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Research Article

Some Results on Strong Edge Geodetic Problem in Graphs

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Abstract. For a graph $G(V(G),E(G))$, the problem to find a $S \subseteq V(G)$ where every edge of the graph G is covered by a unique fixed geodesic between the pair of vertices in S is called the strong edge geodetic problem and the cardinality of the smallest such S is the strong edge geodetic number of G. In this paper the strong edge geodetic problem for product graphs are studied and also some results for general graphs are derived.

Keywords. Strong edge geodetic number; Strong geodetic number; Edge geodetic number; Geodetic set

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1. Introduction

Consider a graph $G(V(G),E(G))$, with order $|V(G)|$ and size $|E(G)|$. An $(x-y)$ geodesic is the length of the shortest path between the vertices x and y. For a graph G, the length of the maximum geodesic is called the graph diameter, denoted as diam(G). Harary *et al*. introduced a graph theoretical parameter in [2] called the geodetic number of a graph and it was further studied in [1]. In [2] the geodetic number of a graph is defined as follows, let $I[u,v]$ be the set of all vertices lying on some $u-v$ geodesic of G, and for some non empty subset S of $V(G)$, $I[S] = \bigcup_{u,v\in S} I[u,v]$. The set S of vertices of G is called a geodetic set of G, if $I[S] = V$. A geodetic set of minimum cardinality is called minimum geodetic set of G. The cardinality of the minimum geodetic set of G is the geodetic number $g(G)$ of G. The geodetic set decision problem is NP-complete [12]. The set $S \subseteq V(G)$ is an edge geodetic cover of G if every edge of G is contained in the geodesic between some pair of vertices in S, and the cardinality of minimum edge geodetic cover is called the edge geodetic number of G denoted as $g_1(G)$ [13]. Strong geodetic problem is a variation of geodetic problem and is defined in [11] as follows. For a graph $G(V(G),E(G))$, given a set $S \subseteq V(G)$, for each pair of vertices $(x,y) \subseteq S$, $x \neq y$, let $\widetilde{g}(x,y)$ be a selected fixed shortest path between x and y. Let $\widetilde{I}(S) = \{\widetilde{g}(x,y)\colon x,y \in S\}$ and $V(\widetilde{I}(S)) = \bigcup\limits_{\widetilde{P} \in \widetilde{I}(S)} V(\widetilde{P})$. If $V(\widetilde{I}(S)) = V$ for some $\widetilde{I}(S)$, then S is called a strong geodetic set. The cardinality of the minimum strong geodetic set is the strong geodetic number of G and is denoted by $sg(G)$. The edge version of the strong geodetic problem is defined in [10] i.e. for a graph $G(V(G),E(G))$, a set $S \subseteq V(G)$ is called a strong edge geodetic set if for any pair $x,y \in S$ a shortest path P_{xy} can be assigned such that $\cup_{\{x,y\} \in \binom{S}{2}} E(P_{xy}) = E(G)$. The cardinality of the smallest strong edge geodetic set of G is called the strong edge geodetic number and is denoted as $sg_e(G)$.

2. Strong Edge Geodetic Number of Graphs

Theorem 1. *For a graph $G(V(G),E(G))$ with diameter $d \geq 2$, the strong edge geodetic number,* $sg_e(G) \geq \left\lceil \frac{d+\sqrt{d^2+8d|E|}}{2d} \right\rceil$.

Proof. Consider a graph $G(V(G),E(G))$ with diameter $d \geq 2$ and assume that the set $S \subseteq V(G)$ forms a minimum strong edge geodetic set for G. Then the edges of G are covered with $\binom{|S|}{2}$ geodesics and each geodesic covers at most d edges. Thus $|E| \leq \binom{|S|}{2} d$. Considering only the positive and the integer roots, the inequality reduces to $|S| \geq \left\lceil \frac{d+\sqrt{d^2+8d|E|}}{2d} \right\rceil$. □

The above bound attains equality when $G \cong P_n$ $(n \geq 2)$.

Theorem 2. *For a graph G with order $n \geq 2$, $sg_e(G) = 2$ if and only if $G \cong P_n$.*

Proof. For $P_n = v_0 v_1 \dots v_n$, the geodesic between the vertices v_o and v_n covers all the edges of P_n and thus $sg_e(P_n) = 2$. Conversely let G be a graph with $sg_e(G) = 2$ and let $S = \{u,v\}$ be the minimum strong edge geodetic set of G. Then all the edges of G lies on the $u-v$ geodesic. Thus $G \cong P_n$. □

A graph is geodetic if for every pair for vertices the shortest path between them is unique. An undirected graph $G = (V,E)$ is said to be geodetic, if between any pair of vertices $x,y \in V(G)$ there is a unique shortest path [5].

