4] of A_λ, ν_λ the topological isomorphism from A_λ onto a dense subalgebra of $\tilde{A}_\lambda$ (defined by the completion of A_λ), $\tilde{q}_\lambda$ the extension of $\overline{q}_\lambda \circ \nu_\lambda^{-1}$ to $\tilde{A}_\lambda$ and $\tilde{\tau}_\lambda$ the topology on $\tilde{A}_\lambda$, defined by $\tilde{q}_\lambda$. Then

$$\tilde{q}_\lambda[(\nu_\lambda \circ \pi_\lambda)(a)] = \overline{q}_\lambda(\pi_\lambda(a)) = q_\lambda(a)$$

for each $a \in A$. Therefore, $\tilde{q}_\lambda$ is an F-norm on $\tilde{A}_\lambda$ because of which $\tilde{A}_\lambda$ is metrizable (see, for example, [8, p. 40]). Hence, $(\tilde{A}_\lambda, \tilde{\tau}_\lambda)$ is a Fréchet algebra for each $\lambda \in \Lambda$.

For each $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$ we define the map $h_{\lambda\mu}$ by $h_{\lambda\mu}(\pi_\mu(a)) = \pi_\lambda(a)$ for each $a \in A$. Then $h_{\lambda\mu}$ is a continuous homomorphism from A_μ onto A_λ, $h_{\lambda\lambda}$ is the identity mapping on A_λ for each $\lambda \in \Lambda$ and $h_{\lambda\mu} \circ h_{\mu\gamma} = h_{\lambda\gamma}$ for each $\lambda, \mu, \gamma \in \Lambda$ with $\lambda \prec \mu \prec \gamma$. Since $\nu_\lambda \circ h_{\lambda\mu} \circ \nu_\mu^{-1}$ is a continuous homomorphism from $\nu_\mu(A_\mu)$ into $\tilde{A}_\lambda$, then there exists a continuous extension $\tilde{h}_{\lambda\mu}$ from $\tilde{A}_\mu$ into $\tilde{A}_\lambda$, which is linear (by Proposition 5 in [7]) and submultiplicative by the continuity of multiplication in $\tilde{A}_\mu$ (similarly as in the proof of Proposition 1, pp. 4–5, in [9] or in the proof of Proposition 3 in [1]). Moreover,

$$\tilde{h}_{\lambda\mu}[\nu_\mu(\pi_\mu(a))] = \nu_\lambda[h_{\lambda\mu}(\pi_\mu(a))] = \nu_\lambda[\pi_\lambda(a)]$$

for each $a \in A$ and $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$. Since $\tilde{h}_{\lambda\lambda}$ is the identity map on $\tilde{A}_\lambda$ for each $\lambda \in \Lambda$ and $\tilde{h}_{\lambda\mu} \circ \tilde{h}_{\mu\gamma} = \tilde{h}_{\lambda\gamma}$, whenever $\lambda, \mu, \gamma \in \Lambda$ and $\lambda \prec \mu \prec \gamma$, then $\{\tilde{A}_\lambda; \tilde{h}_{\lambda\mu}, \Lambda\}$ is a projective system of Fréchet algebras A_λ with continuous homomorphisms $\tilde{h}_{\lambda\mu}$ from $\tilde{A}_\mu$ into $\tilde{A}_\lambda$ and

$$\varprojlim \tilde{A}_\lambda = \left\{ (\nu_\lambda[\pi_\lambda(a)])_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} \tilde{A}_\lambda : \tilde{h}_{\lambda\mu}[\nu_\mu(\pi_\mu(a))] = \nu_\lambda(\pi_\lambda(a)), \text{ whenever } \lambda \prec \mu \right\}$$

is the projective limit of this system.

Let $\tilde{e}$ be the mapping from A into $\prod_{\mu\in\Lambda} \tilde{A}_\mu$, defined by $\tilde{e}(a) = (\nu_\lambda[\pi_\lambda(a)])_{\lambda\in\Lambda}$ for each $a \in A$, and pr_λ the projection of $\prod_{\mu\in\Lambda} \tilde{A}_\mu$ onto $\tilde{A}_\lambda$ for each $\lambda \in \Lambda$. Since $\mathrm{pr}_\lambda(\tilde{e}(a)) = \nu_\lambda[\pi_\lambda(a)]$ for each $a \in A$ and $\lambda \in \Lambda$ and $\nu_\lambda \circ \pi_\lambda$ is continuous for each $\lambda \in \Lambda$, then $\tilde{e}$ is a continuous map from A into $\prod_{\mu\in\Lambda} \tilde{A}_\mu$ (see, for example, [12, Theorem 8.8]). Moreover, if $a, b \in A$ and $\tilde{e}(a) = \tilde{e}(b)$, then $\nu_\lambda \circ \pi_\lambda(a) = \nu_\lambda \circ \pi_\lambda(b)$ for each $\lambda \in \Lambda$. Therefore,