Theorem 3. *For a connected non-complete geodetic graph G on n vertices with minimum cutset of independent vertices $C \subset V(G)$, $sg_e(G) \leq n - c$ where $|C| = c$.*

Proof. Consider a connected non-complete geodetic graph G with minimum cutset of independent vertices $C = \{c_1, c_2, \dots c_c\}$, such that the $V(G) \backslash C$ splits G into at least two components. It is clear that every vertex in C is adjacent to at least one vertex in each component of $G \backslash C$. Let $S = V(G) \backslash C$. Consider an edge $xy \in E(G)$. This edge xy can be of two forms.

Case i: Suppose xy is an edge in any one component of $G\backslash C$. Then xy itself forms a unique fixed geodesic joining x and y, where $\{x, y\} \in S$.

Case ii: Suppose xy is an edge such that $y \in C$ and $x \in S$. Since G is a geodetic graph, there exists a vertex $z \in G\backslash C$ such that the edge xy lies on a unique fixed geodesic xyz. It is clear that the vertices x and z lies on two different components of $G\backslash C$. Thus S forms a strong edge geodetic set of G and therefore $sg_e(G) \le n - c$. □

Theorem 4 ([13])**.** *For a graph G of order n and exactly one vertex of degree* $(n-1)$*,* $g_1(G) = n-1$.

Theorem 5. *For a graph G of order n and exactly one vertex of degree* $(n-1)$*,* $sg_e(G) = n-1$.

Proof. For any connected graph G, $g_1(G) \le sg_e(G)$ and from Theorem 4, $sg_e(G) \ge n-1$. Let $v \in V(G)$ be the unique vertex of degree $n-1$. Let $vx \in E(G)$. This implies that there exists at least at least one vertex x' where x and x' are not adjacent. Let $Q = V(G)\backslash\{v\}$. Clearly, the edge vx lies on the unique fixed geodesic xvx'. Also, any edge of G which are not incident with v lies on a fixed geodesic between the vertices of Q. Thus $sg_e(G) \le n-1$ Hence for a graph G of order n and exactly one vertex of degree $(n-1)$, $sg_e(G) = n-1$.
The converse of the above theorem need not be true. For $G = C_4$, $sg_eG = 3$, but G does not have any vertex of degree 3. □

Corollary 1. *For a graph G of order* $n \ge 3$ *with a cut vertex of degree* $n-1$*,* $sg_e(G) = n-1$.

Proof. Consider a connected graph G of order $n \ge 3$ with a cut vertex x of degree $n-1$. This implies that x is the only vertex of degree $n-1$. Then it follows from Theorem 5 that $sg_e(G) = n-1$. □

Theorem 6. *For a split graph* $G(V,E)$ *with complete set K, maximum stable set T,* $sg_e(G) \ge \frac{2|E|+s_1^2+2s_2^2+3s_1+2s_1s_2}{2(2s_1+s_2)}$*, where* $s_1 = |T|$ *and* s_2 *denotes number of simplicial vertices in complete set K.*

Proof. Let S be a minimum strong edge geodetic set of G and let A be set of simplicial vertices in K with $|A| = s_2$. Also, let B be set of non simplicial vertices in S, $|B| = s_3$. Clearly, $T \subseteq S$, $A \subseteq S$ and $|S| = s_1 + s_2 + s_3$.

The strong edge geodetic set contains three components including the stable set, the set of simplicial vertices in K and the set of non-simplicial vertices. It is clear that s_1 and s_2 are fixed, hence to get the minimum strong edge geodetic set, s_3 should be minimized. The geodesics between any pair of vertices in the stable set T cover a maximum of three edges. Also, the geodesics between any pair of vertices with one vertex in T and other vertex in A cover at most two edges. Thus the optimization problem reduces to

minimize(s_3)

subject to: $0 \le (s_1 + s_2 + s_3) \le n$;

$$|E| \le 3\binom{s_1}{2} + 2s_2s_1 + 2s_3s_1 + s_3s_2.$$

From this we obtain $S \geq \frac{2|E|+s_1^2+2s_2^2+3s_1+2s_1s_2}{2(2s_1+s_2)}$.

Therefore, $sg_e(G) \geq \frac{2|E|+s_1^2+2s_2^2+3s_1+2s_1s_2}{2(2s_1+s_2)}$. □

Corollary 2. *Suppose $A = \phi$. Then $sg_e(G) \geq \frac{2|E|+s_1(s_1+3)}{4s_1}$.*

Theorem 7 ([13])**.** *For any graph G with atleast two vertices of degree $n-1$, $g_1(G) = n$.*

Theorem 8. *If G is a graph with order n and atleast two vertices of degree $n-1$, then $sg_e(G) = n$.*

Proof. For any graph G, $g_1(G) \leq sg_e(G)$ and from Theorem 7. For G with order n and atleast two vertices of degree $n-1$, $sg_e(G) = n$. □

Corollary 3. *For a complete graph K_n with order $n \geq 4$ and edge $e_1 \in E(G)$, $sg_e(K_n - e_1) = n$.*

Corollary 4. *For a complete graph K_n with order $n \geq 6$ and edges $\{e_1, e_2\} \in E(G)$, $sg_e(K_n - \{e_1, e_2\}) = n$.*

Result 1. Let G be a cactus graph with m simplicial vertices and r cycles then, $sg_e(G) \leq m + 2r$. Also, the bound is sharp (refer Figure 2.1).

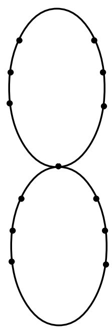

Figure 2.1. Cactus graph with $sg_e(G) = 4$

Result 2. Let G be a block cactus graph with m simplicial vertices and r cycles then, $sg_e(G) \leq m + 2r$. The bound is sharp for the graph (refer Figure 2.2).

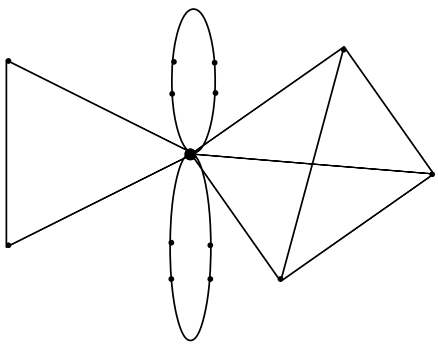

Figure 2.2. Block cactus graph with $sg_e(G) = 9$

Let G and H be two graphs of order n and m, respectively. The corona product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. The order of $G \circ H$ is $o(G)o(H) + o(G)$ [12].

Theorem 9. *For graphs G and H with $o(G) \geq 2$, $sg_e(G \circ H) = o(G)o(H)$.*

Proof. Consider connected graphs G and H with $o(G) = n$ and $o(H) = m$. The corona product of graphs G and H i.e. $(G \circ H)$ contains n copies of H. Let S be the set that contains all vertices in each copies of H_i in $G \circ H$. Since G and H are assumed to be connected, there exists a fixed geodesics between the vertices in each H_i that covers all the edges of H_i. The unique fixed geodesics between the vertices of H_i and H_j, $i \neq j$ covers the edges of G and the edges connecting H_i and G. Thus $sg_e(G \circ H) \leq o(G)o(H)$.

Suppose there exists a set S' such that $o(S') < o(G)o(H)$ is a strong edge geodetic set of $(G \circ H)$. Then there exists H_j such that $v \in H_j$ $(1 \leq j \leq n)$ and $v \notin S'$. Clearly, $e = uv$ where $u \in G$ is not covered by any $u - v$ geodesic of S'. Therefore $sg_e(G \circ H) = o(G)o(H)$. □

Definition 1 ([16])**.** Let G be a connected graph. An ordered set $S = \{u_1, u_2, \ldots u_k\}$ of vertices of G is a linear geodetic set of G if for each x in G, there exists an index i, $1 \leq i < k$ such that x lies on $u_i - u_{i+1}$ geodesic in G and the minimum cardinality of the linear geodetic set is the linear geodetic number, $g_l(G)$.

Definition 2. Consider a graph G with an ordered set of vertices, $S = \{x_1, x_2, \ldots x_k\}$. Then S is a strong linear geodetic if for each $x \in V \setminus S$, there exists an index i, $1 \leq i < k$ such that x lies on the unique fixed geodesic between x_i and x_{i+1}. The minimum cardinality of a strong linear geodetic set is called the strong linear geodetic number, denoted as $sg_l(G)$.