$$a - b \in \bigcap_{\lambda\in\Lambda} \ker q_\lambda = \bigcap_{O\in\mathscr{L}_A} O = \theta_A,$$

because A is a Hausdorff space. It means that $a = b$. Hence, $\tilde{e}$ is a one-to-one map.

[4] Here $\tilde{A}_\lambda$ is a topological algebra with jointly continuous multiplication because the multiplication in A_λ is jointly continuous.

Let now O be an open subset in A, o a point in O, α a fixed index in Λ and

$$U = \left[\prod_{\lambda\in\Lambda} U_\lambda\right] \cap \tilde{e}(A),$$

where $U_\alpha = \nu_\alpha \circ \pi_\alpha(O)$ and $U_\lambda = \tilde{A}_\lambda$, if $\lambda \neq \alpha$. Then U is a neighbourhood of $\tilde{e}(o)$ in $\tilde{e}(A)$. Since

$$\mathrm{pr}_\alpha(U) \subset \nu_\alpha \circ \pi_\alpha(O) = \mathrm{pr}_\alpha(\tilde{e}(O))$$

and α is arbitrary, then $U \subset \tilde{e}(O)$. Hence, $\tilde{e}$ is an open map. Taking this into account, $\tilde{e}$ is a topological isomorphism from A into $\prod_{\lambda\in\Lambda} \tilde{A}_\lambda$.

To show that $\tilde{e}(A)$ is dense in $\varprojlim \tilde{A}_\lambda$, let $(\tilde{a}_\lambda)_{\lambda\in\Lambda} \in \varprojlim \tilde{A}_\lambda$ be an arbitrary element and O an arbitrary neighbourhood of $(\tilde{a}_\lambda)_{\lambda\in\Lambda}$ in $\varprojlim \tilde{A}_\lambda$. Then there is a neighbourhood U of $(\tilde{a}_\lambda)_{\lambda\in\Lambda}$ in $\prod_{\lambda\in\Lambda} \tilde{A}_\lambda$ such that $O = U \cap \varprojlim \tilde{A}_\lambda$. Now, there is a finite subset $H \subset \Lambda$ such that $\prod_{\lambda\in\Lambda} U_\lambda \subset U$, where U_λ is a neighbourhood of $\tilde{a}_\lambda$ in $\tilde{A}_\lambda$, if $\lambda \in H$, and $U_\lambda = \tilde{A}_\lambda$, if $\lambda \in \Lambda \setminus H$. Let $\mu \in \Lambda$ be such that $\lambda \prec \mu$ for every $\lambda \in H$ and

$$V = \bigcap_{\lambda\in H} \tilde{h}_{\lambda\mu}^{-1}(U_\lambda).$$

Then V is a neighbourhood of $\tilde{a}_\mu$ in $\tilde{A}_\mu$. Take an element $a \in (\nu_\mu \circ \pi_\mu)^{-1}(V)$. Then $\nu_\mu \circ \pi_\mu(a) \in V$. Therefore, $\nu_\lambda \circ \pi_\lambda(a) = \tilde{h}_{\lambda\mu}(\nu_\mu \circ \pi_\mu(a)) \in U_\lambda$ for each $\lambda \in H$. It means that $\tilde{e}(a) \in U \cap \tilde{e}(A)$. Consequently, $\tilde{e}(A)$ is dense in $\varprojlim \tilde{A}_\lambda$.

If now A is a locally pseudoconvex (in particular locally convex) Hausdorff algebra with jointly continuous multiplication, then every $\tilde{A}_\lambda$ is a locally pseudoconvex (respectively, locally convex) Fréchet algebra with jointly continuous multiplication. Therefore, in the present case, A is topologically isomorphic to a dense subalgebra of the projective limit $\varprojlim \tilde{A}_\lambda$ of locally pseudoconvex (respectively, locally convex) Fréchet algebras.

Moreover, A and $\varprojlim \tilde{A}_\lambda$ are topologically isomorphic, if A is complete. □

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