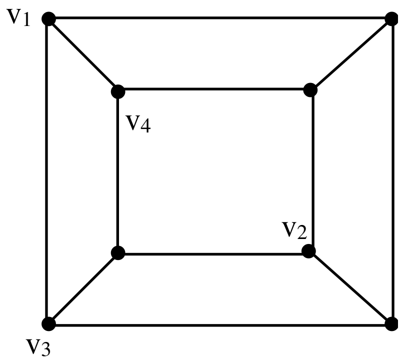

Figure 2.3. Graph G with linear geodetic set $\{v_1, v_2\}$ and strong linear geodetic set $\{v_1, v_2, v_3, v_4\}$

In [8] an upper bound of strong geodetic problem for Cartesian product of two graphs is given as follows: *If G and H are graphs, then* $sg(G \square H) \leq \min\{sg(H)o(G) - sg(H) + 1, sg(G)o(H) - sg(G) + 1\}$. Later in [3], the bound is improved and is given as: *If G and H are graphs, then* $sg(G \square H) \leq \min\{sg(H)o(G) - o(H) + 1, sg(G)o(H) - o(G) + 1\}$. We give a new bound for $sg(G \square H)$ using strong linear geodetic numbers of G and H.

Theorem 10. *For connected graphs G and H,*

$$sg(G\square H) \leq \min\left\{sg_l(H)+(o(G)-1)\left\lfloor\frac{sg_l(H)}{2}\right\rfloor, sg_l(G)+(o(H)-1)\left\lfloor\frac{sg_l(G)}{2}\right\rfloor\right\}.$$

Proof. Let $o(G)=n$, $o(H)=m$ and $V(G\square H)=\{(u_i,v_j)\backslash u_i \in V(G), v_j \in V(H)\}$, where $(i=1,2,\dots n)$ and $(j=1,2,\dots m)$. Let m layers of G in $G\square H$ be: $G^1,G^2\dots G^m$. Let $P \subseteq V(G)$ where $P=\{g_1,g_2,\dots g_k\}$ be the strong linear geodetic set of G. Define a set $S=\{(g_i,h)\backslash i=1,2,\dots k, h \in V(H)\}$. Clearly, $\widetilde{I}(S)=V$. Let $v \in V(H)$. Define $S'=\{(g_i,v) i=1,2,\dots k\} \cup \{(g_j,h), h \in V(H)\backslash v$ where $j=2,4,\dots k$ if $k \cong 0 \bmod 2$ or $j=2,4,\dots(k-1)$ if $k \cong 1 \bmod 2\}$. Let $(g,h) \in V(G\square H)$. If $h=v$ then $(g,h) \in \widetilde{I}(S_v)$ where $S_v \subseteq S$ and $S_v=\{(g_i,v)\backslash i=1,2,\dots k\}$. Now, if $h \neq v$ then there exists vertices g_i and g_{i+1} such that $(g,h) \in \widetilde{I}((g_i,h),(g_{i+1},h))$. Without loss of generality assume $i+1$ is odd. Consider the shortest path $(g_i,h) \to (g_{i+1},h) \to (g_{i+1},v)$. Then it is straightforward to see that $(g_i,h) \in \widetilde{I}((g_i,h),(g_{i+1},v))$. This implies $I(S)=I(S')$. Thus S' forms a strong geodetic for $G\square H$. Therefore, $sg(G\square H) \leq sg_l(G)+(o(H)-1)\lfloor\frac{sg_l(G)}{2}\rfloor$. By symmetry, $sg(G\square H) \leq sg_l(H)+(o(G)-1)\lfloor\frac{sg_l(H)}{2}\rfloor$. Thus for connected graphs G and H, $sg(G\square H) \leq \min\left\{sg_l(H)+(o(G)-1)\lfloor\frac{sg_l(H)}{2}\rfloor, sg_l(G)+(o(H)-1)\lfloor\frac{sg_l(G)}{2}\rfloor\right\}$. □

Definition 3 (Strong shortest path union cover, [18])**.** Let $G(V,E)$ be a connected graph. For a fixed vertex $u \in V(G)$, let $I_{u,v}$ be the selected fixed shortest path between u and v denoted by $\widetilde{P}_{u,v}$. A set $S \subseteq V(G)$ is a strong shortest path union cover of G if for all $e \in E(G)$ there exists $u \in S$ such that e lies on a fixed $\widetilde{P}_{u,v}$ for some $v \in V(G)$. The cardinality of the minimum strong shortest path union cover set is the strong shortest path union cover number denoted by $SSP_uC(G)$.

Theorem 11. *For connected graphs G and H, $sg_e(G\square H) \leq \min\{SSP_uC(H)o(G)+(o(H)-SSP_uC(H))(sg_e(G)-1), SSP_uC(G)o(H)+(o(G)-SSP_uC(G))(sg_e(H)-1)\}$.*

Proof. Let $o(G)=n$, $o(H)=t$. Then $V(G\square H)=\{(g,h)\backslash g \in V(G), h \in V(H)\}$. Let t layers of G in $G\square H$ be: $G^1,G^2\dots G^t$. Let $T \subset V(H)$ as $T=\{h_1,h_2,\dots h_k\}$ be a minimum strong shortest path union cover set of H and and $T^c=V(H)\backslash T$. Also, let $A=\{g_1,g_2,\dots g_m\}$ be a minimum strong edge geodetic set for G and consider $B=\{g_1,g_2,\dots g_{m-1}\}$. Now, we will prove that $S=\{(g,h_i)\backslash g \in V(G), h_i \in T\} \cup \{(g_j,h)\backslash g_j \in A, h \in T^c\}$ forms a strong edge geodetic cover for $G\square H$. Let $e \in E(G\square H)$. Suppose $e=(gh)(gh')$ of $G\square H$. Since T is a strong shortest path union cover set for H, there exists $h_i \in S$ such that the edge hh' lies on the selected fixed shortest path $\widetilde{P}_{h_iv}$ for some $v \in V(H)$. Consider the vertex (g,v) in (G^v) layer and let $S_v \subseteq S$ where $S_v=\{(g_j,v), j=1,2,\dots m\}$ is a strong edge geodetic set for the G^v layer. But it is also a strong geodetic set. Therefore, there exists $(g_r,v),(g'_r,v) \in S_v$ such that the vertex (g,v) lies on the selected fixed geodesic between (g_r,v) and (g'_r,v). Now clearly, $e=(gh)(gh')$ lies on the fixed geodesic $(g,h_i) \to (g,v) \to (g_r,v)$. Suppose e lies on some G_k of $G\square H$. Clearly, $S_k \subset S$ where $S_k=\{(g_i,v_k)\backslash i=1,,\dots m, v_k \in V(H)\backslash T\}$ is a strong geodetic set for G_k layer. Then there exist vertices (g_r,v_k) and (g_s,v_k) where $r,s=1,2,\dots m$ such that e lies on a unique fixed shortest path between them. Suppose $S_k=\{(g,v_k)\backslash g \in V(G), v_k \in T\}$, then $e \in \widetilde{I}(S_k)$. Therefore S is strong edge geodetic set for $G\square H$. Consider, $S'=\{(g,h_i)\backslash g \in V(G), h_i \in T\} \cup \{(g_j,h)\backslash g_j \in B, h \in T^c\}$. Suppose

$e' = (g_m,h')(g_m,h'') \in E(G\square H)$. Since the edge $h'h'' \in E(H)$ there exists $h_j \in T$ and $u \in V(H)$ such that $h'h'' \in \widetilde{P}_{h_j,u}$. Clearly, $e' \in \widetilde{I}((g_m,h_j),(g_m,u))$. Since $S^u = (g_1,u),(g_2,u),\ldots,(g_m,u)$ is a strong geodetic set G^u layer in $G\square H$, there exists a geodesic between (g_k,u) and (g_m,u) where $1 \le k \le m-1$. It is straightforward to see that e' lies on the fixed geodesic $(g_m,h_j) \to (g_m,u) \to (g_k,u)$. Suppose $e' \in (g_s,h)-(g_m,h), 1 \le s \le m-1$ geodesic in G^h layer. Since $h \in V(H)$, there exits an edge hx, $x \in V(H)$. Also, since T is a strong shortest path union cover set of H, there exits $h_l \in T$, $1 \le l \le k$ and $z \in V(H)$ such that $hx \in \widetilde{P}_{h_l,z}$. Clearly, $(g_m,h_l) \in S'$ and e' lies on the fixed geodesic $(g_m,h_l) \to (g_m,h) \to (g_s,h)$. Thus $\widetilde{I}(S) = \widetilde{I}(S')$ and S' is strong edge geodetic set for $G\square H$. This implies that $sg_e(G\square H) \le SSP_uC(H)o(G)+(o(H)-SSP_uC(H))(sg_e(G)-1)$. By symmetry, $sg_e(G\square H) \le SSP_uC(G)o(H)+(o(G)-SSP_uC(G))(sg_e(H)-1)$. Thus for connected graphs G and H, $sg_e(G\square H) \le \min\{SSP_uC(H)o(G)+(o(H)-SSP_uC(H))(sg_e(G)-1), SSP_uC(G)o(H)+(o(G)-SSP_uC(G))(sg_e(H)-1)\}$. □

Remark 1. In the above theorem the strong shortest path union cover number cannot be replaced by the shortest path union cover number.

Definition 4 ([15])**.** Let G be a connected graph. An ordered set $S = \{u_1,u_2,\ldots u_k\}$ of vertices of G is a linear edge geodetic set of G if for each edge $e = xy$ in G, there exists an index i, $1 \le i < k$ such that e lies on $u_i - u_{i+1}$ geodesic in G and the minimum cardinality of the linear edge geodetic set is the linear geodetic number.

Definition 5. Consider a graph G with an ordered set of vertices of G, $S = \{x_1,x_2,\ldots x_k\}$. Then S is a strong linear edge geodetic if for every edge $e \in E(G)$, there exists an index i, $1 \le i < k$ such that e lies on unique fixed $x_i - x_{i+1}$ geodesic. The minimum cardinality of a strong linear edge geodetic set is called the strong linear edge geodetic number, denoted as $sg_{le}(G)$.
A graph G is called strong linear edge geodetic graph,if G has a strong linear edge geodetic set. The graphs in Figure 2.4 and Figure 2.5 are strong linear edge geodetic graphs.

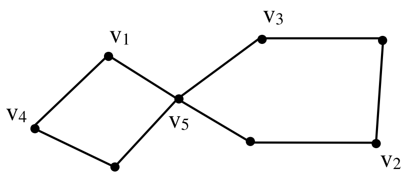

Figure 2.4. Strong linear edge geodetic graph

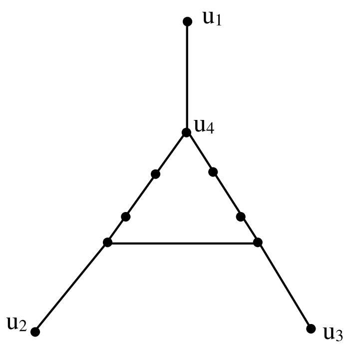

Figure 2.5. Strong linear edge geodetic graph

A graph may have linear edge geodetic set but not strong strong linear edge geodetic set (refer Figure 2.6).

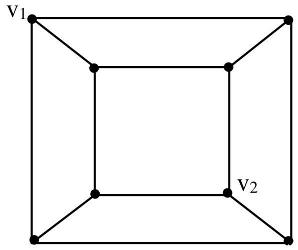

Figure 2.6. A graph G with linear edge geodetic set $\{v_1, v_2\}$ but no strong linear edge geodetic set

Theorem 12. *For connected strong linear edge geodetic graphs G and H,*

$$sg_e(G\Box H) \le \min\left\{SSP_uC(H)o(G) + (o(H) - SSP_uC(H))\left\lfloor \frac{sg_{le}(G)}{2}\right\rfloor,\right.$$
$$\left. SSP_uC(G)o(H) + (o(G) - SSP_uC(G))\left\lfloor \frac{sg_{le}(H)}{2}\right\rfloor\right\}.$$

Proof. Let $o(G) = n$, $o(H) = m$. Then $V(G\Box H) = \{(g,h)\backslash g \in V(G), h \in V(H)\}$. Let m layers of G in $G\Box H$ be: $G^1, G^2 \dots G^m$. Let $T \subset V(H)$ as $T = \{h_1, h_2, \dots h_k\}$ be a minimum strong shortest path union cover set of H and and $T^c = V(H)\backslash T$. Also, let $A = \{g_1, g_2, \dots g_k\}$ be a minimum strong linear edge geodetic set for G and consider a set $B = \{g_2, g_4, \dots g_j$ where $j = m$ if $m \cong 0 \bmod 2$ or $j = (m-1)$ if $m \cong 1 \bmod 2\}$. The set $S = \{(g,h_i)\backslash g \in V(G), h_i \in T\} \cup \{(g_j,h)\backslash g_j \in A, h \in T^c\}$ forms a strong edge geodetic set for $G\Box H$ (by first part of Theorem 11). Let $S' = \{(g,h_i)\backslash g \in V(G), h_i \in T\} \cup \{(g_j,h)\backslash g_j \in B, h \in T^c\}$. The proof of $\widetilde{I}(S) = \widetilde{I}(S')$ is similar to that of Theorem 10. Therefore, we omit the proof. □

Theorem 13. *For complete graphs K_m and K_n, where $m \ge n \ge 2$, $sg_e(K_m\Box K_n) = mn - n$.*

Proof. For any connected graph G, $g_1(G) \le sg_e(G)$. From [14], $g_1(K_m\Box K_n) = mn - n$. Thus $sg_e(G) \ge mn - n$.

Define n layers of K_m in $K_m\Box K_n$ as follows: $K_m^1, K_m^2, K_m^3, \dots K_m^n$. Let $S = V(K_m) = \{g_1, g_2, \dots g_m\}$ and $T = V(K_n) = \{h_1, h_2, \dots h_n\}$. Define the set $T = S \times T\backslash\{(g_i,h_i)\}$ where $i = 1, 2, \dots n$. We prove that T forms a strong edge geodetic set for $K_m\Box K_n$. The set T contains $(m-1)$ vertices from each n layers of K_m in $K_m\Box K_n$. The geodesics between these $(m-1)$ vertices in K_m^i, $1 \le i \le n$ will leave the edges having one endpoint at (g_i,h_i) uncovered. But the diameter of $K_m\Box K_n$ is equal to 2. Thus these uncovered edges can be covered by the unique fixed geodesic between $(g_l,h_i) \to (g_i,h_k)$ where $i \ne l \ne k$. Hence T forms a strong edge geodetic set for $K_m\Box K_n$. This implies that $sg_e(G) \le mn - n$. Thus for complete graphs K_m and K_n where $m \ge n \ge 2$, $sg_e(K_m\Box K_n) = mn - n$. □

Consider $K_n\Box P_m$. Let $K_n^1, K_n^2, \dots, K_n^m$ be the m layers of K_n. Let S be the strong edge geodetic set of $K_n\Box P_m$. Observe that each geodesic between the vertices of S can cover at most one edge of K_n^i, $1 \le i \le m$. Suppose $(k,h),(k,h') \in S$, then the geodesic between $(k,h)$and (k,h') where $k \in K_n$; $h,h' \in P_m$ will not cover any edge of K_n^i, $1 \le i \le m$. If $|S| = nr$, $1 < r < m$, then the geodesic between the vertices of S can cover at most $\binom{rn}{2} - n\binom{r}{2}$ edges of $K_n^i, 1 \le i \le m$. Therefore $m\binom{n}{2} \le \binom{rn}{2} - n\binom{r}{2}$. On simplification, $m \le r^2$. By taking S as the vertices of $r-$ layers of K_n $(1,4,9,\ldots,r^2$ layers of $K_n)$, it is straightforward to see that S is a strong edge geodetic set of $K_n \square P_{r^2}$. Therefore, we have the following theorem:

Theorem 14. *For $K_n \square P_{r^2}$, $sg_e(K_n \square P_{r^2}) = nr$, $r \ge 2$.*

Corollary 5. *If $r^2 \le m \le (r+1)^2$, then $nr \le sg_e(K_n \square P_m) \le n(r+1)$.*

Theorem 15. *For two positive integers a,b where $a \le b \le 2a-3$ and $a \ge 2$, there exists a connected graph G with $sg(G) = a$ and $sg_e(G) = b$.*

Proof. For $G = K_{1,n}$, $sg = a = sg_e$. If $a < b \le 2a-3$, let G be a graph obtained from $P4 : yv_1xu_a$. Add new vertices $u_1, u_2, \ldots, u_{a-1}$ and join each u_i to the vertex y. Also, add vertices v_i and w_j where $2 \le i \le 2a-b-1$ and $1 \le j \le b-a$. Now in G each v_i and w_j is joined to both y and x. Also, the vertices $w_1, w_2, \ldots, w_{b-a}$ induces a complete graph on b-a vertices (refer Figure 2.7). Thus, the graph G is obtained. It is clear that $S = \{u_1, u_2, \ldots, u_{a-1}, u_a\}$ forms a strong geodetic set for G. But S does not form a strong edge geodetic set for G. It can be easily seen that $T = S \cup \{w_1, w_2, \ldots, w_{b-a}\}$ forms a strong edge geodetic set for G. Thus, $sg = a$ and $sg_e = b - a + a - 1 + 1 = b$. □

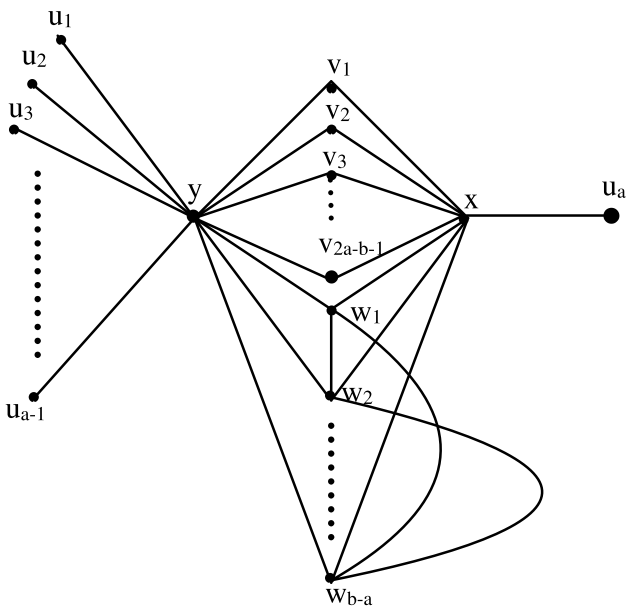

Figure 2.7. Graph G with $sg(G) = a$ and $sg_e(G) = b$

The strong geodetic problem was first studied by Manuel *et al.* in [11] and it is proved to be NP-complete for general graphs. Later, in [7], the computational complexity for strong geodetic problem is given for bipartite and multipartite graphs where it is proved to be NP-complete for both bipartite and multipartite graphs. We prove that the strong geodetic problem is NP-complete for chordal graphs. The proof of NP-completeness of strong geodetic problem for chordal graphs is by polynomial reduction from dominating set problem.

Theorem 16. *Strong geodetic problem for chordal graphs is NP-complete.*

Proof. Given a chordal graph $G(V,E)$, construct $\bar{G}(\bar{V},\bar{E})$ as follows: $\bar{V} = V \cup V' \cup V'' \cup \{a\}$ where V' is a set of independent vertices and V'' induces another set of independent vertices disjoint from V'. The edge set of $\bar{G}$ is $\bar{E} = E \cup E' \cup E''$ where $E' = \{uu' \backslash u \in V, u' \in V'\} \cup \{u'u'' \backslash u' \in V', u'' \in V''\}$ and $E'' = \{ua \backslash u \in V\} \cup \{u'a \backslash u' \in V'\}$. Clearly $\bar{G}$ is a chordal graph.

Since the vertices of V'' forms a set of simplicial vertices and they are the elements of any strong geodetic set in $\bar{G}$. Let D be any domination set of G. We will prove that $D \cup V''$ forms a strong geodetic set for $\bar{G}$. Consider the geodesics between a vertex $y'' \in V''$ and $x \in D$. If $x'' = y''$, then there exists unique geodesic $y'' - x$. Suppose $y \in N(x)$, then there can be two different $y'' - x$ geodesics each of length 3. The $y'' - x$ geodesics can be $y''y'yx$ or $y''y'ax$. Clearly, these geodesics cover all vertices of $\bar{G}$ in a unique fixed path. Thus $D \cup V''$ forms a Strong geodetic set for $\bar{G}$.

Conversely, assume that $D \cup V''$ is a strong geodetic set of $\bar{G}$. Let $D' = D \backslash V'$, where $D' \subset V(G)$. Consider $u' \in V' \cap D$ and $x \in D'$. It is straightforward to see that, the vertices covered by the geodesic $u' - x$ can be covered by the $(u'' - x)$ geodesic. Therefore, the vertices of $V' \cap D$ are redundant. This implies that $D' \cup V''$ is a strong geodetic set for $\bar{G}$. Clearly, D' is a dominating set for G. □

3. Conclusion

This paper contains some general bounds for strong edge geodetic problem, strong edge geodetic problem for specific graph classes and bounds for corona and Cartesian product of graphs. In future, we hope to study the strong edge geodetic problem for other graph products like lexicographic product, strong product and rooted product. Further, the strong edge geodetic problem for some networks can also be found.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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